DIFFERENTIABILITY PROPERTIES OF THE SOLUTION OPERATOR TO AN ELLIPTIC VARIATIONAL INEQUALITY OF THE SECOND KIND

C. CHRISTOF[§] AND C. MEYER[§]

Abstract. We study the stability of solutions to H_0^1-elliptic variational inequalities of the second kind that contain a non-differentiable Nemytskii operator. The local Lipschitz continuity of the solution map with respect to perturbations of the right-hand side and perturbations of the coefficient of the Nemytskii operator is proved for a large class of problems and directional differentiability results are obtained under suitable structural assumptions. It is further shown that directional derivatives of the solution map are typically characterized by elliptic variational inequalities in weighted Sobolev spaces whose bilinear forms contain surface integrals and whose right-hand sides depend on the direction of the derivative. Our work extends results recently obtained by De los Reyes and Meyer and demonstrates that fine properties of the solution and (pull backs of) distributional derivatives have to be taken into account when it comes to the sensitivity analysis for elliptic variational inequalities of the second kind in Sobolev spaces.

Key words. Elliptic Variational Inequalities of the Second Kind, Directional Differentiability, Sensitivity Analysis, Lipschitz Stability, Optimal Control of Variational Inequalities, Distributional Pull Back, Control to State Map, Variational Analysis, Regularity, Parameter Integrals

AMS subject classifications. 34E05 35B30 35J20 35R45 47J20 49J40 49K40 65K15

1. Introduction. While the continuity and differentiability of solution operators to elliptic variational inequalities of the first kind have been studied in a multitude of papers (cf. [3,4,10,13,14,18,19,23]), sensitivity results for elliptic variational inequalities of the second kind can only hardly be found in the literature (see [6,24] for two of the few contributions). This is in particular unsatisfying regarding the increasing interest in the analysis and numerical solution of optimal control problems governed by this type of variational inequality (cf. [5,6]). With this paper we hope to make a further step towards the extension of the known continuity and differentiability results to elliptic variational inequalities of the second kind by analyzing in detail problems of the form

$$w \in H_0^1(\Omega), \;\; a(w, v-w) + \int_\Omega cj(v)\mathrm{d}\lambda - \int_\Omega cj(w)\mathrm{d}\lambda \geq \langle f, v-w\rangle \;\; \forall v \in H_0^1(\Omega), \quad \text{(P)}$$

where a is a continuous coercive bilinear form and $j : \mathbb{R} \to [0,\infty)$ is a convex function satisfying $j(0) = 0$. The outline of this paper is the following:

In Section 2 we clarify the notation, make precise our assumptions and discuss basic results concerning the existence, uniqueness and regularity of solutions to (P) that are needed throughout this paper.

In Section 3 we prove the local H^1- and L^∞-Lipschitz continuity of the solution map $S : (c, f) \mapsto w$ associated with the variational inequality (P) (see Theorem 3.1).

In Section 4 we give an overview of the strategy that we use in the subsequent section to study the (directional) differentiability of the solution map S.

[§]Faculty of Mathematics, Technische Universität Dortmund, Dortmund-Germany.

Section 5 is concerned with the differentiability of S for functions j which are twice continuously differentiable away from the origin. It will be seen here that fine properties of the solution w and the second distributional derivative of j are relevant for the differentiability properties of the solution operator S and that directional derivatives of S are typically characterized by elliptic variational inequalities in weighted Sobolev spaces. See Theorem 5.15 for our main result.

Lastly, in Section 6 we conclude our investigation, interpret our findings and address open questions.

It should be noted that the (weak) directional differentiability of the solution operator $S : f \mapsto w$ to a problem of the form

$$w \in H_0^1(\Omega), \;\; a(w, v-w) + \int_\Omega |v| \mathrm{d}\lambda - \int_\Omega |w| \mathrm{d}\lambda \geq \langle f, v-w\rangle \;\; \forall v \in H_0^1(\Omega) \qquad (1.1)$$

has already been studied by De los Reyes and Meyer in [6]. These two authors used the idea to reformulate the variational inequality (1.1) as a variational equality by introducing a slack variable q and to analyze the convergence behavior of the difference quotients associated with q and the solution w to obtain information about the regularity of the solution map S. This primal-dual approach yields results similar to ours if right-hand sides f are considered where the solution $w = S(f)$ is continuous, where the sets $\{w > 0\}$ and $\{w < 0\}$ are at a positive distance from each other, and where the level set $\{w = 0\}$ does not have $(d-1)$-dimensional components (cf. [6, Assumptions 3.13, 3.16]).

