

Scalar Multivariate Subdivision Schemes and Box Splines

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Abstract

We study convergent scalar d -variate subdivision schemes satisfying sum rules of order $k \in \mathbb{N}$, with dilation matrix $2I$. Using the results of Möller and Sauer in [18], stated for general expanding dilation matrices, we characterize the structure of the mask symbols of such schemes by showing that they must be linear combinations of shifted box spline generators of a quotient polynomial ideal \mathcal{J}^k . The directions of the corresponding box splines are $\theta \in \{0, 1\}^d \setminus \{(0, \dots, 0)\}$. The quotient ideal \mathcal{J}^k , as shown in [18], is determined by the given order of the sum rules or, equivalently, by the order of the Strang–Fix conditions.

Our results open a way to a systematic study of subdivision schemes. For example, in the bivariate case, if the mask symbol of any convergent subdivision scheme is in \mathcal{J}^k , then the mask is an affine combination of smoothed versions of three-directional box splines. Many special cases, including affine combinations of convergent schemes, can be looked at this way; see, e.g., [7] and the references given therein.

As in the univariate case, this characterization seems to be the proper way of matching the smoothness, as determined in [1], of the box spline building blocks with the order of polynomial reproduction of the corresponding scheme. Due to the interaction of the building blocks, the convergence and smoothness, however, are usually destroyed, if several convergent schemes are combined in this way.

We illustrate our results with several examples.

Keywords: subdivision schemes, box splines, quotient ideals

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Introduction and notation

A (scalar) d -variate subdivision scheme is given by a scalar \mathbb{Z}^d -indexed sequence $\mathbf{a} = (a_\alpha)_{\alpha \in \mathbb{Z}^d}$, the so-called mask, defining the subdivision operator $\mathcal{S}_\mathbf{a}$ on data sequences $\mathbf{d} = (d_\alpha)_{\alpha \in \mathbb{Z}^d} \in \ell(\mathbb{Z}^d)$ as follows:

$$(\mathcal{S}_\mathbf{a} \mathbf{d})_\alpha = \sum_{\beta \in \mathbb{Z}^d} d_\beta a_{\alpha - 2\beta}, \quad \alpha \in \mathbb{Z}^d. \quad (1)$$

We assume that the mask is finite, *i.e.*, only finitely many coefficients are non-zero.

In our study we use the corresponding symbol notation. For a finitely supported sequence $\mathbf{c} = (c_\alpha)_{\alpha \in \mathbb{Z}^d}$, the symbol is given by the Laurent polynomial

$$c(z) = \sum_{\alpha \in \mathbb{Z}^d} c_\alpha z^\alpha \quad (2)$$

with $z = (z_1, \dots, z_d) \in (\mathbb{C} \setminus \{0\})^d$ and, in the multi-index notation,

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}, \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d.$$

In the symbol notation, the subdivision step in (1) is described by the identity

$$(\mathcal{S}_a d)(z) = d(z^2) a(z). \quad (3)$$

The first factor in (3) refers to an upsampled version of the data d .

Equation (3) can also be written as follows using convolution with the submasks. Let

$$\mathbf{E} = \{0, 1\}^d = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{2^d}\} \quad (4)$$

be the set of representatives of $\mathbb{Z}^d/2\mathbb{Z}^d$, given by the vertices of the unit cube $[0, 1]^d$, with

$$\mathbf{e}_1 = \mathbf{0} = (0, 0, \dots, 0) \quad \text{and} \quad \mathbf{e}_{2^d} = \mathbf{1} = (1, 1, \dots, 1). \quad (5)$$

Then, the 2^d submasks $\mathbf{a}_\mathbf{e}$ and their symbols $\mathbf{a}_\mathbf{e}(z)$ are defined by

$$\mathbf{a}_\mathbf{e} = (a_{\mathbf{e}+2\alpha})_{\alpha \in \mathbb{Z}^d} \quad \text{and} \quad \mathbf{a}_\mathbf{e}(z) = \sum_{\alpha \in \mathbb{Z}^d} a_{\mathbf{e}+2\alpha} z^\alpha, \quad \mathbf{e} \in \mathbf{E}. \quad (6)$$

The standard decomposition

$$a(z) = \sum_{\alpha \in \mathbb{Z}^d} a_\alpha z^\alpha = \sum_{\mathbf{e} \in \mathbf{E}} z^\mathbf{e} \mathbf{a}_\mathbf{e}(z^2), \quad (7)$$

with $z^2 = (z_1^2, z_2^2, \dots, z_d^2)$ yields identity (3) in the equivalent form

$$(\mathcal{S}_a d)(z) = \sum_{\mathbf{e} \in \mathbf{E}} z^\mathbf{e} d(z^2) \mathbf{a}_\mathbf{e}(z^2). \quad (8)$$

This shows that the subdivision step can also be thought of as the result of interleaving the outcomes of the convolutions of the original data with all submasks.

Sometimes we also look at the symbols restricted to the d -variate torus, i.e. $|z_j| = 1$, $j = 1, \dots, d$, and we change to the real variable $\xi = (\xi_1, \dots, \xi_d)$ via the transformation $z_j = e^{-i\pi\xi_j}$, $j = 1, \dots, d$. The set of extreme points $\mathbf{E} = \{0, 1\}^d$ then transforms into

$$Z = Z_{\mathbf{E}} = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2^d}\} = \{-1, +1\}^d. \quad (9)$$

These are the vertices of the cube $[-1, +1]^d$, and we have

$$\varepsilon_1 = e^{-i\pi\mathbf{e}_1} = \mathbf{1} \quad \text{and} \quad \varepsilon_{2^d} = e^{-i\pi\mathbf{e}_{2^d}} = -\mathbf{1}.$$

We say that the subdivision scheme \mathcal{S}_a is convergent, if for any starting sequence $d \in \ell_\infty(\mathbb{Z}^d)$ there exists a uniformly continuous function f_d such that

$$\lim_{r \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}^d} |(\mathcal{S}_a^r d)_\alpha - f_d(2^{-r}\alpha)| = 0$$

and $f_d \neq 0$ for some initial data d . Our results are also valid, if we consider L_p -convergent subdivision schemes, $1 \leq p < \infty$, see [10] for the precise definition of such convergence. What kind of convergence we assume is not critical for our study as we mostly work with the symbol in (7).

The necessary conditions, see [4, Proposition 2.1] and [15, Theorem 3.1], for the convergence of the subdivision scheme \mathcal{S}_a is known to be the so-called sum rule of degree 0 (or order 1) referring to the submasks:

$$\mathbf{a}_\mathbf{e}(\mathbf{1}) = \sum_{\alpha \in \mathbb{Z}^d} a_{\mathbf{e}+2\alpha} = 1, \quad \mathbf{e} \in \mathbf{E}. \quad (10)$$

Equivalently, for the mask symbol we have

$$a(\mathbf{1}) = 2^d \quad \text{and} \quad a(\varepsilon) = 0 \quad \text{for } \varepsilon \in Z \setminus \{\mathbf{1}\}. \quad (11)$$

For this reason we call Z the zero set.

