SPACE ADAPTIVE FINITE ELEMENT METHODS FOR DYNAMIC OBSTACLE PROBLEMS

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Abstract. The necessity to approximate dynamic contact problems arises in many engineering processes. Because of the local effects in the contact zone, adaptive techniques are suited to improve the finite element discretisation of such problems. In this article, the Newmark method in time and finite elements in space are used to approximate the solution numerically. A spatial error estimator is derived from the semi-discretised problem. The approach relies on auxiliary problem, which is a variational equation. An adaptive refinement process is based on this error control. Numerical results illustrate the performance of the presented method.

Key words. dynamic obstacle problem, a posteriori errror estimation, mesh refinement, finite element method

AMS subject classifications. 35L85, 65M50, 65M60

1. Introduction. Dynamic contact problems arise in many engineering processes, e.g. in milling and grinding processes, vehicle design and ballistics. The main effects in these processes result from the contact between the considered bodies. Dynamic obstacle problems provide a model problem for this kind of contact problems. In obstacle problems the contact takes place inside the domain; in contrast to Signorini problems, where the contact occurs on the boundary. The influence of the contact on the elastic domain is more direct in obstacle problems than in Signorini problems. Thus, errors in handling the contact are immediately seen in the approximated solution.
In dynamic contact problems, the contact zone depends on time. Hence, numerical simulations have to be adapted on the timedependent conditions to improve the error with minimal additional costs.

An adequate technique, which gives rise to a flexible and efficient finite element discretisation, is a posteriori error control and mesh refinement. In general, a posteriori error estimates for second order hyperbolic problems are based on two different discretisation approaches. One approach uses space time Galerkin methods to discretise the problems and applies similar techniques for error control as in the static case ([1, 2, 3, 4]). The other one is based on finite difference in time and finite elements in space. Here, separate error estimators are used for space and time direction ([5, 6, 7]) or error estimates for the whole problem ([8, 9]) are derived.
In this article, finite difference methods in time and finite elements in space are used to discretise the dynamic obstacle problems. Because only the data of the current time step comes into play the error estimator can be evaluated efficiently. However, the separation of time and space direction complicates the consideration of space time effects. The aim of this article is to derive an error estimator for the finite element discretisation of the space direction. Therefore, an error control technique for static contact problems is applied to the semi discrete spatial problem. This technique goes back to Braess [10] and Schröder [11]. Other approaches to a posteriori error control for static contact problems are discussed in [12, 13, 14, 15, 16, 17, 18, 19]. In particular, an adaptive scheme for two-body contact is contained in [20]. Convergence results for adaptive algorithms in the context of obstacle problems are proven in [21].

Adaptivity for the time direction is not taken into account in this article for notational simplicity, although it is easy to incorporate. One can do this on the basis of error estimators, which are known from the literature of second order hyperbolic problems [5].
The temporal discretisation of dynamic contact problems is difficult. A lot of approaches based

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on different problem formulations have been presented in literature. In [22] the Penalty-method is used to solve the discrete problems. Special contact elements in combination with Lagrange multipliers are presented in [23]. Other techniques for smoothing and stabilizing the computation with special finite elements, e.g. Mortar finite elements, are presented in [24, 25, 26]. In [27] an additional L^2 projection is used to stabilise the Newmark scheme. Algorithms for dynamic contact/impact problems based on the energy- and momentum conservation are derived in [28, 29]. An additive splitting of the acceleration into two parts, representing the interior forces and the contact forces, is the basis of the methods introduced in [30, 31]. In [32, 33, 34] algorithms based on variational inequalities and optimisation algorithms are presented. Detailed surveys of this topic can be found in the monographs [35, 36].

The article is organised as follows: In Section 2 the strong and the weak continuous formulation of the dynamic obstacle problem is discusssed. The discretisation with Rothe's method is developed by formulating the problem as a hyperbolic variational inequality. An error estimator is derived in Section 4. The next section is concerned with the practical issues of dynamic meshes. The error estimator and an adaptive algorithm is tested by an example in Section 6. The article concludes with some remarks on the presented method and further developments.