In this paper, we use a methodology that does not require the introduction and analysis of slack variables but solely works with primal quantities. This allows us to study (1.1) (and similar problems) in the situation where $(d-1)$-dimensional parts of the set $\{w = 0\}$ are present and, moreover, enables us to show that these lower dimensional components of the active set manifest themselves as surface integrals in the variational inequalities that characterize the directional derivatives of the solution operator S.

2. Preliminaries and Notation.

In what follows, we use the standard notation $H_0^1(\Omega)$, $W^{k,p}(\Omega)$, $C^{k,\gamma}(\overline{\Omega})$, $k \in \mathbb{N}$, $1 \leq p \leq \infty$, $\gamma \in (0,1]$, for the Sobolev and Hölder spaces on a domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$. We refer to [1,8] for details on these spaces. The dual of $H_0^1(\Omega)$ w.r.t. $L^2(\Omega)$ and the associated dual pairing are denoted with $H^{-1}(\Omega)$ and $\langle .,. \rangle$, respectively. With λ^k and $\mathcal{H}^k$ we denote the k-dimensional Lebesgue and Hausdorff measure (where $\mathcal{H}^k$ is assumed to be scaled as in [9, Definition 2.1] such that it coincides with the surface measure on sufficiently regular sets). When the dimension is clear from the context, we drop the index k and simply write λ. Further, we define $x^+ := \max(0, x)$, $x^- := \min(0, x)$ and $L^p_+(\Omega) := \{c \in L^p(\Omega) : c \geq 0 \text{ a.e. in } \Omega\}$, $1 \leq p \leq \infty$. With C we denote a generic constant. If we want to emphasize that C depends on a quantity α, we write $C = C(\alpha)$. The topological closure and interior of a set S are denoted with $\mathrm{cl}(S)$ and $\mathrm{int}(S)$, respectively. For subsets of the Euclidean space we also use the notation $\overline{S} := \mathrm{cl}(S)$.

As already mentioned, the purpose of this paper is to study variational inequalities of the form

$$w \in H_0^1(\Omega), \;\; a(w, v-w) + \int_\Omega c j(v) \mathrm{d}\lambda - \int_\Omega c j(w) \mathrm{d}\lambda \geq \langle f, v-w\rangle \;\; \forall v \in H_0^1(\Omega). \quad \text{(P)}$$

Our standing assumptions on the quantities a, j and Ω appearing in (P) are as follows:

Assumption 2.1 (Standing Assumptions).

- *$\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded (strong) Lipschitz domain,*
- *$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is a (not necessarily symmetric) bilinear form defined by $a(v_1, v_2) := \langle A v_1, v_2 \rangle$, where $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is a second order partial differential operator of the type*

$$Av := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left(\alpha_{ij} \frac{\partial}{\partial x_i} v \right) + \beta v$$

with $\alpha_{ij} \in C^1(\overline{\Omega})$ and $\beta \in L_+^\infty(\Omega)$ such that there exists a $C > 0$ with

$$\sum_{i,j=1}^{d} \alpha_{ij}(x) \zeta_i \zeta_j \geq C|\zeta|^2 \quad \forall \zeta \in \mathbb{R}^d \quad \forall x \in \Omega,$$

- *$j : \mathbb{R} \to [0, \infty)$ is a convex function satisfying $j(0) = 0$.*

We point out that in Section 5 we tighten the assumptions on j and confine ourselves to the situation where j can be written in the form $j(x) = j_1(x^+) + j_2(-x^-)$ with non-negative, convex functions $j_1, j_2 \in C^2([0, \infty))$ satisfying $j_1(0) = j_2(0) = 0$ and $j_1'(0), j_2'(0) > 0$. Up to then, however, the minimal regularity in Assumption 2.1 is sufficient for our needs. Note that standard arguments from convex analysis (see, e.g., [7]) yield the following:

