Space Adaptive Finite Element Methods for Dynamic Signorini Problems in the Simulation of the NC-Shape Grinding Process

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Abstract

A simulation of the NC-shape grinding process of free formed surfaces is considered. A central issue in the simulation is the numerical solution of dynamic Signorini problems. Here, space adaptive techniques for such problems are discussed. The Newmark method in time and low order finite elements in space are used for discretisation. For the global discretisation error in space, an a posteriori error estimate is derived on the basis of the semi-discrete problem in mixed form. This approach relies on an auxiliary problem, which takes the form of a variational equation. An adaptive method based on the estimate is applied to improve the finite element approximation. Numerical results illustrate the performance of the presented method.

Keywords
A Posteriori Error Estimation, Dynamic Signorini Problem, Finite Element Method, Mesh Refinement, NC-Shape Grinding

1 INTRODUCTION

Shape grinding with toroid grinding wheels is favoured for achieving high material removal rates while grinding free formed surfaces. The contact area between grinding wheel and workpiece is therefore complex and varying. Without detailed knowledge about the contact area, which is influenced by many factors, the shape grinding process cannot be performed optimally. To improve this flexible production process and in order to ensure a suitable process strategy, we are developing a simulation tool, which contains a geometric-kinematic process simulation and a finite element simulation. In the geometric-kinematic simulation, a process force model is provided, which is based on the contact surface between the grinding-wheel and the workpiece. Using a dexel model of the workpiece, this simulation calculates the material-removal and reproduces the movement of the grinding-wheel and the workpiece according to the NC data. In the finite element simulation, the grinding-wheel and the spindle are explicitly included. The remaining parts of the machine are modelled by bearings. Since only small deformations are observed, a linear elastic material law is assumed. The process machine interactions are included by contact conditions, leading to a dynamic Signorini problem. The two parts of the simulation are coupled in such a way, that the displacements of the grinding-wheel determined by the finite element simulation are transferred to the geometric-kinematic simulation, which calculates the removal on the basis of these displacements. A detailed presentation of the simulation approach and the engineering process is found in [1]. In this note, we will focus on the adaptive discretisation of the dynamic Signorini problem.

In the simulation, we have to take into account that the workpiece interacts with the grinding wheel only in a small contact zone. However, the behaviour of the grinding machine is strongly affected by the resulting contact forces. For a reliable simulation, a precise prediction is required of the contact forces, the contact zone and their effects onto the whole body. Furthermore, the contact zone and the contact forces are strongly depending on time. Hence, the precise consideration of these dependences is essential in the numerical simulation. An adequate technique, which gives rise to a flexible and efficient finite element discretisation, is based on a posteriori error control and resulting adaptive mesh refinement. To integrate the adaptive refinement into the grinding simulation, we estimate the error by using the data of the current time step, only. The time step size is prescribed by the geometric-kinematic simulation. Thus, adaptivity in time is not needed. We use finite differences in time and finite elements in space to discretise the dynamic Signorini problems. The aim of this article is to derive an error estimator for the finite element discretisation in space direction. Therefore, we apply an error control technique for static contact problems, which goes back to Braess [2] and Schröder [3], to the semi discrete spatial problem. In [4,5] similar ideas are used to derive a posteriori error estimates for dynamic obstacle and dynamic simplified Signorini problems, which are model problems for the problem under consideration. In these articles, adaptive refinement techniques based on the a posteriori error estimates are described. For a further survey concerning the discretisation of dynamic contact problems and adaptive methods see [4,5] and the references therein.

The article is organised as follows: In the next two sections, the strong and weak formulations of the dynamic Signorini problem are introduced and the discretisation of the problem is discussed. In Section 4, the spatial error estimator is derived. Then, an example illustrates the application of the mentioned techniques. The article concludes with a discussion of the results.

2 CONTINUOUS FORMULATION

In this section, we present the strong and the weak formulation of the dynamic Signorini problem. The domain Ω is assumed to be a subset of $\Re^3$ and $I := [0,T] \subset \Re$ is the time interval. The boundary of Ω is given by $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_C \cup \Gamma_N$, where the decomposition has to be mutually disjoint. Homogeneous Dirichlet boundary conditions are prescribed on Γ_D and homogeneous Neumann boundary conditions on Γ_N. The contact takes place on Γ_C.