1 Convergent subdivision schemes and the ideal \mathcal{J}

We use the results in [18] that interpret the zero conditions in (11) as the property that the mask symbol $a(z)$ belongs to a certain ideal \mathcal{J} of the ring of Laurent polynomials (with real coefficients). In addition in [18], in the bivariate case, the authors determine a special set of generating functions for this ideal. The ideal \mathcal{J} is defined as the set of all Laurent polynomials satisfying

$$a(z) \in \mathcal{J} \iff a(\varepsilon) = 0 \text{ for } \varepsilon \in Z \setminus \{\mathbf{1}\}. \quad (12)$$

The necessary condition for the convergence selects those elements from \mathcal{J} which, in addition, satisfy the condition $a(\mathbf{1}) = 2^d$. We denote $\langle r_1, \dots, r_n \rangle$ to be the ideal generated by the set $\{r_1, \dots, r_n\}$.

In this section we determine a set of generators for the ideal \mathcal{J} which—for the purpose of studying properties of subdivision schemes—turns out to be a more suitable one than the Groebner basis given in [18]. These generators are the symbols of appropriately chosen box splines. The main result of this section, Theorem 1.4, then shows that the symbol of any convergent subdivision scheme is an affine combination of the translates of these box splines.

We start by a simple observation that extends the result given by Möller and Sauer in [18] to the multivariate case.

Proposition 1.1. *The ideal \mathcal{J} in (12) is generated by the system*

$$1 - z_1^2, 1 - z_2^2, \dots, 1 - z_d^2 \quad \text{and} \quad \pi(z) := \prod_{i=1}^d \frac{1 + z_i}{2}. \quad (13)$$

Proof. \mathcal{J} is the quotient ideal characterized by $(1 - z_i)a(z) \in \mathcal{I}$, $i = 1, \dots, d$, with \mathcal{I} denoting the ideal of Laurent polynomials vanishing on the zero set Z . Since Z is the set of common zeros of the polynomials $(1 - z_i)^2$, $i = 1, \dots, d$, we see that these functions form a basis of \mathcal{I} , see [9, p. 4] for a definition of a basis for an ideal. Thus, any Laurent polynomial $\ell \in \mathcal{I}$ can be written as a combination

$$\ell(z) = \sum_{i=1}^d p_i(z) (1 - z_i)^2$$

with suitable Laurent polynomials $p_i(z)$. This representation is not necessarily unique. The claim follows due to $\pi(z) \in \mathcal{J}$, $\pi(\mathbf{1}) = 1$ and

$$a(z) \in \mathcal{J} \iff \ell(z) = a(z) - a(\mathbf{1}) \pi(z) \in \mathcal{I}.$$

□

In the scalar univariate case, it is a well-known fact that $a(z) \in \mathcal{J}$ if and only if $a(-1) = 0$ if and only if $a(z) = (1 + z) b(z)$ for some Laurent polynomial $b(z)$. The latter property is equivalent to

$$a(z) (1 - z) = (1 - z^2) b(z). \quad (14)$$

For the multivariate case, Proposition 1.1 tells us that $a(z) \in \mathcal{J}$ if and only if we have a representation of type

$$a(z) \begin{pmatrix} 1 - z_1 \\ 1 - z_2 \\ \vdots \\ 1 - z_d \end{pmatrix}^T = \begin{pmatrix} 1 - z_1^2 \\ 1 - z_2^2 \\ \vdots \\ 1 - z_d^2 \end{pmatrix}^T \begin{pmatrix} b_{11}(z) & b_{12}(z) & \cdots & b_{1d}(z) \\ b_{21}(z) & b_{22}(z) & \cdots & b_{2d}(z) \\ \vdots & \vdots & \ddots & \vdots \\ b_{d1}(z) & b_{d2}(z) & \cdots & b_{dd}(z) \end{pmatrix}, \quad (15)$$

with some matrix Laurent polynomial $\mathbf{B}(z)$. This has been already observed in [4, 19]. The representation in (15) does not have to be unique, see [5].

1.1 Box splines generators of the ideal \mathcal{J}

In the following, we will use the notation from the theory of box splines. Let

$$X_d = (\mathbf{e}_i)_{i=2}^{2^d} \quad (16)$$

be the $d \times (2^d - 1)$ -matrix given by the non-zero elements of the set \mathbf{E} in (4). We treat these as directional vectors from which we build the box splines of degree zero, the characteristic functions of parallelepipeds of d -dimensional volume equal to 1. This means that we take any d columns from X_d to produce a square submatrix Θ of X_d such that $\det \Theta = \pm 1$, *i.e.*, Θ is unimodular. Such an integer matrix has an integer inverse and its columns generate the integer grid \mathbb{Z}^d .

In what follows the Laurent polynomials

$$r_\alpha(z) = \frac{1}{2}(1 + z^\alpha) \quad \text{and} \quad s_\alpha(z) = \frac{1}{2}(1 - z^\alpha), \quad \alpha \in \mathbb{Z}^d, \quad (17)$$

play a prominent role. The polynomials are normalized so that $r_\alpha(\mathbf{1}) = 1$ and

$$r_\alpha(z) + s_\alpha(z) = 1, \quad \alpha \in \mathbb{Z}^d.$$

We give another trivial, but useful identity

$$s_\beta(z) r_\alpha(z) + s_\alpha(z) r_\beta(z) = s_{\alpha+\beta}(z) = 1 - r_{\alpha+\beta}(z), \quad \alpha, \beta \in \mathbb{Z}^d, \quad (18)$$

which holds identically in the variable z . This identity (18) shows that the polynomials r_α, r_β and $r_{\alpha+\beta}$ generate the ring of Laurent polynomials (since they generate a unit). This holds for any space dimension d .

With each $d \times d$ -submatrix Θ of X_d we associate the normalized polynomial

$$q_\Theta(z) = \prod_{\theta \in \Theta} r_\theta(z), \quad (19)$$

where θ runs through all the columns of Θ . In case the matrix Θ is unimodular, the polynomial $q_\Theta(z)$ is the symbol of the corresponding degree zero box spline.

Proposition 1.2. *The ideal \mathcal{J} is generated by the elements $q_\Theta(z)$, where Θ runs through the family $\mathcal{U}^{(d)}$ of all unimodular $d \times d$ -submatrices of X_d at least one of whose columns is a standard unit vector of \mathbb{R}^d .*

Proof. We first show that $q_\Theta \in \mathcal{J}$ for each unimodular Θ . Given Θ , we have to show that $q_\Theta(\varepsilon) = 0$ for all $\varepsilon \in \mathbb{Z} \setminus \{\mathbf{1}\}$. In other words, given such Θ and ε , there is a $\theta \in \Theta$ such that $r_\theta(\varepsilon) = 0$. Writing $\varepsilon = e^{-i\pi \mathbf{e}}$, with $\mathbf{e} \in \mathbb{E} \setminus \{\mathbf{0}\}$, we have to show that for given Θ and $\mathbf{e} \in \mathbb{E} \setminus \{\mathbf{0}\}$ there is a $\theta \in \Theta$ such that

$$(e^{-i\pi \mathbf{e}})^\theta = e^{-i\pi \mathbf{e}^T \theta} = -1.$$

If this property does not hold, *i.e.*, if for all $\theta \in \Theta$ the result is $+1$, we would have, for any $\alpha \in \mathbb{Z}^d$,

$$e^{-i\pi \mathbf{e}^T \Theta \alpha} = 1^{\alpha_1} \dots 1^{\alpha_d} = +1.$$

This is impossible, since \mathbf{e} has at least one component equal to 1, say, at position k , and we can choose $\alpha \in \mathbb{Z}^d$ such that $\Theta \alpha$ is the k -th standard unit vector.