Lemma 2.2. *If $j : \mathbb{R} \to [0, \infty)$ satisfies the conditions in Assumption 2.1, then it is true that:*

a) j is Lipschitz on bounded sets,

b) j is directionally differentiable in every $x \in \mathbb{R}$ in all directions $h \in \mathbb{R}$ and the directional derivative $j'(x; h)$ satisfies

$$j'(x; h) = \inf_{t>0} \left(\frac{j(x + th) - j(x)}{t} \right),$$

c) j is Hadamard differentiable in every $x \in \mathbb{R}$ in all directions $h \in \mathbb{R}$, i.e., if $(h_n) \subset \mathbb{R}$, $(t_n) \subset (0, \infty)$ are sequences satisfying $h_n \to h$ and $t_n \to 0$, then it holds

$$j'(x; h) = \lim_{n \to \infty} \left(\frac{j(x + t_n h_n) - j(x)}{t_n} \right),$$

d) $j|_{[0,\infty)}$ is monotonically increasing and $j|_{(-\infty,0]}$ is monotonically decreasing,

e) for all $x, y \in \mathbb{R}$, it holds

$$j'(x; y - x) + j'(y; x - y) \leq 0.$$

We emphasize that we do not impose any conditions on the growth of the function j for $x \to \pm\infty$. Consequently, the functional

$$v \mapsto \int_\Omega c j(v) \mathrm{d}\lambda$$

appearing in (P) may take the value $+\infty$ if $c \geq 0$ holds a.e. in Ω. To circumvent problems with this mapping behavior, we will often make use of the following results that go back to Stampacchia:

Lemma 2.3 (Stampacchia).

a) For all $v \in H_0^1(\Omega)$ it holds $v^+, v^- \in H_0^1(\Omega)$ and

$$\nabla(v^+) = \begin{cases} \nabla v & \text{a.e. in } \{v > 0\} \\ 0 & \text{a.e. in } \{v \leq 0\} \end{cases}, \qquad \nabla(v^-) = \begin{cases} \nabla v & \text{a.e. in } \{v < 0\} \\ 0 & \text{a.e. in } \{v \geq 0\} \end{cases}.$$

Moreover, the map $H_0^1(\Omega) \ni v \mapsto (v^+, v^-) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is continuous.

b) If $v \in H_0^1(\Omega)$ is a function such that

$$v_k := v - \min(k, \max(v, -k)) = \begin{cases} v - k & \text{a.e. in } \{v \geq k\} \\ 0 & \text{a.e. in } \{|v| < k\} \\ v + k & \text{a.e. in } \{v \leq -k\} \end{cases}, \quad k \geq 0, \tag{2.1}$$

satisfies

$$\|v_k\|_{H^1}^2 \leq \int_\Omega f |v_k| \mathrm{d}\lambda \quad \forall k \geq 0 \tag{2.2}$$

for some $f \in L^p(\Omega)$ with $p > \max(d/2, 1)$, then there exists a constant C depending only on Ω, p and d such that

$$\|v\|_{L^\infty} \leq C \|f\|_{L^p}.$$

Proof. Part a) of Lemma 2.3 is classical and can be found, e.g., in [17, Theorem II.A1] (see also [2, Theorem 5.8.2] and [12, Corollary 2.1]). The L^∞-bound in b) is obtained along the lines of [17, Lemma II.B2] (cf. [6, Lemma 3.8]). Since we will use b) several times, we briefly recall the proof for the convenience of the reader: Let us assume that $d > 2$ and define $L(h) := \{|v| \geq h\}$, where, as usual, $\{|v| \geq h\}$ is shorthand for $\{x \in \Omega : |v(x)| \geq h\}$ (defined up to sets of measure zero). Then it follows from the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{2d/(d-2)}(\Omega)$ that there exists a $C > 0$ which depends only on Ω and d such that for all $h \geq k \geq 0$ we have

$$\|v_k\|_{H^1}^2 \geq C^2 \|v_k\|_{L^{2d/(d-2)}}^2 \geq C^2 \left(\int_{L(h)} (|v| - k)^{\frac{2d}{d-2}} \mathrm{d}\lambda \right)^{\frac{d-2}{d}} \geq C^2 (h-k)^2 \lambda(L(h))^{\frac{d-2}{d}}. \tag{2.3}$$