2. Continuous Formulation. In this section, the strong formulation and the weak formulation of the dynamic obstacle problem are presented. The domain Ω is assumed to be a subset of $\mathbb{R}^2$ and $I := [0,T] \subset \mathbb{R}$ is the time interval. Homogeneous Dirichlet boundary conditions are assumed for notational convenience. The functions u_0 and v_0 represent the start displacement and the start velocity, respectively. The right hand side is denoted by f. The time dependent obstacle is given by the function $\psi : \Omega \times I \to \mathbb{R} \cup \{-\infty\}$. Here, the restriction $u \geq \psi$ is considered, $u \leq \psi$ can be treated analogously.

The start displacement u_0 should be an element of $H_0^1(\Omega)$ and the start velocity v_0 should be an element of $L_0^2(\Omega)$. The right hand side f has to be contained in $L^\infty\left(I; L^2(\Omega)\right)$. For the precise definition of weak derivatives and the corresponding function spaces see [37]. The gradient of the displacement u in space direction is denoted by ∇u and Δu is the usual laplace operator. The first time derivative is denoted by $\dot{u}$ and the second one by $\ddot{u}$. In the following, all relations have to be understood almost everywhere.

The unconstrained trial space V is given by

$$V := W^{2,\infty}\left(I; L^2(\Omega)\right) \cap L^\infty\left(I; H_0^1(\Omega)\right).$$

It should be remarked, that the space V is choosen in this way to simplify the notation. In general, this definition is too strong to guarantee the existince of a solution [37].
The parametrisation of the obstacle ψ has to be contained in $L^\infty\left(I; L^2(\Omega)\right)$. The set of admissible displacements is

$$K := \{\varphi \in V \,|\, \varphi \geq \psi \text{ a.e. on } \Omega \times I\}.$$

The L^2 scalar product is defined by $(u,v) = \int_\Omega uv\,dx$ for $u, v \in L^2(\Omega)$. The weak formulation of the dynamic obstacle problem, which is similar to the formulation of dynamic Signorini problems (see e.g. [38]), reads

PROBLEM 2.1. *Find a function $u \in K$ with $u(t=0) = u_0$ and $\dot{u}(t=0) = v_0$ for which*

$$(\ddot{u}(t), \varphi(t) - u(t)) + (\nabla u(t), \nabla(\varphi(t) - u(t))) \geq (f(t), \varphi(t) - u(t))$$

holds for all $\varphi \in K$ and all $t \in I$.

If the solution is sufficiently smooth, we obtain the equivalent formulation (see [39])

$$\begin{aligned} \ddot{u} - \Delta u &\geq f \\ u - \psi &\geq 0 \\ (\ddot{u} - \Delta u - f)(u - \psi) &= 0. \end{aligned}$$

3. Discretisation. We use Rothe's method to discretise the dynamic obstacle problem. First, the temporal direction is discretised by an adequate time stepping scheme. Here, the Newmark method (see [40]) is applied. The obtained spatial problems are numerically solved by the finite element method.

3.1. Temporal Discretization. The time interval I is split into N equidistant subintervals $I_n := (t_{n-1}, t_n]$ of length $k = t_n - t_{n-1}$ with $0 =: t_0 < t_1 < \ldots < t_{N-1} < t_N := T$. The value of a function u at a time instance t_n is given by u^n. The velocity is given by $v = \dot{u}$. The acceleration a is the second time derivative of u.

In the Newmark method, v and a are approximated as:

$$\begin{aligned} a^n &= \frac{1}{\beta k^2}\left(u^n - u^{n-1}\right) - \frac{1}{\beta k} v^{n-1} - \left(\frac{1}{2\beta} - 1\right) a^{n-1}, \\ v^n &= v^{n-1} + k\left[(1-\gamma)\, a^{n-1} + \gamma a^n\right]. \end{aligned} \tag{3.1}$$

Here, β and γ are free parameters in the interval $[0, 2]$. For second order convergence, the parameter γ should be choosen as $\frac{1}{2}$. Furthermore, the inequality $2\beta \geq \gamma \geq \frac{1}{2}$ has to be valid for unconditional stability (see [41]). For dynamic contact problems, the choice $\gamma = \beta = \frac{1}{2}$ is recommanded (see [23, 34]). The semi discrete problem reads as follows:

PROBLEM 3.1. *Find u with $u^0 = u_0$ and $v^0 = v_0$, such that in every time step $n \in \{1, 2, \ldots, N\}$, the function $u^n \in K^n$ is the solution of the variational inequality*

$$\left(a^n, \varphi - u^n\right) + \left(\nabla u^n, \nabla\left(\varphi - u^n\right)\right) \geq \left(f^n, \varphi - u^n\right), \tag{3.2}$$

which must hold for all $\varphi \in K^n$.
The set $K^n := \left\{\varphi \in H_0^1(\Omega) \,|\, \varphi \geq \psi^n \text{ a.e. on } \Omega\right\}$ is the time discretised set of admissible displacements. Substituting the equation (3.1) in the inequality (3.2) leads to

$$\left(u^n, \varphi - u^n\right) + \frac{1}{2}k^2 \left(\nabla u^n, \nabla\left(\varphi - u^n\right)\right) \geq \left(\frac{1}{2}k^2 f^n + u^{n-1} + k\, v^{n-1}, \varphi - u^n\right).$$

This can be written as

$$c\left(u^n, \varphi - u^n\right) \geq \left(F^n, \varphi - u^n\right), \tag{3.3}$$

where c is defined by

$$c(\omega, \varphi) := (\omega, \varphi) + \frac{1}{2}k^2 \left(\nabla\omega, \nabla\varphi\right)$$

and F as

$$F^n := \frac{1}{2}k^2 f^n + u^{n-1} + k\, v^{n-1}.$$

The bilinearform c is uniformly elliptic and symmetric. Thus, an elliptic variational inequality has to be solved in every time step. An efficient way for solving variational inequalities is given by their mixed formulation. Especially, the included Lagrange parameters are interpretable as contact forces. The variational inequality (3.3) is equivalent to the following mixed problem:

PROBLEM 3.2. *Find (u, λ) with $u^0 = u_0$ and $v^0 = v_0$, such that $(u^n, \lambda^n) \in V^n \times \Lambda^n$ is the solution of the system*

$$c\left(u^n, \varphi\right) + \langle \lambda^n, \varphi \rangle = \left(F^n, \varphi\right) \tag{3.4}$$

$$\langle \mu - \lambda^n, u - \psi^n \rangle \leq 0 \tag{3.5}$$

which must hold for all $\varphi \in V^n$, all $\mu \in \Lambda^n$ and all $n \in \{1, 2, \ldots, N\}$.
Here, Λ^n is the dual space of the closed and convex cone $G^n := \left\{\mu \in H_0^1(\Omega) \middle| \, \mu \leq 0\right\}$. The dual pairing is expressed by $\langle \cdot, \cdot \rangle$.

The equivalence of the two formulations is a well-known conclusion from the general theory of minimisation problems in Hilbert spaces, which is presented e.g. in [42, 43].

3.2. Spatial Discretization. A finite element approach is applied to discretise the mixed problem 3.2. We use adaptive algorithms with dynamic meshes. Therefore, the trial spaces V_h^n and Λ_H^n vary from time step to time step. Bilinear basis functions on the mesh $\mathbb{T}^n$ are used for the finite element space V_h^n. The discrete Lagrange Multipliers are piecewise constant and are contained in the set Λ_H^n. The index H points out, that coarser meshes are used for the Lagrange Multipliers. Here, we use $H = 2h$. A detailed study of the stability properties of this discretisation can be found in [11].

If the meshes differ between two time steps, the data of the previous time step have to be transfered from the previous mesh to the current one. This is done by a L^2-projection and is marked in the formulas with a prefix I_h. The transfer can also be done by interpolation, which needs less effort, but can lead to instabilities. The space and time discrete problem is