It remains to show that all polynomials $1 - z_i^2$, $i = 1, \dots, d$, can be generated from the set of polynomials $\{q_\Theta : \Theta \in \mathcal{U}^{(d)}\}$. We prove a slightly stronger statement

$$1 - z_i^2 \in \langle q_\Theta : \Theta \in \mathcal{U}_i^{(d)} \rangle, \quad i = 1, \dots, d,$$

with $\mathcal{U}_i^{(d)} \subset \mathcal{U}^{(d)}$ denoting the subfamily of unimodular $d \times d$ -matrices one of whose columns is the i -th standard unit vector of \mathbb{R}^d . The proof is by induction on d . For $d = 1$, we have

$$\frac{1 - z_1^2}{4} = \frac{1 - z_1}{2} \cdot \frac{1 + z_1}{2}.$$

In the induction step $d \rightarrow d + 1$, it is sufficient to consider the case $i = d + 1$, since the remaining cases can be reduced to this case by cyclic permutations of the variables. Without loss of generality we have

$$X_{d+1} = \begin{pmatrix} & & & & & & 0 \\ & X_d & & X_d & & & \vdots \\ & & & & & & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

The identity (18) with $\alpha = (0, \dots, 0, 1, 1)$, $\beta = (0, \dots, 0, -1, 0) \in \mathbb{Z}^{d+1}$ implies

$$\frac{1 - z_{d+1}}{2} = z_d^{-1} \left\{ \frac{1 - z_d z_{d+1}}{2} \cdot \frac{1 + z_d}{2} - \frac{1 - z_d}{2} \cdot \frac{1 + z_d z_{d+1}}{2} \right\}. \quad (20)$$

By the induction hypothesis we have

$$\frac{1 - z_d^2}{4} = \sum_{\Theta \in \mathcal{U}_d^{(d)}} p_{\Theta}(z_1, \dots, z_d) q_{\Theta}(z_1, \dots, z_d)$$

with some Laurent polynomials p_{Θ} . Dividing the above by $\frac{1+z_d}{2}$ yields

$$\frac{1 - z_d}{2} = \sum_{\Theta \in \mathcal{U}_d^{(d)}} p_{\Theta}(z_1, \dots, z_d) \tilde{q}_{\Theta}(z_1, \dots, z_d)$$

with $q_{\Theta} = \frac{1+z_d}{2} \tilde{q}_{\Theta}$, $\Theta \in \mathcal{U}_d^{(d)}$. Hence, replacing z_d by $z_d z_{d+1}$, we also get

$$\frac{1 - z_d z_{d+1}}{2} = \sum_{\Theta \in \mathcal{U}_d^{(d)}} p_{\Theta}(z_1, \dots, z_{d-1}, z_d z_{d+1}) \tilde{q}_{\Theta}(z_1, \dots, z_{d-1}, z_d z_{d+1}).$$

Substituting the two above identities in (20) und multiplying the result by $\frac{1+z_{d+1}}{2}$ yields the representation

$$\begin{aligned} \frac{1 - z_{d+1}^2}{4} &= z_d^{-1} \left\{ \sum_{\Theta \in \mathcal{U}_d^{(d)}} p_{\Theta}(z_1, \dots, z_{d-1}, z_d z_{d+1}) \tilde{q}_{\Theta}(z_1, \dots, z_{d-1}, z_d z_{d+1}) \frac{1 + z_d}{2} \frac{1 + z_{d+1}}{2} \right. \\ &\quad \left. - \sum_{\Theta \in \mathcal{U}_d^{(d)}} p_{\Theta}(z_1, \dots, z_d) \tilde{q}_{\Theta}(z_1, \dots, z_d) \frac{1 + z_d z_{d+1}}{2} \frac{1 + z_{d+1}}{2} \right\}. \end{aligned}$$

Writing $\Theta = \begin{pmatrix} & & & & 0 \\ & \tilde{\Theta} & & & \vdots \\ & & & & 0 \\ & & & & 1 \end{pmatrix} \in \mathbb{Z}^{d \times d}$, we see that for $d+1$, the index Θ in $p_{\Theta} \tilde{q}_{\Theta}$ above refers to unimodular $(d+1) \times (d+1)$ -matrices of the type

$$\begin{pmatrix} & & & & 0 & 0 \\ & \tilde{\Theta} & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & 1 & 0 \\ * & \cdots & * & 0 & 1 & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & & & & 0 & 0 \\ & \tilde{\Theta} & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & 1 & 0 \\ 0 & \cdots & 0 & 1 & 1 & \end{pmatrix},$$

respectively. The *-entries in the first matrix are just a copy of the last row in $\tilde{\Theta}$. Since both types of matrices are from the class $\mathcal{U}_{d+1}^{(d+1)}$, the induction is complete. \square

We list the generators q_Θ of \mathcal{J} for low-dimensional cases.

- For $d = 1$ the ideal \mathcal{J} is the principal ideal generated by $r(z) = \frac{1+z}{2}$.
- For $d = 2$, we have

$$X_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and the ideal \mathcal{J} is generated by the three elements

$$\begin{aligned} q_1(z) &= \frac{1}{4}(1+z_1)(1+z_2), \\ q_2(z) &= \frac{1}{4}(1+z_1)(1+z_1z_2) \text{ and} \\ q_3(z) &= \frac{1}{4}(1+z_2)(1+z_1z_2). \end{aligned}$$

- In case $d = 3$, we have

$$X_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The non-unimodular Θ are obtained by selecting the columns $(1, 2, 4)$, $(1, 3, 5)$, $(1, 6, 7)$, $(2, 3, 6)$, $(2, 5, 7)$, $(3, 4, 7)$, where $\det \Theta = 0$, and $(4, 5, 6)$, where $\det \Theta = -2$. The set of generators q_Θ , referred to in Proposition 1.2, thus consists of $\binom{7}{3} - 7 = 28$ elements. However, this system is highly redundant.

The last example, for $d = 3$, raises the question how to characterize all $d \times d$ -submatrices Θ of X such that $q_\Theta \in \mathcal{J}$. The answer to this question is given in the following proposition.

Proposition 1.3. *For any $d \times d$ -submatrix Θ of X_d in (16), we have*

$$q_\Theta \in \mathcal{J} \iff \det \Theta \equiv 1 \pmod{2}.$$

Proof. The proof modifies and extends the argument of the first part of the proof of Proposition 1.2. With the same notation, for given Θ the statement $q_\Theta \in \mathcal{J}$ is equivalent to the statement that for each $\mathbf{e} \in E \setminus \{\mathbf{0}\}$ there is a $\theta \in \Theta$ such that $\mathbf{e}^T \theta$ is odd. In other words: The linear map

$$L : \mathbb{R}^d \rightarrow \mathbb{R}^d, \mathbf{x}^T \mapsto \mathbf{x}^T \Theta$$

has the property that the images of all $\mathbf{e} \in E \setminus \{\mathbf{0}\}$ have at least one odd component each.