We assume a linear elastic material model with small deformations. The linearized strain operator is given by $2\varepsilon(u) := (\nabla u + \nabla u^T)$, where ∇u is the gradient of the displacement u in space direction. The first and second time derivatives are denoted by u' and u'', respectively. The stress operator of linear elasticity, which is defined by the modulus of elasticity E and by Poisson's number ν, is $\sigma(u)$. The density of the material is given by ρ. The functions u_0 and v_0 represent the initial displacement and the initial velocity, respectively. The unconstrained trial space is given by V. The precise definition of the function space may be found in [5].

For the description of the contact conditions, the techniques presented in [6], Chapter 2, are used. The time dependent gap function is denoted by g. The displacement in normal direction is given by $\delta_n(u)$. Here, the restriction $\delta_n(u) \le g$ is considered, $\delta_n(u) \ge g$ can be treated analogously. The set of admissible displacements is $K := \{\varphi \in V \,|\, \delta_n(\varphi) \le g \text{ on } \Gamma_C \times I\}$. The L^2-scalar product is defined by $(u,v) = \int_\Omega uv\,dx$. The function f represents the volume forces. Eventually, the weak formulation of the dynamic Signorini problem, see e.g. [7], reads

Problem 2.1

Find a function $u \in K$ with $u(t=0) = u_0$ and $u'(t=0) = v_0$, for which

$$(\rho u''(t), \varphi(t) - u(t)) + (\sigma(u(t)), \varepsilon(\varphi(t) - u(t))) \ge (f(t), \varphi(t) - u(t)) \tag{1}$$

holds for all $\varphi \in K$ and all $t \in I$.

If the solution is sufficiently smooth, we obtain the equivalent strong formulation (see [7])

$$\rho u'' - div(\sigma(u)) = f \quad \text{in } \Omega \times I \tag{2}$$

$$u = 0 \quad \text{on } \Gamma_D \times I \tag{3}$$

$$\sigma_{nn}(u) = 0 \quad \text{on } \Gamma_N \times I \tag{4}$$

$$\delta_n(u) - g \leq 0 \quad \text{on } \Gamma_C \times I \tag{5}$$

$$-\sigma_{nn}(u) \geq 0 \quad \text{on } \Gamma_C \times I \tag{6}$$

$$\sigma_{nn}(\delta_n(u) - g) = 0 \quad \text{on } \Gamma_C \times I \tag{7}$$

3 DISCRETISATION

We use Rothe's method to discretise the dynamic Signorini problem. First, the temporal direction is discretised by the Newmark method. The resulting spatial problems are approximately solved by low order finite elements.

3.1 Temporal Discretisation

The time interval I is split into N equidistant subintervals $I_n := (t_{n-1}, t_n]$ of length $k = t_n - t_{n-1}$ with $0 := t_0 < t_1 < ... < t_{N-1} < t_N := T$. The value of a function w at a time instance t_n is denoted by w^n . We use the notation $v = u'$ and $a = u''$ for the velocity and the acceleration, respectively. In the Newmark method, v and a are approximated by

$$a^n = \frac{1}{\beta k^2}\left(u^n - u^{n-1}\right) - \frac{1}{\beta k} v^{n-1} - \left(\frac{1}{2\beta} - 1\right) a^{n-1} \tag{8}$$

$$v^n = v^{n-1} + k\left[(1-\alpha) a^{n-1} + \alpha a^n\right] \tag{9}$$

Here, α and β are free parameters in the interval $[0, 2]$. For second order convergence, the parameter α has to be chosen as 0.5. Furthermore, the inequality $2\beta \geq \alpha \geq 0.5$ has to be valid for unconditional stability. For dynamic contact problems, the choice $\alpha = \beta = 0.5$ is recommended to guarantee conservation of energy and momentum (see [8]).

An efficient way for solving variational inequalities is given by their mixed formulation. Especially, the included Lagrange parameters may be interpreted as contact forces. Following the ideas of [5], the semi-discrete problem in mixed form reads as follows:

Problem 3.1

Find (u, λ) with $u^0 = u_0$ and $v^0 = v_0$ such that $(u^n, \lambda^n) \in V^n \times \Lambda^n$ is the solution of the system

$$c(u^n, \varphi) + \langle \lambda^n, \delta_n(\varphi) \rangle = (F^n, \varphi) \tag{10}$$

$$\langle \mu - \lambda^n, \delta_n(u^n) - g^n \rangle \leq 0 \tag{11}$$

for all $\varphi \in V^n$, all $\mu \in \Lambda^n$ and all $n \in \{1, 2, ..., N\}$.