PROBLEM 3.3. *Find* $(u_h^n, \lambda_H^n) \in V_h^n \times \Lambda_H^n$ *with* $u_h^0 = I_h u_0$ *and* $v_h^0 = I_h v_0$*, such that the system*

$$c\,(u_h^n, \varphi_h) + \langle \lambda_H^n, \varphi_h \rangle = (F_h^n, \varphi_h) \tag{3.6}$$

$$\langle \mu_H - \lambda_H^n, u_h - \psi^n \rangle \le 0 \tag{3.7}$$

is valid for all $\varphi_h \in V_h^n$*, all* $\mu_H \in \Lambda_H^n$ *and all* $n \in \{1, 2, \ldots, N\}$.
Here, F_h^n is given by

$$F_h^n := \frac{1}{2} k^2\, f^n + I_h u_h^{n-1} + k\, I_h v_h^{n-1}.$$

The system (3.6-3.7) leads to the following saddlepoint problem in $\mathbb{R}^m$:

$$A^n \bar{u}^n + B^n \bar{\lambda}^n = \bar{F}^n$$
$$\left(\bar{\mu} - \bar{\lambda}^n\right)^T \left(\left(B^n\right)^T \bar{u}^n - \bar{\psi}^n\right) \le 0,$$

which must hold for all $\bar{\mu} \in \Lambda^n \subset \mathbb{R}^{\tilde{m}}$. Here, $A^n := M^n + \frac{1}{2} k^2 K^n$ is the generalised stiffness matrix, $M^n \in \mathbb{R}^{m\times m}$ is the mass matrix and $K^n \in \mathbb{R}^{m\times m}$ is the stiffness matrix. The matrix $B^n \in \mathbb{R}^{m\times \tilde{m}}$ represents the dual pairing in (3.7). Notice, that all matrices can change from time step to time step.
The saddlepoint problem can be written as a quadratic optimisation problem, which can be solved e.g. by SQP methods, by substituting $\bar{u}^n := \left(A^n\right)^{-1}\left(\bar{F}^n - B^n \bar{\lambda}^n\right)$. More details can be found in [11].

4. Spatial Error Estimation. In this section, an error estimation is derived for the spatial error in every time step. The estimation is easy to implement and can be evaluated fast. The temporal error is not considered. The idea of the error estimation goes back to Braess [10], who presented it for static obstacle problems. This idea was extended by Schröder [11] to static Signorini problems even with friction by introducing a general framework for error control in Banach spaces. In order to apply this framework, we consider the following saddlepoint problem:

PROBLEM 4.1. *Find* $\left(\tilde{u}^n, \tilde{\lambda}^n\right) \in V^n \times \Lambda^n$*, so that*

$$c\,(\tilde{u}^n, \varphi) + \langle \tilde{\lambda}^n, \varphi \rangle = (F_h^n, \varphi)$$
$$\langle \mu - \tilde{\lambda}^n, \tilde{u}^n - \psi^n \rangle \le 0$$

is valid for all $\varphi \in V^n$ *and all* $\mu \in \Lambda^n$.

An intregal part of the general framework in [11] is the formulation of the auxiliary problem:

PROBLEM 4.2. *Find* $u_\star^n \in V^n$*, so that the variational equation*

$$c\,(u_\star^n, \varphi) = (F_h^n, \varphi) - \langle \lambda_H^n, \varphi \rangle$$

holds for all $\varphi \in V^n$.

Problem 4.2 corresponds to the first line of Problem 4.1, but with the discrete Lagrangian multiplier λ_H^n instead of $\tilde{\lambda}^n$. Applying Lemma IV.2 in [11] yields

LEMMA 4.1. *There are constants $C', C'' \in \mathbb{R}_{>0}$, so that*

$$\|\tilde{u}^n - u_h^n\|^2 + \left\|\tilde{\lambda}^n - \lambda_H^n\right\|^2 \leq C' \|u_\star^n - u_h^n\|^2 + C''\langle \lambda^n - \lambda_H^n, u_h - \psi^n\rangle.$$

Here, $\|\cdot\|$ denotes the norm correponding to the function spaces. We use the $H^1(\Omega)$-norm for V^n. Taking into account, that the discrete solution u_h^n is also a discrete solution of problem 4.2, we are able to get rid of the term $\|u_\star^n - u_h^n\|$ by using an appropriate error estimator for the auxiliary problem: Let $\eta_\star^n > 0$ be an error estimator of the Problem 4.2. I.e., there exists a constant $C_\star > 0$ independent of V_h^n and Λ_H^n, so that

$$\|u_\star^n - u_h^n\|^2 \leq C_\star (\eta_\star^n)^2.$$

Then, Lemma 4.1 leads to

$$\|\tilde{u}^n - u_h^n\|^2 + \left\|\tilde{\lambda}^n - \lambda_H^n\right\|^2 \leq C' C_\star (\eta_\star^n)^2 + C''\langle \lambda^n - \lambda_H^n, u_h - \psi^n\rangle.$$