If this property does not hold, then there is an $\mathbf{e} \in E \setminus \{\mathbf{0}\}$ which is mapped onto a vector with all components even. By adding all rows of Θ indexed by the position of the non-zero entries of this \mathbf{e} shows that, when replacing one of these rows by the sum of them, we will produce a modified row with even entries without changing the determinant. Expanding the determinant along that row shows that $\det \Theta$ is even as well.

On the other hand, if the property holds, we can interpret the map, restricted to the set E , as an endomorphism \bar{L} of E considered as a vector space over the field \mathbb{Z}_2 . The property then tells that the kernel of \bar{L} is just the zero element $\mathbf{0} \in E = (\mathbb{Z}_2)^d$. Whence, \bar{L} is an automorphism, and its determinant equals the nonzero element in \mathbb{Z}_2 . This means that $\det \Theta$ is odd. \square

The main result of this section, Theorem 1.4, follows from Proposition 1.2 and the necessary condition for convergence of subdivision schemes.

Theorem 1.4. *The mask symbol of any convergent d -variate subdivision scheme \mathcal{S}_a can be written in the form*

$$a(z) = 2^d \sum_{\Theta} \lambda_{\Theta} \sigma_{\Theta}(z) q_{\Theta}(z),$$

where $\sigma_{\Theta}(z)$ are Laurent polynomials satisfying $\sigma_{\Theta}(\mathbf{1}) = 1$, and λ_{Θ} are real numbers subject to $\sum_{\Theta} \lambda_{\Theta} = 1$. The sum runs over all unimodular $d \times d$ -submatrices Θ of X_d from (16).

Remark. Since $q_{\Theta}(z)$ are the masks of certain box splines, the mask $a(z)$ is an affine combination of the masks each of which originates from a degree zero box spline convolved with some (smoothing) factor. Proposition 1.2 shows that it suffices to consider the generators $q_{\Theta} \in \mathcal{U}^{(d)}$ only, i.e., the corresponding parallelepipeds all have at least one unit vector as an edge.

1.2 Sum rules of higher order and the ideals \mathcal{J}^k

The Strang–Fix conditions on the mask symbol and the higher-order sum rules are known to be the proper extensions of the necessary condition for the convergence of scalar multivariate subdivision schemes. To state these we define the normalized trigonometric polynomial

$$a^{\wedge}(\xi) = \frac{1}{2^d} a(e^{-i\xi}) = \frac{1}{2^d} \sum_{\alpha \in \mathbb{Z}^d} a_{\alpha} e^{-i\alpha \cdot \xi} \quad (21)$$

representing the mask symbol on the d -dimensional torus.

Definition 3. *We say that the mask symbol satisfies the Strang–Fix conditions of order k (or degree $k - 1$), $k \in \mathbb{N}$, if*

$$\begin{aligned} a^{\wedge}(\mathbf{0}) &= 1 \quad \text{and} \\ D^{\beta} a^{\wedge}(\pi \mathbf{e}) &= 0 \quad \text{for } \beta \in \mathbb{N}_0^d \quad \text{with } |\beta| < k, \quad \mathbf{e} \in E \setminus \{\mathbf{0}\}. \end{aligned}$$

Here, $D^{\beta} = \frac{\partial^{|\beta|}}{\partial \xi_1^{\beta_1} \dots \partial \xi_d^{\beta_d}}$. Equivalently, the Strang–Fix condition of order k on the mask symbol is given by the zero conditions of order k

$$\left(D^{\beta} a \right) (\boldsymbol{\varepsilon}) = 0 \quad \text{for } \beta \in \mathbb{N}_0^d \quad \text{with } |\beta| < k, \quad \text{for } \boldsymbol{\varepsilon} \in Z \setminus \{\mathbf{1}\}, \quad (22)$$

combined with the normalizing condition $\frac{1}{2^d} a(\mathbf{1}) = a^{\wedge}(\mathbf{0}) = 1$. The partial derivatives D^{β} in (22) are taken with respect to the variables z_1, \dots, z_d . The condition in (22) generalizes (11). In terms of the ideal \mathcal{J} , the zero condition (22) is equivalent to the property that $a(z) \in \mathcal{J}^k$. One of the main contributions of [18] is that the authors established this and other connections between the mask properties and the structure of the ideals \mathcal{J}^k .

There is an equivalent form of Strang–Fix conditions on the mask symbol called the sum rules of order k . The equivalence between the two is shown in what follows. For any algebraic polynomial q and its corresponding partial differential operator $q(iD)$, we have

$$q(iD) a^{\wedge}(\xi) = \frac{1}{2^d} \sum_{\alpha \in \mathbb{Z}^d} a_{\alpha} q(\alpha) e^{-i\alpha \cdot \xi} = \frac{1}{2^d} \sum_{\mathbf{e} \in E} \sum_{\alpha \in \mathbb{Z}^d} a_{\mathbf{e}+2\alpha} q(\mathbf{e}+2\alpha) e^{-i(\mathbf{e}+2\alpha) \cdot \xi}.$$

An inductive argument with respect to the order of the differential operator thus tells us that the zero conditions in (22) are equivalent to the sum rules of the same order: *For any polynomial q of total degree less than k , we have*

$$\sum_{\alpha \in \mathbb{Z}^d} a_{\mathbf{e}+2\alpha} q(\mathbf{e}+2\alpha) = \sum_{\alpha \in \mathbb{Z}^d} a_{2\alpha} q(2\alpha) \quad \text{for all } \mathbf{e} \in E. \quad (23)$$

We refer also to [16], Section 4.3.3.

The following result is the generalization of Theorem 1.4.

Theorem 1.5. *A convergent d -variate subdivision scheme \mathcal{S}_a satisfies the sum rules of order k , if and only if its mask symbol can be written in the form*

$$a(z) = 2^d \sum_j \lambda_j \sigma_j(z) B_j(z)$$

where $\sum_j \lambda_j = 1$, $\sigma_j(z)$ are Laurent polynomials normalized by $\sigma_j(\mathbf{1}) = 1$, and $B_j(z)$ are k -fold products of Laurent polynomials q_Θ with unimodular $d \times d$ -submatrices Θ of X_d from (16).

Note that, due to (17) and (19), all symbols $B_j(z)$ are normalized to satisfy $B_j(\mathbf{1}) = 1$, whence $a(\mathbf{1}) = 2^d$.

We also point out one important difference between the univariate and the multivariate cases: While the order k of sum rules is carried over from the generators of the ideal to all of \mathcal{J}^k , this is no longer the case for the smoothness; see the example of the butterfly scheme in section 2.2.4.