Here, c is defined by $c(\omega,\varphi) := (\rho\omega,\varphi) + 0.5k^2(\sigma(\omega),\varepsilon(\varphi))$ and F^n as $F^n := 0.5k^2 f^n + u^{n-1} + kv^{n-1}$. The bilinear form c is uniformly elliptic, continuous and symmetric. The dual cone of the set $G^n := \{\mu \in H^{1/2}(\Gamma_C) \mid \mu \le 0\}$ is given by Λ^n. The dual pairing is expressed by $\langle \cdot,\cdot \rangle$.

3.2 Spatial Discretisation

A finite element approach is applied to discretise the mixed Problem 3.1. We use adaptive algorithms with dynamic meshes. Therefore, the trial spaces V_h^n and Λ_H^n may vary from time step to time step. Trilinear basis functions on the mesh T^n are used for the finite element space V_h^n. The discrete Lagrange multipliers are piecewise constant and are contained in the set Λ_H^n. The index H indicates that coarser meshes may be chosen for the Lagrange multipliers. In our calculations, we use $H = 2h$ for stability reasons. A detailed study of the stability properties of this discretisation can be found in [3].

If the meshes differ between two time steps, the data of the previous time step have to be transferred from the previous mesh to the current one. This is done by an L^2-projection and is denoted, below, by a prefix I_h. The transfer could also be done by interpolation, which needs less effort, but can lead to instabilities. The space and time discrete problem is

Problem 3.2

Find $(u_h^n, \lambda_H^n) \in V_h^n \times \Lambda_H^n$ with $u_h^0 = I_h u_0$ and $v_h^0 = I_h v_0$, such that the system

$$c(u_h^n, \varphi_h) + \langle \lambda_H^n, \delta_n(\varphi_h) \rangle \quad = (F_h^n, \varphi_h) \tag{12}$$

$$\langle \mu_H - \lambda_H^n, \delta_n(u_h^n) - g^n \rangle \quad \le 0 \tag{13}$$

is valid for all $\varphi_h \in V_h^n$ and $\mu_H \in \Lambda_H^n$, $n \in \{1, 2, ..., N\}$.

Here, F_h^n is given by $F_h^n := 0.5k^2 f^n + I_h u_h^{n-1} + kI_h v_h^{n-1}$. The system (12-13) can be rewritten as a quadratic optimisation problem, which can be solved, e.g., by SQP methods. More details can be found in [3].

4 SPATIAL ERROR ESTIMATION

In this section, an error estimate is derived for the spatial error in every time step. The estimation is easy to implement and can be evaluated fast. The temporal error is not considered. The idea of the error estimation goes back to Braess [2], who presented it for static obstacle problems. This idea was extended by Schröder [3] to static Signorini problems even with friction by introducing a general framework for error control of variational inequalities in Hilbert spaces. In order to apply this framework here, we consider the following saddle point problem:

Problem 4.1

Find $(\tilde{u}^n, \tilde{\lambda}^n) \in V^n \times \Lambda^n$, such that

$$c(\tilde{u}^n, \varphi) + \langle \tilde{\lambda}^n, \delta_n(\varphi) \rangle \quad = (F_h^n, \varphi) \tag{14}$$

$$\langle \mu - \tilde{\lambda}^n, \delta_n(\tilde{u}^n) - g^n \rangle \quad \le 0 \tag{15}$$

are valid for all $\varphi \in V^n$ and all $\mu \in \Lambda^n$.

An essential part of the general framework in [3] is the formulation of the auxiliary problem:

Problem 4.2

Find $u_*^n \in V^n$, such that the variational equation

$$c(u_*^n, \varphi) = (F_h^n, \varphi) - \langle \lambda_H^n, \delta_n(\varphi) \rangle \tag{16}$$

holds for all $\varphi \in V^n$.