On the other hand, our assumption $p > \max(d/2, 1)$ implies

$$\frac{p(d+2)}{2d} > \frac{d+2}{4} \geq 1$$

and we may use Hölder's inequality, Young's inequality and the Sobolev embeddings to obtain (with the same constant C as in (2.3))

$$\begin{aligned} \int_\Omega f |v_k| \mathrm{d}\lambda &\leq \frac{1}{C} \left(\int_{L(k)} |f|^{\frac{2d}{d+2}} \mathrm{d}\lambda \right)^{\frac{d+2}{2d}} \|v_k\|_{H^1} \\ &\leq \frac{1}{C} \left(\left(\int_{L(k)} |f|^p \mathrm{d}\lambda \right)^{\frac{2d}{p(d+2)}} \lambda(L(k))^{1 - \frac{2d}{p(d+2)}} \right)^{\frac{d+2}{2d}} \|v_k\|_{H^1} \\ &\leq \frac{1}{C} \|f\|_{L^p} \lambda(L(k))^{\frac{d+2}{2d} - \frac{1}{p}} \|v_k\|_{H^1} \\ &\leq \frac{1}{2C^2} \|f\|_{L^p}^2 \lambda(L(k))^{\frac{d+2}{d} - \frac{2}{p}} + \frac{1}{2} \|v_k\|_{H^1}^2. \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4) with (2.2) yields

$$C^2(h-k)^2\lambda(L(h))^{\frac{d-2}{d}} \le \frac{1}{C^2}\|f\|_{L^p}^2\lambda(L(k))^{\frac{d+2}{d}-\frac{2}{p}},$$

i.e.,

$$(h-k)\lambda(L(h))^{\frac{d-2}{2d}} \le \frac{1}{C^2}\|f\|_{L^p}\lambda(L(k))^{\frac{d+2}{2d}-\frac{1}{p}} = \frac{1}{C^2}\|f\|_{L^p}\left[\lambda(L(k))^{\frac{d-2}{2d}}\right]^s \tag{2.5}$$

for all $h \ge k \ge 0$ with

$$s := \left(\frac{2d}{d-2}\right)\left(\frac{d+2}{2d}-\frac{1}{p}\right) > \left(\frac{2d}{d-2}\right)\left(\frac{d+2}{2d}-\frac{2}{d}\right) = 1.$$

From (2.5) and [17, Lemma II.B1] (a simple growth result for real valued functions), it now follows $\lambda(L(k_0)) = 0$ for

$$k_0 := 2^{\frac{s}{s-1}}\frac{1}{C^2}\lambda(\Omega)^{\frac{(s-1)(d-2)}{2d}}\|f\|_{L^p}.$$

This proves the claim for $d > 2$. For $d = 2$ the argumentation used above has to be altered slightly to take into account that the Sobolev conjugate $2d/(d-2)$ degenerates in two dimensions. The modifications, however, are straightforward, so we leave the proof to the reader. For $d = 1$ the claim is trivially true. This completes the proof. ∎

We are now in the position to analyze the solvability of the variational inequality (P):

THEOREM 2.4. *Let Assumption 2.1 hold. Then it is true that:*

a) For all $(c, f) \in L^1_+(\Omega) \times H^{-1}(\Omega)$ there exists a unique solution $w \in H^1_0(\Omega)$ to the variational inequality (P).

b) If $(c, f) \in L^1_+(\Omega) \times L^p(\Omega)$ with $p > \max(d/2, 1)$, then there exists a $C > 0$ which depends only on d, p, Ω and A such that $\|w\|_{L^\infty} \le C\|f\|_{L^p}$.