2 Bivariate schemes

Throughout this section, we consider the bivariate case

$$d = 2, \quad \text{where} \quad X_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Here, the generators of the ideal \mathcal{J}^k are the symbols of certain three-directional box splines. Using the three-directional notation, we define

$$B_{a,b,c}^\#(z_1, z_2) = 2^{-(a+b+c)} (1+z_1)^a (1+z_2)^b (1+z_1 z_2)^c \quad (24)$$

for some non-negative integers a, b and c . The superscript $\#$ allows us to distinguish between the normalized symbol $B_{a,b,c}^\#$ and the symbol $B_{a,b,c} = 4B_{a,b,c}^\#$. The directions refer to the corresponding columns of X_2 , since in terms of the notation of (17),

$$r_{(1,0)}(z_1, z_2) = \frac{1+z_1}{2}, \quad r_{(0,1)}(z_1, z_2) = \frac{1+z_2}{2}, \quad \text{and} \quad r_{(1,1)}(z_1, z_2) = \frac{1+z_1 z_2}{2}.$$

An important property of such box spline symbols is shown in the following Lemma.

Lemma 2.1. *For a given triple (a, b, c) , the ideal generated by the three symbols $B_{a+1,b,c}^\#$, $B_{a,b+1,c}^\#$ and $B_{a,b,c+1}^\#$ is the principal ideal generated by $B_{a,b,c}^\#$.*

Proof. Since each of the symbols $B_{a+1,b,c}^\#$, $B_{a,b+1,c}^\#$ or $B_{a,b,c+1}^\#$ is a multiple of $B_{a,b,c}^\#$, we just have to show that the latter can be generated from the other three. It follows from the identity (18) with α, β being the first two columns of X_2 that

$$\frac{1}{2} \left\{ (1-z_2) B_{1,0,0}^\#(z_1, z_2) + (1-z_1) B_{0,1,0}^\#(z_1, z_2) \right\} + B_{0,0,1}^\#(z_1, z_2) = 1.$$

Multiplying both sides of this identity by $B_{a,b,c}^\#$ proves the lemma. \square

2.1 Three-directional box spline generators for \mathcal{J}^k

Theorem 2.2. *In the bivariate case, the k -th power \mathcal{J}^k , $k \in \mathbb{N}$, of the ideal \mathcal{J} is generated by the set of three-directional box spline symbols*

$$\mathcal{L}_k := \left\{ B_{k-a, k-a, a}^\#, B_{k-a, a, k-a}^\#, B_{a, k-a, k-a}^\# : a = 0, 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor \right\}.$$

Proof. We use a shorthand notation for the box spline symbols by identifying the symbol with its index triple

$$B_{a,b,c}^\# \leftrightarrow (a, b, c).$$

In this notation, the set \mathcal{L}_k corresponds to the list

$$L_k := \left\{ (k-a, k-a, a), (k-a, a, k-a), (a, k-a, k-a) : a = 0, 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor \right\}. \quad (25)$$

If $(a, b, c) \in L$ and if $(a', b', c') \geq (a, b, c)$ componentwise, then $B_{a', b', c'}^\# \in \langle L \rangle$. We call (a', b', c') a multiple of (a, b, c) in this case.

The proof of the theorem is by induction on k . For $k = 1$, we have

$$L_1 = \left\{ (1, 1, 0), (1, 0, 1), (0, 1, 1) \right\} \quad (26)$$

and the claim is nothing else but Proposition 1.2.

In the induction step $k \rightarrow k+1$, using the induction hypothesis for $\mathcal{J}^{k+1} = \mathcal{J} \cdot \mathcal{J}^k$, we see that \mathcal{J}^{k+1} is generated by the box spline symbols from \tilde{L}_{k+1} , which consists of the corresponding triples

$$\left\{ \begin{array}{l} (k+1-a, k+1-a, a), \quad (k+1-a, a+1, k-a), \quad (a+1, k+1-a, k-a), \\ (k+1-a, k-a, a+1), \quad (k+1-a, a, k+1-a), \quad (a+1, k-a, k+1-a), \\ (k-a, k+1-a, a+1), \quad (k-a, a+1, k+1-a), \quad (a, k+1-a, k+1-a), \end{array} \quad a = 0, 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor \right\}.$$

On the other hand,

$$L_{k+1} = \left\{ (k+1-b, k+1-b, b), (k+1-b, b, k+1-b), (b, k+1-b, k+1-b) : b = 0, 1, \dots, \left\lfloor \frac{k+1}{2} \right\rfloor \right\}.$$

Therefore, we have to show that the lists \tilde{L}_{k+1} and L_{k+1} both generate the same ideal. We do this in two steps. We show firstly that each element from L_{k+1} is in the ideal generated by \tilde{L}_{k+1} , and, secondly, we verify that each element in \tilde{L}_{k+1} is a multiple of some element in L_{k+1} .

Firstly, note that the elements of L_{k+1} are the diagonal elements in the list \tilde{L}_{k+1} with $b = a$ with the exception of $\lfloor \frac{k}{2} \rfloor < \lfloor \frac{k+1}{2} \rfloor = b$. In this case $k = 2\ell + 1$ is odd and $b = \ell + 1$. For this value of b the list L_{k+1} contains just one element, viz. $(\ell + 1, \ell + 1, \ell + 1)$, and this element, by Lemma 2.1, can be generated by the three elements above the main diagonal from \tilde{L}_{k+1} by letting $k = 2\ell + 1$ and $a = \ell$.

Conversely, the diagonal elements in \tilde{L}_{k+1} are all listed in L_{k+1} . A typical non-diagonal element in \tilde{L}_{k+1} is given by a permutation of the components of the triple $(k+1-a, a+1, k-a)$. This triple is a multiple of $((k+1)-(a+1), a+1, (k+1)-(a+1))$ and appears in L_{k+1} for $b = a+1$ except for $a = \lfloor \frac{k}{2} \rfloor = \lfloor \frac{k+1}{2} \rfloor$. In this case $k = 2\ell$ is even, $a = \ell$ and $(k+1-a, a+1, k-a) = (\ell + 1, \ell + 1, \ell)$. Note that this element is the first element of L_{k+1} with $k = 2\ell$ and $b = \ell$.

Since both lists are invariant under permutations of the components of the triples, the induction step is complete and the claim follows. \square

The following conjecture has been computer checked for $k \leq 10$.

Conjecture 2.3. *The set \mathcal{L}_k generating \mathcal{J}^k is minimal in the following sense: For any $B_{a,b,c}^\# \in \mathcal{L}_k$, the reduced set $\mathcal{L}_k \setminus \{B_{a,b,c}^\#\}$ does no longer generate the ideal \mathcal{J}^k .*

In the bivariate case, Theorem 1.5 reads as follows.

Theorem 2.4. *A convergent bivariate subdivision scheme \mathcal{S}_a satisfies the sum rules of order k if and only if its mask symbol can be written in the form*

$$a(z) = 2^d \sum_{(a,b,c) \in L_k} \lambda_{(a,b,c)} \sigma_{(a,b,c)}(z) B_{a,b,c}^\#(z),$$

where $\sum_{(a,b,c) \in L_k} \lambda_{(a,b,c)} = 1$ and $\sigma_{(a,b,c)}$ are Laurent polynomials normalized by $\sigma_{(a,b,c)}(\mathbf{1}) = 1$.