Problem 4.2 corresponds to the first line of Problem 4.1, but with the discrete Lagrangian multiplier λ_H^n instead of $\tilde{\lambda}^n$. Applying Lemma IV.2 in [3] yields

Lemma 4.3

There are constants $C', C'' \in \Re_{>0}$, such that

$$\left\| \tilde{u}^n - u_h^n \right\|^2 + \left\| \tilde{\lambda}^n - \lambda_H^n \right\|^2 \le C' \left\| u_*^n - u_h^n \right\|^2 + C'' \left\langle \tilde{\lambda}^n - \lambda_H^n, \delta_n(u_h^n) - g^n \right\rangle. \tag{17}$$

Here, $\| \cdot \|$ denotes the norm corresponding to the related function spaces. We use the $H^1(\Omega)^3$-norm for V^n. Taking into account, that the discrete solution u_h^n is also a discrete solution of Problem 4.2, we are able to get rid of the term $\left\| u_*^n - u_h^n \right\|$ by using an appropriate error estimator for the auxiliary problem: Let $\eta_*^n > 0$ be an error estimator of Problem 4.2, i.e., there exists a constant $C_* > 0$ independent of V_h^n and Λ_H^n, such that $\left\| u_*^n - u_h^n \right\|^2 \le C_* (\eta_*^n)^2$ holds. In our calculations, we used a residual based error estimator, see, e.g., [9].

The estimation of the remaining term $\left\langle \tilde{\lambda}^n - \lambda_H^n, \delta_n(u_h^n) - g^n \right\rangle$ by the techniques presented in [5] leads to

Proposition 4.4

There exists a constant $C > 0$ independent of V_h^n and Λ_H^n, such that

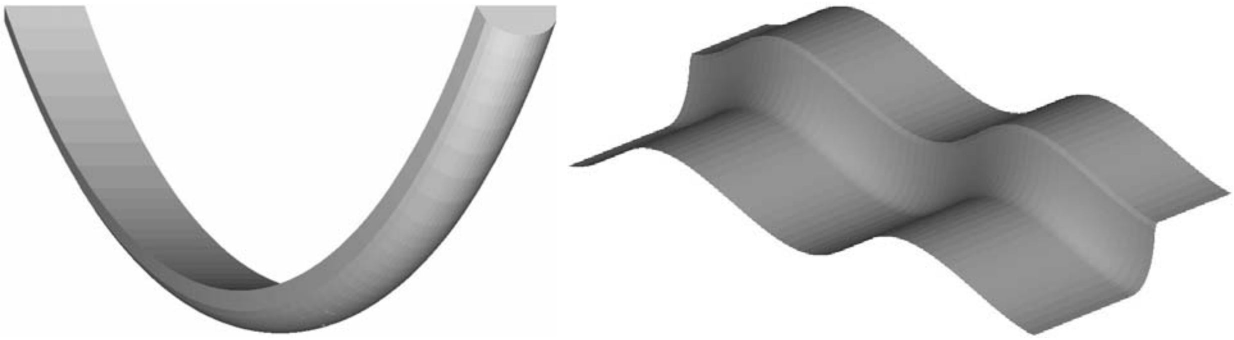
$$\left\| \tilde{u}^n - u_h^n \right\|^2 + \left\| \tilde{\lambda}^n - \lambda_H^n \right\|^2 \le C\Big((\eta_*^n)^2 + \left\| (g^n - \delta_n(u_h^n))_+ \right\|^2 + \left| \left(\lambda_H^n, (g^n - \delta_n(u_h^n))_+ \right) \right| \Big) \tag{18}$$

holds. Here, ω_+ denotes the positive part of a function ω.

All terms in the error estimate of Proposition 4.4 can be interpreted as typical sources of errors in contact problems. The term $\left\| (g^n - \delta_n(u_h^n))_+ \right\|$ measures the error of the geometrical contact condition and the term $\left| \left(\lambda_H^n, (g^n - \delta_n(u_h^n))_+ \right) \right|$ measures the violation of the complementarity condition.

In our numerical tests, the term $\left\| (g^n - \delta_n(u_h^n))_+ \right\|$ always turned out to be of higher order in h, see [3]. Since it is difficult to split this term into its element-wise contributions, it is neglected in the numerical realisation.