c) If $(c, f) \in L^{p_1}_+(\Omega) \times L^{p_2}(\Omega)$ with $p_1 > \max(2d/(d+2), 1)$, $p_2 > \max(d/2, 1)$, then for all $v \in H^1_0(\Omega)$ it holds

$$a(w, v) + \int_\Omega cj'(w; v)\mathrm{d}\lambda \ge \langle f, v\rangle . \tag{2.6}$$

d) If $(c, f) \in L^{p_1}_+(\Omega) \times L^{p_2}(\Omega)$ with $p_1 > \max(2d/(d+2), 1)$, $p_2 > \max(d/2, 1)$ and if $j|_{(-\infty,0]}) \in C^1((-\infty, 0])$, $j|_{[0,\infty)} \in C^1([0,\infty))$, then there exists a slack variable $q \in L^\infty(\Omega)$ such that

$$Aw + cq = f \quad \textit{and} \quad q = j'(w) \textit{ a.e. in } \{j(w) \ne 0\}.$$

e) If $(c, f) \in L^p_+(\Omega) \times L^p(\Omega)$, $\max(d/2, 1) < p < \infty$, $j|_{(-\infty,0]}) \in C^1((-\infty, 0])$ and $j|_{[0,\infty)} \in C^1([0,\infty))$ and if Ω has a $C^{1,1}$-boundary, then w is in $W^{2,p}(\Omega)$. In particular, $w \in C^1(\overline{\Omega})$ for all $(c, f) \in L^p_+(\Omega) \times L^p(\Omega)$ with $p > d$.

Proof.

Ad a) The bilinear form a is H^1_0-elliptic and continuous due to Assumption 2.1 and Friedrichs' inequality. Further, we obtain from Fatou's lemma and the properties of j that the functional

$$H^1_0(\Omega) \ni v \mapsto \int_\Omega cj(v)\mathrm{d}\lambda \in \mathbb{R} \cup \{\infty\}$$

is convex, proper and lower semicontinuous for a fixed $c \in L^1_+(\Omega)$. This allows us to apply standard results as, e.g., [12, Theorem 4.1]) to obtain the unique solvability of (P).

Ad b) Let w be the unique solution to (P) and let w_k, $k \geq 0$, be defined as in (2.1). Then we may choose the test function $v := w - w_k$ in (P) to obtain (using Lemma 2.3 a))

$$\begin{aligned}\int_\Omega \sum_{i,j=1}^{d} -\alpha_{ij}\partial_i w_k \partial_j w_k - \beta w w_k \mathrm{d}\lambda \\ + \int_{\{w>k\}} c\Big(j(k) - j(w)\Big)\mathrm{d}\lambda \\ + \int_{\{w<-k\}} c\Big(j(-k) - j(w)\Big)\mathrm{d}\lambda \geq - \int_\Omega f w_k \mathrm{d}\lambda. \end{aligned} \tag{2.7}$$

Since j is monotonically increasing on $[0, \infty)$ and monotonically decreasing on $(-\infty, 0]$, and since $\beta \geq 0$ a.e. in Ω, it follows from (2.7) (in combination with Assumption 2.1 and Friedrichs' inequality) that there exists a constant C depending only on d, Ω and A such that

$$\|w_k\|_{H^1}^2 \leq \int_\Omega C|f||w_k|\mathrm{d}\lambda \quad \forall k \geq 0.$$

Using Lemma 2.3 b) now yields the claim.

Ad c) Let $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $t \in (0,1)$ be arbitrary. Then we may choose the test function $w + tv$ in (P) to obtain

$$a(w,v) + \int_\Omega c\,\frac{j(w+tv) - j(w)}{t}\mathrm{d}\lambda \geq \langle f, v\rangle\,.$$

Note that it follows from $f \in L^{p_2}(\Omega)$, $p_2 > \max(d/2, 1)$, that w is essentially bounded in Ω (see b)). Further, we obtain from Lemma 2.2 a) that j is Lipschitz on the interval $[-\|w\|_{L^\infty} - \|v\|_{L^\infty}, \|w\|_{L^\infty} + \|v\|_{L^\infty}]$. Thus, there exists a $C > 0$ with

$$\left|\frac{j(w+tv) - j(w)}{t}\right| \leq C|v| \in L^\infty(\Omega) \quad \forall t \in (0,1).$$

Using the dominated convergence theorem, we may now deduce

$$a(w,v) + \int_\Omega c j'(w;v)\mathrm{d}\lambda \geq \langle f, v\rangle\,. \tag{2.8}$$