The remaining term $\langle \tilde{\lambda}^n - \lambda_H^n, u_h^n - \psi^n\rangle$ is estimated by

LEMMA 4.2. *Let $\nu_0 > 0$ be the constant of continuity of c. Furthermore, let $d \in \tilde{K}^n := \{v \in V^n | \psi^n - u_h^n + v \geq 0\}$ and $\varepsilon > 0$. Then, there holds*

$$\langle \tilde{\lambda}^n - \lambda_H^n, u_h^n - \psi^n\rangle \leq \frac{\varepsilon}{2} \|\tilde{u}^n - u_h^n\|^2 + \frac{(1+\varepsilon)\nu_0^2}{2\varepsilon} \|d\|^2 + \frac{1}{2} \|u_\star^n - u_h^n\|^2 + |(\lambda_H^n, d)|.$$

Proof. Inserting 0 and $2\lambda_H^n$ in (3.7) yields $\langle \lambda_H^n, u_h^n - \psi^n\rangle = 0$. Furthermore, we get

$$\begin{aligned}
\langle \tilde{\lambda}^n, u_h^n - \psi^n\rangle &= -\langle \tilde{\lambda}^n, \psi^n - (u_h + d)\rangle - \langle \tilde{\lambda}^n, d\rangle \\
&\leq c(\tilde{u}^n, d) - (F_h^n, d) \\
&= c(\tilde{u}^n - u_h^n, d) + c(u_h^n, d) - (F_h^n, d) \\
&\leq \nu_0 \|\tilde{u}^n - u_h^n\| \|d\| + c(u_h^n, d) - (F_h^n, d) \\
&\leq \frac{\varepsilon}{2}\|\tilde{u}^n - u_h^n\|^2 + \frac{\nu_0^2}{2\varepsilon}\|d\|^2 + c(u_h^n, d) - (F_h^n, d).
\end{aligned}$$

Here, we use Young's inequality. The term $c(u_h^n, d) - (F_h^n, d)$ is estimated as follows:

$$\begin{aligned}
c(u_h^n, d) - (F_h^n, d) &= c(u_h^n - u_\star^n, d) - (\lambda_{0,H}, d) \\
&\leq \nu_0 \|u_\star^n - u_h^n\| \|d\| - (\lambda_{0,H}, d) \\
&\leq \frac{1}{2} \|u_\star^n - u_h^n\|^2 + \frac{\nu_0^2}{2} \|d\|^2 + |(\lambda_H, d)|.
\end{aligned}$$

□

Eventually, we obtain an a posteriori error estimation by the following proposition:

PROPOSITION 4.3. *There exists a constant $C > 0$ independent of V_h^n and Λ_H^n, so that*

$$\|\tilde{u}^n - u_h^n\|^2 + \left\|\tilde{\lambda}^n - \lambda_H^n\right\|^2 \leq C\left((\eta_\star^n)^2 + \left\|(\psi^n - u_h^n)_+\right\|^2 + \left|(\lambda_H^n, (\psi^n - u_h^n)_+)\right|\right)$$

holds. Here, f_+ denotes the positive part of a function f, which means

$$f_+(x) = \begin{cases} f(x), & f(x) \geq 0, \\ 0, & f(x) < 0. \end{cases}$$

Proof. Combining Lemma 4.1 and Lemma 4.2 yields

$$\begin{aligned}
&\|\tilde{u}^n - u_h^n\|^2 + \left\|\tilde{\lambda}^n - \lambda_H^n\right\|^2 \\
&\leq C'C_\star(\eta_\star^n)^2 + C''\langle\tilde{\lambda}^n - \lambda_H^n, u_h^n - \psi^n\rangle \\
&\leq (C' + \frac{1}{2}C'')C_\star(\eta_\star^n)^2 + C''\left(\frac{\varepsilon}{2}\|\tilde{u}^n - u_h^n\|^2 + \frac{(1+\varepsilon)\nu_0^2}{2\varepsilon}\|d\|^2 + |(\lambda_H^n, d)|\right).
\end{aligned}$$