We have used the discrete value F_h^n instead of F^n in Problem 4.1, i.e., Proposition 4.4 specifies a temporally local error estimator for the spatial discretisation error. This technique is commonly used in the derivation of error estimators for numerical methods for ordinary differential equations, see, e.g., [10]. The presented error estimator expresses the spatial error distribution in the single time steps. But it only provides information about the global error under the assumption $F_h^n \approx F^n$, which should hold for small k and h. An a priori error analysis of the Newmark method in the context of dynamic contact problems is needed to make a precise statement. To the best of the authors' knowledge, this analysis does not exist and cannot be derived by standard techniques due to the low regularity of the continuous solution.



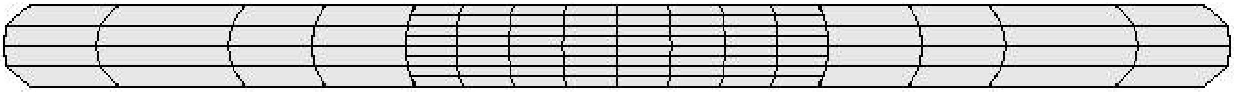
(a) Grinding-wheel (b) Workpiece

Figure 1: Geometry of the grinding-wheel and the workpiece

5 NUMERICAL RESULTS

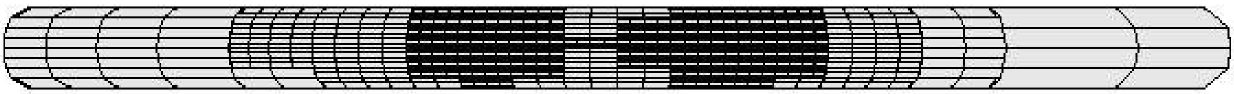
In this example, we concentrate on the decisive part of the grinding-wheel-spindle-system for the contact simulation, i.e., we only discretise one quarter of the outer grinding-wheel. In Figure 1(a), the domain is depicted. The radius of the grinding-wheel is $100\,\mathrm{mm}$ and the radius of the torus is $4.2\,\mathrm{mm}$. Homogeneous Dirichlet boundary conditions are assumed at both ends. The following material parameters are chosen: $\rho = 7.85\,\mathrm{kg/dm^3}$, $E = 2.1 \cdot 10^5\,\mathrm{MPa}$ and $\nu = 0.29$. In this example, we neglect the rotation of the grinding-wheel. The workpiece is shown in Figure 1(b), where the image was scaled by a factor of 5 for viewing purposes. It has a sinusoidal profile. The material removal is carried out line by line. Here, we simulate one line without calculating the removal of material. The horizontal and the vertical infeed are $0.5\,\mathrm{mm}$. The length of the workpiece is $100\,\mathrm{mm}$. We use a rate of feed of $7200\,\mathrm{mm/min}$. The time interval I is $[0,1]$. The adaptive algorithm presented in [5] is used for the refinement, where a fixed fraction strategy with $25\,\%$ refinement fraction und no coarsening is chosen. Three refinement cycles are performed. The time step length is $k = 0.005\,\mathrm{s}$.

In Figure 2, meshes for different time steps are shown. Furthermore, the number of Lagrange multipliers $M_H^n = \dim \Lambda_H^n$ is noted. The mesh in Figure 2(a) is the initial mesh. It is not refined, since no contact occurs in the first time steps and no outer forces are applied. Afterwards, the workpiece hits the grinding-wheel from the right side. The contact area is well resolved in Figure 2(b). The second refined zone corresponds to a region of great displacements due to the symmetric behaviour of the grinding-wheel. In the Figures 2(c) and (d), more and more cells are located in the left refined zone, as the first maximum of the workpiece passes the grinding-wheel. The grinding-wheel expands between the two maxima. Thus no contact occurs, which leads to the mesh presented in Figure 2(e). Then, the second maximum of the workpiece passes the grinding-wheel. When comparing the corresponding mesh sequences Figure 2(b)-(d) and Figure 2(f)-(h), one observes similar behaviour of the adaptive refinement as expected. In Figure 2(i), the last mesh is depicted. Since contact does not occur anymore, no extra refinement in the contact area is performed. The adaptive refinement is governed by the elastic expand of the grinding wheel. Summing up, we can state that the adaptive algorithm based on the presented error estimator resolves the contact zone in the expected manner and thus increases the accuracy.