This characterization of convergent bivariate subdivision schemes opens a systematic way for their study, since the properties of three-directional box splines are well-known; see [2] or also [3, 6, 17].

We point out only the following three facts:

1. The set of generators is symmetric in the sense that it is invariant under permutations of the variables. Equivalently, the indices of the generators can be interchanged cyclically.
2. For even k , the $k/2$ -th power of (the symbol of) the Courant hat function $B_{1,1,1}$ appears among the generators of \mathcal{J}^k .
3. Most interesting for us, however, is the fact that the smoothness of these generators matches perfectly with the order of polynomial reproduction, viz. the generators are elements of

$$L_\infty^{(\kappa-1)} \subset C^{(\kappa-2)}$$

with

$$\kappa = a + 2(k - a) - (k - a) = k,$$

see [2]. This is a property of the generators which generalizes the related univariate result.

It may also be of interest that the total degree of the box spline with the symbol $B_{k-a,k-a,a}^\#$ is $\mu_a = 2k - a - 2 = 2\kappa - a - 2$.

2.2 Examples

In this section we illustrate the result of Theorem 2.4 with some examples. We would like to emphasize that this result does not only simplify the study of the properties of subdivision schemes, but also yields a way for enhancing certain properties of existing schemes by combining them appropriately.

In the following, the set \mathcal{L}_k of the box spline symbols is the set of generators for \mathcal{J}^k as in Theorem 2.4.

2.2.1 Three-directional box splines

Theorem 2.4 allows us to find, for a given three-directional box spline symbol

$$B_{a,b,c}^\#(z_1, z_2) = 2^{-(a+b+c)} (1+x)^a (1+z_2)^b (1+z_1 z_2)^c,$$

the maximal integer k such that $B_{a,b,c}^\# \in \mathcal{J}^k$.

Proposition 2.5. *Given the triple (a, b, c) , the maximal k such that $B_{a,b,c}^\# \in \mathcal{J}^k$, is given by*

$$k := a + b + c - \max\{a, b, c\}. \quad (27)$$

Proof. We first show that $B_{a,b,c}^\# \in \mathcal{J}^k$ for k in (27). Choose $\nu = \min\{a, b, c\}$. Then take the minimum μ of the remaining two entries in the triple (a, b, c) and let $k' = \nu + \mu$. Then (a, b, c) is a multiple of some permutation of $(k' - \nu, k' - \nu, \nu)$. Due to $B_{k'-\nu, k'-\nu, \nu}^\# \in \mathcal{L}^{k'}$, we have $B_{a,b,c}^\# \in \mathcal{J}^{k'}$. It is easy to see that $k' = k$, and it remains to show that $B_{a,b,c}^\# \notin \mathcal{J}^{k+1}$.

W.l.o.g. we may assume that $a \geq b \geq c$, since other cases can be reduced to this one by using obvious transformations. In this case $k = b + c$ and

$$D^{(0,b+c)} B_{a,b,c}^\#(1, -1) = D^{(0,b+c)} B_{0,b,c}^\#(1, -1),$$

and we are going to see that the result is not zero. Here and in what follows, $D^{(m,n)} = \frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n}$. Now, by the Leibniz rule

$$D^{(0,b+c)} B_{0,b,c}^\#(1, -1) = \sum_{i=0}^{b+c} \binom{b+c}{i} D^{(0,i)} B_{0,b,0}^\#(1, -1) D^{(0,b+c-i)} B_{0,0,c}^\#(1, -1).$$

The first factor in the sum vanishes except for $i = b$, where it takes the value $2^{-b} b!$, and the entire sum is given by $D^{(0,k)} B_{0,b,c}^\#(1, -1) = 2^{-(b+c)} b! c! \neq 0$. This yields the claim. \square

Remark. The number k in (27) characterizes the smoothness $L_\infty^{(k-1)} \subset C^{(k-2)}$ of the corresponding box spline, see [2]. Since the subdivision scheme with the symbol $a(z_1, z_2) = 4 B_{a,b,c}^\#(z_1, z_2)$ reproduces polynomials of degree at most $k - 1$, smoothness and polynomial reproduction match in the same way as in the univariate case.

2.2.2 Four-directional box splines

Four-directional box spline symbols are of the form

$$B_{a,b,c,d}^\#(z_1, z_2) = 2^{-(a+b+c+d)} (1+z_1)^a (1+z_2)^b (1+z_1 z_2)^c \left(1 + \frac{z_1}{z_2}\right)^d,$$

This is consistent with our earlier notation, since the fourth direction $\alpha = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ comes in through the factor $r_\alpha(z_1, z_2) = \frac{1}{2}(1 + z_1 z_2^{-1}) = z_2^{-1} \frac{z_1 + z_2}{2}$. An alternative version of writing the symbol is

$$z_2^d B_{a,b,c,d}^\#(z_1, z_2) = B_{a,b,c}^\#(z_1, z_2) \left(\frac{z_1 + z_2}{2}\right)^d \quad (28)$$

illustrating the well-known fact that any four-directional box spline is indeed a special convex combination of the shifts of a three-directional box spline. This convex combination uses the normalized binomial weights $\lambda_{j,d} = 2^{-d} \binom{d}{j}$, $j = 0, \dots, d$. The representation in (28), however, is not optimal, if one tries to determine the maximal k such that $B_{a,b,c,d}^\# \in \mathcal{J}^k$. Instead, we use the identity $z_1 + z_2 = (1 + z_1)(1 + z_2) - (1 + z_1 z_2)$ which—in terms of the normalized box spline symbols—reads as

$$z_2 B_{0,0,0,1}^\#(z_1, z_2) = \frac{z_1 + z_2}{2} = 2 B_{1,1,0}^\#(z_1, z_2) - B_{0,0,1}^\#(z_1, z_2),$$

and work with

$$z_2^d B_{a,b,c,d}^\#(z_1, z_2) = \sum_{\ell=0}^d 2^\ell (-1)^{d-\ell} B_{a+\ell, b+\ell, c+d-\ell}^\#(z_1, z_2). \quad (29)$$

Proposition 2.6. *Given the quadruple (a, b, c, d) , the maximal k such that $B_{a,b,c,d}^\# \in \mathcal{J}^k$, is given by*

$$k := a + b + c + d - \max\{a, b, c + d\}. \quad (30)$$

Moreover, $B_{a,b,c,d}^\# \in \mathcal{J}^k$ if and only if $B_{a,b,c+d}^\# \in \mathcal{J}^k$.

Proof. By Proposition 2.5 and (29), the maximal $k(\ell)$ such that $B_{a+\ell, b+\ell, c+d-\ell}^\# \in \mathcal{J}^{k(\ell)}$ is determined using

$$\begin{aligned} k(\ell) &= a + b + c + d + \ell - \max\{a + \ell, b + \ell, c + d - \ell\} \\ &= \min\{b + c + d, a + c + d, a + b + 2\ell\} \\ &\geq \min\{b + c + d, a + c + d, a + b\} \\ &= a + b + c + d - \max\{a, b, c + d\} = k, \quad \ell = 0, \dots, d. \end{aligned}$$

Thus, $B_{a,b,c,d}^\# \in \mathcal{J}^k$ and it is left to show that $B_{a,b,c,d}^\# \notin \mathcal{J}^{k+1}$.