This proves c) for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. If v is unbounded, we can choose the function $v_k := \min(v^+, k) + \max(v^-, -k) \in L^\infty(\Omega)$, $k \geq 0$, in (2.8) and exploit the positive homogeneity of the directional derivative to obtain

$$\begin{aligned} a(w, v_k) + \int_\Omega c j'(w;1)\min(v^+, k)\mathrm{d}\lambda \\ + \int_\Omega c j'(w;-1)\min(-v^-, k)\mathrm{d}\lambda \geq \langle f, v_k\rangle\,. \end{aligned}$$

Letting $k \to \infty$ in the above and using again the dominated convergence theorem (with the majorant $(\|j'(w;1)\|_{L^\infty} + \|j'(w;-1)\|_{L^\infty})c|v| \in L^1(\Omega)$, cf. the Sobolev embeddings, $p_1 > \max(2d/(d+2), 1)$ and the local Lipschitz continuity of the function j) yields (2.6) in the general case.

Ad d) To prove the existence of the slack variable q, we employ an approximation argument: Consider for $\varepsilon > 0$ the regularized variational inequality

$$w_\varepsilon \in H_0^1(\Omega), \quad a(w_\varepsilon, v - w_\varepsilon) + \int_\Omega c j_\varepsilon(v) \mathrm{d}\lambda - \int_\Omega c j_\varepsilon(w_\varepsilon) \mathrm{d}\lambda \geq \langle f, v - w_\varepsilon \rangle \qquad \forall v \in H_0^1(\Omega) \quad (\mathrm{P}_\varepsilon)$$

with

$$j_\varepsilon(x) := \sqrt{\varepsilon + j(x)^2} - \sqrt{\varepsilon} \quad \forall x \in \mathbb{R}.$$

Then $j_\varepsilon : \mathbb{R} \to [0, \infty)$ is a convex function satisfying $j_\varepsilon(0) = 0$ and it follows from a) that there exists a unique solution w_ε to (P_ε). Moreover, we obtain from b) that this solution satisfies $\|w_\varepsilon\|_{L^\infty} \leq C \|f\|_{L^{p_2}}$ for some constant C which does not depend on ε. Note that testing with w in (P_ε) and w_ε in (P) yields

$$a(w - w_\varepsilon, w - w_\varepsilon) \leq \int_\Omega c \left(j_\varepsilon(w) - j(w) + j(w_\varepsilon) - j_\varepsilon(w_\varepsilon) \right) \mathrm{d}\lambda \leq 2\sqrt{\varepsilon} \|c\|_{L^1},$$

i.e., we have $\|w - w_\varepsilon\|_{H^1} \to 0$ as $\varepsilon \to 0$. Further, it follows from the definition of j_ε, $j|_{(-\infty,0])} \in C^1((-\infty, 0])$, $j|_{[0,\infty)} \in C^1([0, \infty))$ and $j(0) = 0$ that j_ε is continuously differentiable with

$$j_\varepsilon' = \frac{jj'}{\sqrt{\varepsilon + j^2}} \text{ in } \mathbb{R} \setminus \{0\}, \quad j_\varepsilon'(0) = 0.$$

This implies in combination with c) that

$$a(w_\varepsilon, v) + \int_\Omega c \frac{j(w_\varepsilon) j'(w_\varepsilon)}{\sqrt{\varepsilon + j(w_\varepsilon)^2}} \mathbb{1}_{\{w_\varepsilon \neq 0\}} v \mathrm{d}\lambda = \langle f, v \rangle \tag{2.9}$$

holds for all $v \in H_0^1(\Omega)$, where $\mathbb{1}_{\{w_\varepsilon \neq 0\}} \in L^\infty(\Omega)$ is the indicator function of the set $\{w_\varepsilon \neq 0\}$. The slack variable

$$q_\varepsilon := \frac{j(w_\varepsilon) j'(w_\varepsilon)}{\sqrt{\varepsilon + j(w_\varepsilon)^2}} \mathbb{1}_{\{w_\varepsilon \neq 0\}} \tag{2.10}$$

appearing in (2.9) satisfies

$$\|q_\varepsilon\|_{L^\infty} \leq \max_{x \in [-\|w_\varepsilon\|_{L^\infty}, \|w_\varepsilon\|_{L^\infty}] \setminus \{0\}} |j'(x)| \leq C < \infty$$