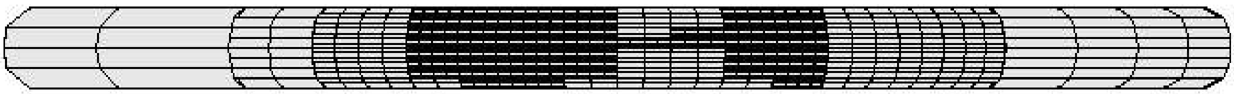


(a) Mesh at $n = 1$, $M_H^n = 20$

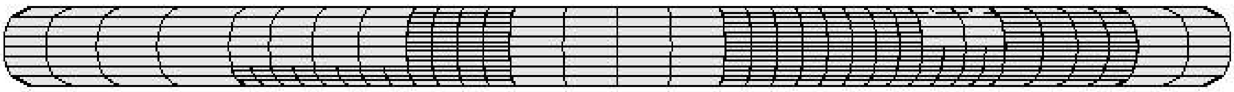
(b) Mesh at $n = 25$, $M_H^n = 263$



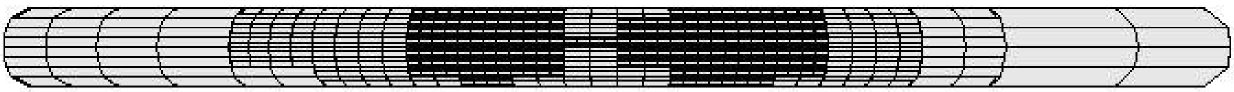
(c) Mesh at $n = 50$, $M_H^n = 263$



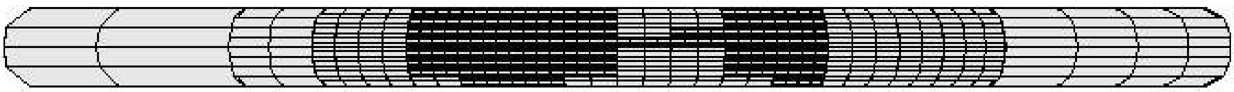
(d) Mesh at $n = 75$, $M_H^n = 257$



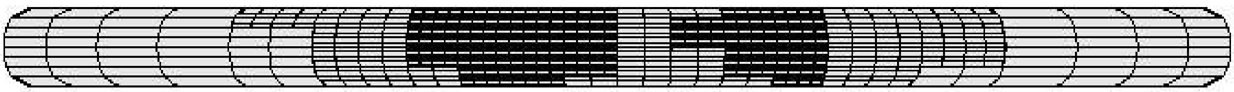
(e) Mesh at $n = 100$, $M_H^n = 71$



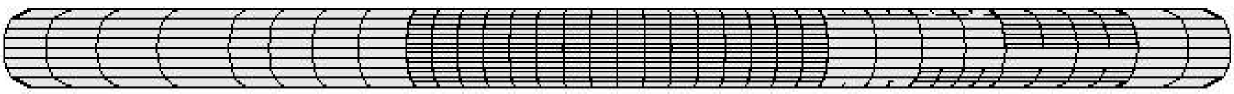
(f) Mesh at $n = 125$, $M_H^n = 263$



(g) Mesh at $n = 150$, $M_H^n = 257$



(h) Mesh at $n = 175$, $M_H^n = 257$



(i) Mesh at $n = 200$, $M_H^n = 71$

Figure 2: Meshes for different time steps of the dynamic Signorini example

6 CONCLUSIONS

The presented adaptive methods significantly improve the discretisation of dynamic Signorini problems. Especially, in the simulation approach described in the introduction for the NC-shape grinding process, they could lead to a significant improvement of the simulation results. The error estimate can be evaluated in every time step independent of the other time steps. Thus, no data have to be stored and the mesh can directly be modified. The integration of the error estimator into the grinding simulation is in preparation.

One has to accept several drawbacks, when using the presented technique: The error estimate provides no quantitative information about the error. Furthermore, the error is measured in the global H^1-norm. In view of the simulation of the NC-shape grinding process, measuring the error in terms of a user-defined functional, e.g. the mean value of the displacement on the contact boundary, is preferable. A way out could be the combination of goal oriented error estimators for static contact problems and the presented technique.

7 ACKNOWLEDGMENTS

This research work was supported by the German Research Foundation (Deutsche Forschungsgemeinschaft, DFG) within the Priority Program 1180 “Prediction and Manipulation of Interaction between Structure and Process”.

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