Choosing $0 < \varepsilon < 2/C''$, we get

$$\begin{aligned}
&(1 - \frac{C''\varepsilon}{2})\|\tilde{u}^n - u_h^n\|^2 + \left\|\tilde{\lambda}^n - \lambda_H^n\right\|^2 \leq \\
&\qquad \max\{(C' + \frac{C''}{2})C_\star, \frac{C''(1+\varepsilon)\nu_0^2}{2\varepsilon}, C''\}((\eta_\star)^2 + \|d\|^2 + |(\lambda_{0,H}, d)|).
\end{aligned}$$

Since $\psi^n - u_h^n + (\psi^n - u_h^n)_+ \geq 0$, we set $d := (\psi^n - u_h^n)_+ \in \tilde{K}^n$ and complete the proof. □

Remark 4.4. *All terms in the error estimation of Proposition 4.3 are interpretable as typical sources of errors in contact problems. The term* $\|(\psi^n - u_h^n)_+\|$ *measures the error of the geometrical contact condition and the term* $|(\lambda_H^n, \psi^n - u_h^n)_+)|$ *measures the violation of the complementary condition.*

Remark 4.5. *The term* $\|(\psi^n - u_h^n)_+\|$ *is of higher order in* h*, which is shown by numerical experiments in [11]. Since the localisation of this term is difficult, it is neglected in the numerical realisation.*

In order to apply the error estimation of Proposition 4.3, we have to specify an appropriate error estimator $\eta_\star^n$ for the Problem 4.2. In principle, each error estimator known from literature of variational equations is possible to be used. See [44] or [45] for an overview. For the sake of completeness, a residual based error estimator for Problem 4.2 is specified:

$$\begin{aligned}
(\eta_\star^n)^2 &:= \sum_{K \in \mathbb{T}^n} \eta_K^2 \\
\eta_K^2 &:= h_K^2 \|r\|_{L^2(K)}^2 + h_K \|R\|_{L^2(\partial K)}^2
\end{aligned}$$

with

$$\begin{aligned}
r &:= F_h^n - \lambda_H^n + \frac{1}{2}k^2 \triangle u_h^n - u_h^n \\
R &:= -\frac{1}{2}k^2 \left[\frac{\partial u_h^n}{\partial \nu}\right].
\end{aligned}$$

The quantity R represents the jump discontinuity in the approximation to the normal flux on the interface. See [44] section 2.2 for more details.

Remark 4.6. *We have used the discrete value* F_h^n *instead of* F^n *in problem 4.1. I.e., proposition 4.3 provides a temporal local error estimator for the spatial discretisation error. This technique is commonly used in the derivation of error estimators for numerical methods for ordinary differential equations, see e.g. [46]. The presented error estimator expresses the spatial error distribution in the single time steps. But it provides only informations about the global error under the assumption* $F_h^n \approx F^n$*, which should hold for small* k *and* h*. An a priori error analysis of the Newmark method in the context of dynamic contact problems is needed to make a precise statement. To the best of the authors' knowledge, this analysis does not exist and cannot be derived by standard techniques due to the low regularity of the analytic solution.*

5. Adaptive Algorithm. In general, adaptive algorithms for dynamic problems are based on refinement strategies, which are known from static problems, see e.g. [45, 47]. Commonly used adaptive algorithms for time dependent problems, see e.g. [2, 48], perform an adaptive refinement process using a prescribed tolerance in every time step. This refinement process is independent of previous and following time steps. Here, the crucial point is, that the time interval is passed only once. The tolerance cannot be reached, if the solution in the previous time step has not been calculated exactly enough. Moreover, the difference of the meshes of two succesive time steps may lead to a significant increase of the error. Usually, rapid changes of the problem parameters are the reason for this behaviour.