In the case $a + b < \min\{a + c + d, b + c + d\}$ we have $k(0) = k$ and $k(\ell) > k$, $\ell = 1, \dots, d$. Therefore, $B_{a,b,c,d}^\# \notin \mathcal{J}^{k+1}$ as the 0-th summand in (29) determines k .

Consider now the case

$$a + b \geq \min\{a + c + d, b + c + d\}.$$

Due to $z_2^d B_{a,b,c,d}^\#(z_1, z_2) = z_1^d B_{b,a,c,d}^\#(z_2, z_1)$, which follows from (28), we may assume w.l.o.g. that $a \geq b$. This implies

$$k(\ell) = k = b + c + d, \quad \ell = 0, \dots, d.$$

We show next that the partial derivative $D^{(c+d,b)}$ of order k of the right-hand side in (29) does not vanish at $(z_1, z_2) = (1, -1)$. This will imply that $z_2^d B_{a,b,c,d}^\#(z_1, z_2)$ and, hence,

$B_{a,b,c,d}^\#(z_1, z_2)$ are not in \mathcal{J}^{k+1} . For $\ell = 0, \dots, d$, the Leibniz rule yields

$$\begin{aligned} & D^{(c+d,b)} B_{a+\ell, b+\ell, c+d-\ell}^\#(z_1, z_2) \\ &= D^{(c+d,0)} B_{a+\ell, 0, 0}^\#(z_1, z_2) \sum_{j=0}^b \binom{b}{j} D^{(0,j)} B_{0, b+\ell, 0}^\#(z_1, z_2) D^{(0, b-j)} B_{0, 0, c+d-\ell}^\#(z_1, z_2) \\ &= \sum_{i=0}^{c+d} \binom{c+d}{i} D^{(i,0)} B_{a+\ell, 0, 0}^\#(z_1, z_2) \\ & \quad \sum_{j=0}^b \binom{b}{j} D^{(0,j)} B_{0, b+\ell, 0}^\#(z_1, z_2) D^{(c+d-i, b-j)} B_{0, 0, c+d-\ell}^\#(z_1, z_2). \end{aligned}$$

Letting $(z_1, z_2) = (1, -1)$, we see that the sum vanishes for $\ell = 1, \dots, d$, since

$$D^{(0,j)} B_{0, b+\ell, 0}^\#(1, -1) = 0, \quad j = 0, \dots, b.$$

For $\ell = 0$, the second sum reduces to the summand with $j = b$ since, again,

$$D^{(0,j)} B_{0, b, 0}^\#(1, -1) = 0, \quad j = 0, \dots, b-1.$$

Thus the second sum equals to

$$D^{(0,b)} B_{0, b, 0}^\#(1, -1) D^{(c+d-i, 0)} B_{0, 0, c+d}^\#(1, -1)$$

which is zero except for $i = 0$. The entire sum takes the value

$$B_{a, 0, 0}^\#(1, -1) D^{(0,b)} B_{0, b, 0}^\#(1, -1) D^{(c+d, 0)} B_{0, 0, c+d}^\#(1, -1) = 2^{-b} b! (c+d)! \left(-\frac{1}{2}\right)^{c+d} \neq 0.$$

Therefore, the right-hand side in (29) is not in \mathcal{J}^{k+1} . \square

Remark. The proof of Proposition 2.6 establishes the connection between the smoothness of the (a, b, c, d) box spline, which can be determined using the results in [1], and the order k in (30). This box spline has the smoothness $L_\infty^{(\kappa-1)} \subset C^{(\kappa-2)}$ with

$$\kappa = \min\{a+b+c, a+b+d, a+c+d, b+c+d\} = a+b+c+d - \max\{a, b, c, d\},$$

while the order k (degree $k-1$) of polynomial reproduction is given by

$$k = \min\{b+c+d, a+c+d, a+b\} = a+b+c+d - \max\{a, b, c+d\}.$$

Corollary 2.7. *For the four-directional (a, b, c, d) box spline we have $\kappa - k \geq 0$, and*

$$\kappa - k > 0 \quad \Leftrightarrow \quad c+d > \max\{a, b\} \quad \text{and} \quad \min\{c, d\} > 0.$$

Proof. By the above remark,

$$\kappa - k = \max\{a, b, c + d\} - \max\{a, b, c, d\} \geq 0.$$

In the case $\max\{a, b, c + d\} = \max\{a, b\}$ we have $\kappa - k = 0$, while in the case $c + d > \max\{a, b\}$ we get

$$\kappa - k = c + d - \max\{a, b, c, d\} = \min\{c + d - a, c + d - b, c, d\}.$$

The first two entries in the latter quadruple are positive, and

$$\kappa - k = 0 \iff \min\{c, d\} = 0.$$

In the case $d = 0$, we are in the three-directional case, while for $c = 0$ we have, using (28),

$$B_{a,b,0,d}^{\#}(z_1, z_2^{-1}) = z_2^{-b} B_{a,b,d}^{\#}(z_1, z_2).$$

This shows that these four-directional splines are actually three-directional ones reflected about the z_1 -axis. \square

We consider some mask symbols examples next and give their representations in terms of the generators from the list \mathcal{L}_k .

The (1,1,1,1) box spline (the Zwart-Powell element) has the mask symbol

$$a(z_1, z_2) = 4 z_2^{-1} \frac{z_1 + z_2}{2} B_{1,1,1}^{\#}(z_1, z_2).$$

In this case $\kappa = 4 - 1 = 3$, $k = 4 - 2 = 2$, and the associated subdivision scheme reproduces polynomials of total degree at most one.

The (2,2,1,1) box spline has the mask symbol

$$a(z_1, z_2) = a z_2^{-1} \frac{z_1 + z_2}{2} B_{2,2,1}^{\#}(z_1, z_2).$$

We have $\kappa = 6 - 2 = 4$ and $k = 6 - 2 = 4$, telling us that polynomials of degree up to 3 are reproduced. The representation of $a(z_1, z_2)$ with the generators from \mathcal{L}_4 is given by

$$a(z_1, z_2) = 4 \left\{ 2 B_{3,3,1}^{\#}(z_1, z_2) - B_{2,2,2}^{\#}(z_1, z_2) \right\}.$$

More interesting are the higher order four-directional splines. For example, the (4,4,1,1) box spline has $\kappa = 10 - 4 = 6$ and the order of polynomial reproduction $k = 10 - 4 = 6$. Its mask symbol can be represented as

$$\begin{aligned} a(z_1, z_2) &= 4 z_2 \cdot 2^{-10} (1 + z_1)^4 (1 + z_2)^4 (1 + z_1 z_2) \left(1 + \frac{z_1}{z_2}\right) \\ &= 4 \left\{ 2 B_{5,5,1}^{\#}(z_1, z_2) - B_{4,4,2}^{\#}(z_1, z_2) \right\}. \end{aligned}$$

(Compare this with the list \mathcal{L}_6 .)