with a constant C independent of ε. Thus, the family $\{q_\varepsilon\}_{\varepsilon > 0}$ is bounded in $L^\infty(\Omega)$ and we can apply the theorem of Banach-Alaoglu to extract a subsequence (unrelabeled) such that q_ε converges to some $q \in L^\infty(\Omega)$ w.r.t. the weak*-topology in $L^\infty(\Omega) \cong L^1(\Omega)^*$ as $\varepsilon \to 0$. Using that $\|w - w_\varepsilon\|_{H^1} \to 0$ holds for $\varepsilon \to 0$ and that we have $cv \in L^1(\Omega)$ due to the Sobolev embeddings and $p_1 > \max(2d/(d + 2), 1)$, we can pass to the limit in (2.9) to obtain that the weak limit q satisfies

$$a(w, v) + \int_\Omega c\, q v \mathrm{d}\lambda = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega).$$

This proves the claim. The fact that $q = j'(w)$ holds a.e. in $\{j(w) \neq 0\}$ follows trivially from the pointwise convergence of (a subsequence of) q_ε to $j'(w)$ a.e. in $\{j(w) \neq 0\}$ (see (2.10) and $\|w - w_\varepsilon\|_{H^1} \to 0$).

Ad e) If $(c, f) \in L^p_+(\Omega) \times L^p(\Omega)$, $\max(d/2, 1) < p < \infty$, $j|_{(-\infty,0])} \in C^1((-\infty, 0])$ and $j|_{[0,\infty)} \in C^1([0, \infty))$, then it follows from part d) of the theorem and $\max(d/2, 1) \geq \max(2d/(d+2), 1)$ for all $d \geq 1$ that there exists a $q \in L^\infty(\Omega)$ with

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial}{\partial x_i} w\right) + \beta w = -cq + f =: \tilde{f}.$$

Thus, w solves an elliptic partial differential equation of second order with right-hand side $\tilde{f} \in L^p(\Omega)$ and we can employ standard regularity results (see [11, Theorem 9.15]) and the Sobolev embeddings to obtain the claim.

□

We remark that part d) of Theorem 2.4 can also be proved with dualization arguments (cf. [5]). Note that in what follows, we will not analyze the sensitivity of the slack variable q with respect to perturbations of c and f (in contrast to the approach in [6]). In our analysis, the slack variable q is only needed to obtain the regularity result in Theorem 2.4 e).

3. Local Lipschitz Continuity of the Solution Map. Having studied the existence, the uniqueness and the regularity of the solution w to (P), we now turn our attention to the mapping properties of the solution operator $S : (c, f) \mapsto w$. In what follows, we first analyze the continuity of S as a function from $L^p_+(\Omega) \times L^p(\Omega)$, $p > \max(d/2, 1)$, to $H^1_0(\Omega)$. The main result of this section is:

THEOREM 3.1. *Let Assumption 2.1 hold, let $p > \max(d/2, 1)$, and let $r > 0$ be arbitrary but fixed. Then there exists a constant C depending only on d, p, Ω, A, r and j such that the solution operator $S : (c, f) \mapsto w$ associated with (P) satisfies*

$$\begin{aligned}\|S(c_1, f_1) - S(c_2, f_2)\|_{H^1} &+ \|S(c_1, f_1) - S(c_2, f_2)\|_{L^\infty}\\ &\leq C\Big(\|c_1 - c_2\|_{L^p} + \|f_1 - f_2\|_{L^p}\Big)\end{aligned}$$

for all $(c_1, f_1), (c_2, f_2) \in L^p_+(\Omega) \times \{f \in L^p(\Omega) : \|f\|_{L^p} \leq r\}$.

Proof. Let $f_1, f_2 \in L^p(\Omega)$ with $\|f_1\|_{L^p}, \|f_2\|_{L^p} \leq r$ and $c_1, c_2 \in L^p_+(\Omega)$ be arbitrary but fixed. Denote with w_1 the solution $S(c_1, f_1)$ and with w_2 the solution $S(c_2, f_2)$. Then we know from Theorem 2.4 b) and c) that $\|w_1\|_{L^\infty}, \|w_2\|_{L^\infty} \leq C$ holds with a constant C depending only on d, p, Ω, A and r and that