In dynamic obstacle problems, the problem parameters changes rapidly. Hence, the mentioned algorithms are not appropriate. An alternative is given by algorithms based on the ideas in [49, 50]. The refinement procedure is split into several cycles. The whole time interval is passed in every cycle. A cycle constists of two steps: In the first step, the approximated solution of whole problem is determined, the error is estimated and the mesh is refined with a usual refinement strategy, e.g. a fixed fraction strategy, see [45, 47]. Multiple hanging nodes in space and time may be generated by this refinement. In a second step, they are removed. Furthermore, it has to be ensured, that the mesh has patch structure. We say, a 2D mesh has patch structure, if four adjacent mesh cells, which have the same size, can be combined to a patch or macro element. The patch structure allows a reasonable coarsening of the mesh. In particular, no instability effects, as checkerboard patterns, are observed in numerical experiments. The removal of hanging nodes in time closely connects the meshes of different time steps. A detailed presentation of this adaptive algorithm and its extensions will be given in [51].

6. Numerical Results. The error estimator and the adaptive algorithm based on it are tested with the following example. We set $\Omega := [0,1]^2$ and $I := [0, 0.03125]$. The start displacement and the start velocity are

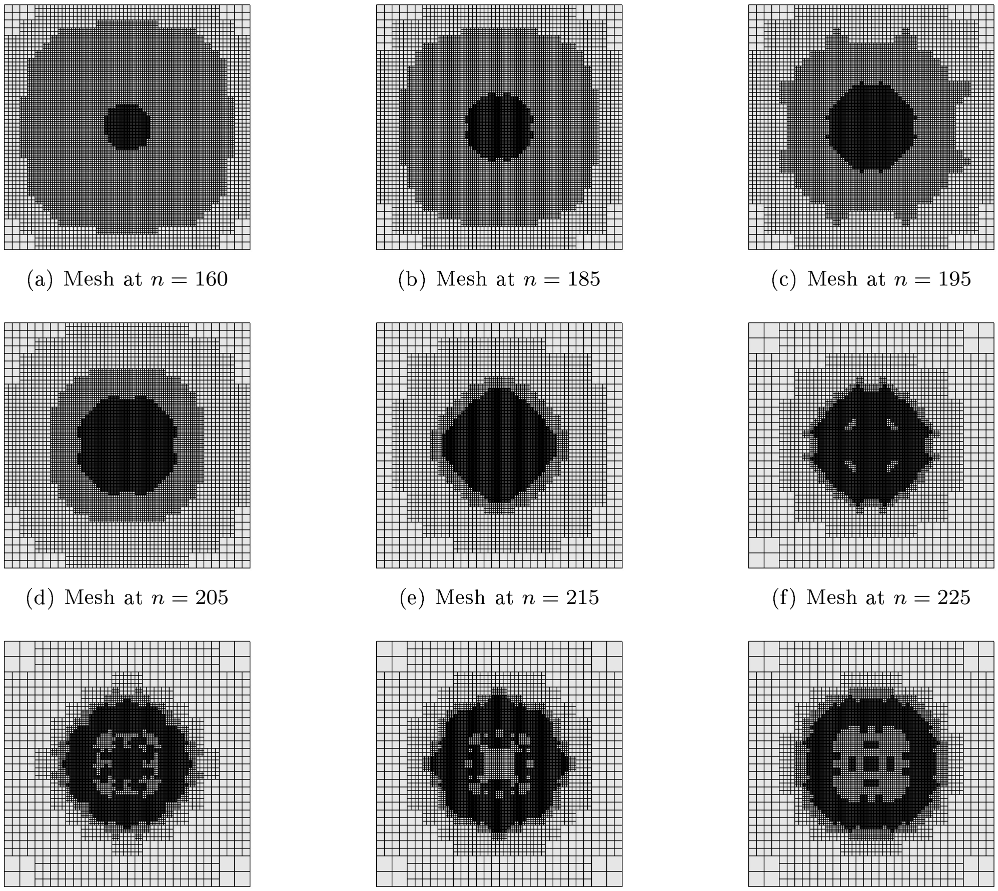
$$\begin{aligned} u_0(x_1,x_2) &:= 0, \\ v_0(x_1,x_2) &:= -\sqrt{2}\pi \sin(\pi x_1)\sin(\pi x_2). \end{aligned}$$