In the masks displayed below the boldface entry at bottom-left position refers to the index (0,0). This assumption is not really important, since we can always multiply the corresponding Laurent polynomial symbol with a unit to produce a corresponding shift of the mask.

2.2.3 A bivariate interpolatory scheme

Interpolatory schemes are characterized by the fact that one of the submask is a δ sequence, or equivalently, some subsymbol is identically one. Theorem 2.4, thus, allows us to present a systematic way for creating interpolatory schemes from our lists of generators by equating the coefficients of their affine combinations and normalizing appropriately.

To provide just one such example, consider the interpolatory scheme in [14, Example 2], a bivariate version of the univariate four-point interpolation scheme given in [11]. Its mask is

$$A = \frac{1}{32} \begin{pmatrix} 0 & 0 & -1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 10 & 18 & 10 & 0 & -1 \\ -2 & 0 & 18 & 32 & 18 & 0 & -2 \\ -1 & 0 & 10 & 18 & 10 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & 0 & -1 & -2 & -1 & 0 & 0 \end{pmatrix}.$$

The scheme reproduces polynomials up to degree $k - 1 = 3$, whence $\mathbf{a}(z) \in \mathcal{J}^4$, and a representation of $\mathbf{a}(z)$ in terms of three-directional box splines from the list \mathcal{L}_4 is given by

$$\begin{aligned} z_1^3 z_2^3 \mathbf{a}(z_1, z_2) &= -2^4 B_{4,4,0}^\#(z_1, z_2) - 2(z_1^2 + z_2^2) B_{2,2,2}^\#(z_1, z_2) + 2^3 (1 + z_1 + z_2) B_{3,3,1}^\#(z_1, z_2) \\ &= 4 \left\{ -4 B_{4,4,0}^\#(z_1, z_2) - \frac{z_1^2 + z_2^2}{2} B_{2,2,2}^\#(z_1, z_2) + 6 \frac{1 + z_1 + z_2}{3} B_{3,3,1}^\#(z_1, z_2) \right\}. \end{aligned}$$

From the second line, the weights λ are recognized as -4 , -1 , and 6 , and the normalized σ -symbols are

$$1, \quad \frac{z_1^2 + z_2^2}{2}, \quad \text{and} \quad \frac{1 + z_1 + z_2}{3},$$

respectively.

2.2.4 The butterfly scheme

The butterfly scheme has been studied in [13] and [14, Example 5]. Its mask is given by

$$A = \frac{1}{16} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 2 & 8 & 8 & 2 & -1 \\ 0 & 0 & 8 & 16 & 8 & 0 & 0 \\ -1 & 2 & 8 & 8 & 2 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ \mathbf{0} & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is again an interpolating scheme, and reproduces polynomials of degree $k - 1 = 3$. The representation of the mask symbol in terms of three-directional box spline symbols from the

list \mathcal{L}_4 is given by

$$\begin{aligned}
& z_1^3 z_2^3 a(z_1, z_2) \\
&= 4 \left\{ 26 \frac{7 + 6 z_1 z_2}{13} B_{3,3,1}^\#(z_1, z_2) - 2 z_2 B_{3,1,3}^\#(z_1, z_2) - 2 z_1 B_{1,3,3}^\#(z_1, z_2) \right. \\
&\quad \left. - 21 \frac{1 + z_1 + z_2}{3} B_{2,2,2}^\#(z_1, z_2) \right\} \\
&= 4 \left\{ 7 z_1 z_2 B_{2,2,2}^\#(z_1, z_2) - 2 z_1 B_{1,3,3}^\#(z_1, z_2) - 2 z_2 B_{3,1,3}^\#(z_1, z_2) - 2 z_1 z_2 B_{3,3,1}^\#(z_1, z_2) \right\}.
\end{aligned}$$

We see that the generators are all multiples of $B_{1,1,1}^\#(z_1, z_2)$. This tells us that the symbol can be factorized as

$$a(z_1, z_2) = B_{1,1,1}^\#(z_1, z_2) b(z),$$

a fact noticed in [13]. We would like to emphasize the following properties of the butterfly scheme. Firstly, a simple computation yields that the symbol $b(z)$ does not define a convergent subdivision scheme, although each of the summands in $b(z)$ by itself does correspond to a convergent scheme. Secondly, butterfly is an interpolatory subdivision scheme, but none of the summands in the affine combination above possess this property.

2.2.5 A convergent scheme

The symbols presented in the above examples all possess a property that is very important for their regularity analysis: they are multiples of one specific box spline symbol of type $B_{a,b,c}^\#$. The regularity analysis of such schemes is given in [12], Section 4.3. The type of factorization used there, however, is a very special situation which does not generally hold for convergent schemes.

A very simple example that comes to mind is the symbol given by

$$\begin{aligned}
a(z_1, z_2) &= 4 \left\{ \frac{1}{2} B_{1,1,0}^\#(z_1, z_2) + \frac{1}{2} B_{0,1,2}^\#(z_1, z_2) \right\} = 4 \frac{1 + z_2}{2} c(z_1, z_2) \\
\text{with } c(z_1, z_2) &= \frac{1}{2} \frac{1 + z_1}{2} + \frac{1}{2} \left(\frac{1 + z_1 z_2}{2} \right)^2.
\end{aligned}$$

By Theorem 2.2, the symbol $a(z_1, z_2)$ is in \mathcal{J} , but none of the generators from the list \mathcal{L}_1 divides the symbol.

In order to check convergence of the scheme, we have to look at representations according to eq. (15). One possible such representation is given by

$$\begin{aligned}
\mathbf{B}(z_1, z_2) &= \begin{pmatrix} b_{11}(z_1, z_2) & b_{12}(z_1, z_2) \\ b_{21}(z_1, z_2) & b_{22}(z_1, z_2) \end{pmatrix} \\
\text{with } b_{11}(z_1, z_2) &= \frac{1}{4} \{ z_1 z_2^3 - z_2^3 + z_1 z_2^2 + z_2^2 + 4z_2 + 2 \} \\
b_{12}(z_1, z_2) &= 0 \\
b_{21}(z_1, z_2) &= \frac{1}{4} \{ z_1 z_2 - z_1 - z_2 + 1 \} \\
\text{and } b_{22}(z_1, z_2) &= \frac{1}{4} \{ z_1^2 z_2^2 + 2z_1 z_2 + 2z_1 + 3 \}.
\end{aligned}$$

We have to verify that the vector subdivision scheme $\mathcal{S}_{\mathbf{B}}$ converges to zero. This has been checked through symbolic calculations; but we had to go as far as to the fifth power of $\mathcal{S}_{\mathbf{B}}$ to yield contractivity.

We note that in this example the two building blocks, with symbols $4 B_{1,1,0}^{\#}(z_1, z_2)$ and $4 B_{0,1,2}^{\#}(z_1, z_2)$, are not symbols of C -convergent subdivision schemes, while the combination yields C -convergence. We also refer to the constructions in [8], where the convex combination of a four-directional, zero order box spline and a C^1 -quadratic box spline are used to obtain the so-called GP pseudo-quadratic box spline. This example shows enhancement with respect to linear independence of the translates, at the expense of reduced joint smoothness.

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