The obstacle ψ is a constant function $\psi := -0.1$. The length of the time steps k is choosen as $1.25 \cdot 10^{-4}$ and the start mesh size h_0 as 0.0625. Five refinement cycles are performed, where a fixed fraction strategy with a constant refinement fraction of 50% without coarsening is used. Meshes of different time steps are displayed in Figure 6.1. Figure 6.1 (a) shows the mesh before the first contact between the membrane and the obstacle takes place. In Figure 6.1 (b), the contact zone is a circle in the middle of the membrane. It moves outside and the membrane is reflected in the middle. One observes, that the meshes are refined in the outer contact zone and in areas, where the membrane is vibrating. In Figure 6.2, the estimated error of the discretisation with adaptive refined meshes is compared to the estimated error of the discretisation with globally refined meshes. The maximum estimated global error of the whole time interval

$$\eta = \max_{1 \le n \le N} \eta^n$$

is displayed on the y-axis and the sum over all number of degrees of freedom is shown on the x-axis. The adaptive refined discretisation only needs a quarter of the unknowns to achieve the same accuracy as the globally refined one.

7. Conclusions. The presented space adaptive scheme for dynamic obstacle problems shows a significant improvement in the numerical experiments. More sophisticated refinement strategies can further reduce the number of unknowns needed to reach a certain tolerance. However, not every strategy seems to be suited for adaptive schemes. E.g. the refinement strategy presented in [50], where the refinement indicators are compared over all time steps, has been tested. The results are not satisfactory. The contact zone is not refined before the first contact. The algorithm is not able to detect the moment of the first contact exactly, which increases the error significantly.
An other method to reduce the number of unknowns is given by time adaptive discretisations;



(a) Mesh at $n = 160$ (b) Mesh at $n = 185$ (c) Mesh at $n = 195$

(d) Mesh at $n = 205$ (e) Mesh at $n = 215$ (f) Mesh at $n = 225$

(g) Mesh at $n = 230$ (h) Mesh at $n = 240$ (i) Mesh at $n = 250$

Fig. 6.1. *Meshes for different time steps*

using error estimators for the Newmark method, e.g., as presented in [5]. This will be considered in future works.

The difficulties discussed in Remark 4.6 and the separation of the spatial and temporal discretisation complicate the deriving of rigorous a posteriori error estimators. A way out could be the application of a space-time Galerkin method [52] and of the DWR technique [47].

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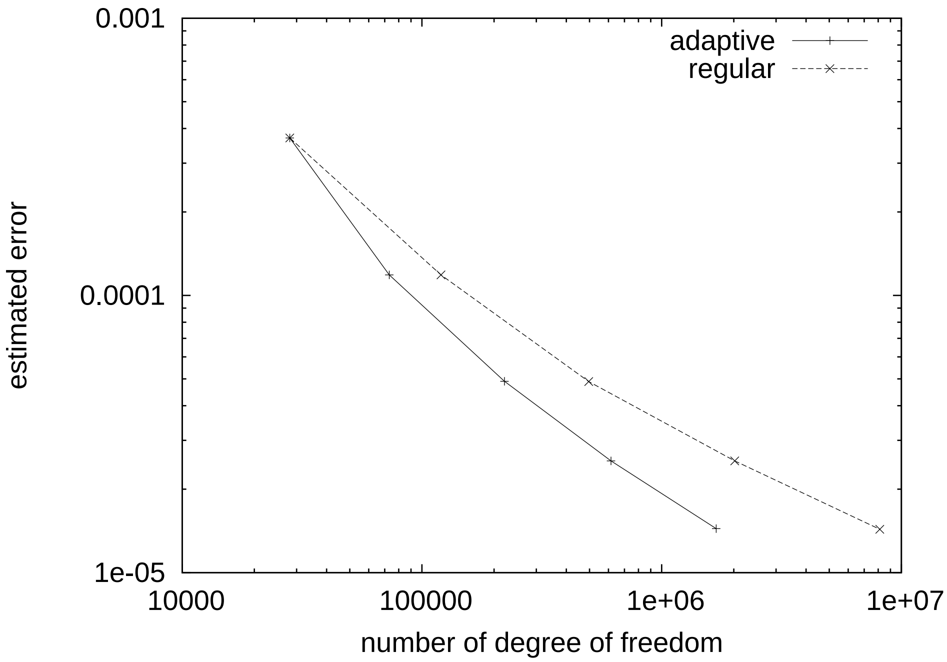

FIG. 6.2. *Trend of the error estimation for adaptive and regular refinement*

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