$$a(w_1, v) + \int_\Omega c_1 j'(w_1; v)\mathrm{d}\lambda \geq \langle f_1, v\rangle$$

and

$$a(w_2, -v) + \int_\Omega c_2 j'(w_2; -v)\mathrm{d}\lambda \geq \langle f_2, -v\rangle$$

holds for all $v \in H^1_0(\Omega)$. Adding the above inequalities yields

$$\begin{aligned}a(w_1 - w_2, v) + \int_\Omega (c_1 - c_2) j'(w_1; v)\mathrm{d}\lambda + \int_\Omega c_2\Big(j'(w_1; v) + j'(w_2; -v)\Big)\mathrm{d}\lambda&\\ \geq \langle f_1 - f_2, v\rangle \quad \forall v \in H^1_0(\Omega).&\end{aligned} \tag{3.1}$$

Choosing $v = w_2 - w_1$, we now obtain (using e) in Lemma 2.2, Sobolev embeddings and the local Lipschitz continuity of j)

$$\begin{aligned} &a(w_1 - w_2, w_1 - w_2) \\ &\quad \leq \langle f_1 - f_2, w_1 - w_2\rangle + \int_\Omega (c_1 - c_2) j'(w_1; \operatorname{sgn}(w_2 - w_1))|w_1 - w_2| \mathrm{d}\lambda \\ &\qquad\qquad + \int_\Omega c_2 \Big(j'(w_1; w_2 - w_1) + j'(w_2; w_1 - w_2)\Big) \mathrm{d}\lambda \\ &\quad \leq C\Big(\|f_1 - f_2\|_{L^p} + \|c_1 - c_2\|_{L^p}\Big)\|w_1 - w_2\|_{H^1} \end{aligned}$$

with a constant C depending only on d, p, Ω, r and j. Consequently,

$$\|w_1 - w_2\|_{H^1} \leq C(d, p, \Omega, A, r, j)\Big(\|c_1 - c_2\|_{L^p} + \|f_1 - f_2\|_{L^p}\Big).$$

This proves the local Lipschitz continuity in $H^1(\Omega)$. It remains to prove the pointwise stability estimate. To this end, we choose the test function $v = -(w_1 - w_2)_k$, $k \geq 0$, in (3.1), where $(w_1 - w_2)_k$ is defined as in (2.1). This yields

$$\begin{aligned} &a((w_1 - w_2)_k, (w_1 - w_2)_k) \\ &\quad \leq \langle f_1 - f_2, (w_1 - w_2)_k\rangle + \int_\Omega (c_1 - c_2) j'(w_1; -(w_1 - w_2)_k) \mathrm{d}\lambda \\ &\qquad\qquad + \int_\Omega c_2 \Big(j'(w_1; -(w_1 - w_2)_k) + j'(w_2; (w_1 - w_2)_k)\Big) \mathrm{d}\lambda. \end{aligned} \tag{3.2}$$

Note that the positive homogeneity of the directional derivative, the definition of $(w_1 - w_2)_k$ and e) in Lemma 2.2 imply

$$\begin{aligned} &\int_\Omega c_2 \Big(j'(w_1; -(w_1 - w_2)_k) + j'(w_2; (w_1 - w_2)_k)\Big) \mathrm{d}\lambda \\ &\quad = \int_{\{|w_1 - w_2| > k\}} c_2 \Big(j'(w_1; -\operatorname{sgn}(w_1 - w_2)) + j'(w_2; \operatorname{sgn}(w_1 - w_2))\Big)(|w_1 - w_2| - k) \mathrm{d}\lambda \\ &\quad = \int_{\{|w_1 - w_2| > k\}} c_2 \Big(j'(w_1; w_2 - w_1) + j'(w_2; w_1 - w_2)\Big) \frac{|w_1 - w_2| - k}{|w_1 - w_2|} \mathrm{d}\lambda \\ &\quad \leq 0. \end{aligned}$$

Thus, it follows from (3.2) that

$$\|(w_1 - w_2)_k\|_{H^1}^2 \leq C(d, p, \Omega, A, r, j) \int_\Omega \Big(|f_1 - f_2| + |c_1 - c_2|\Big)|(w_1 - w_2)_k| \mathrm{d}\lambda$$

holds for all $k \geq 0$. Lemma 2.3 b) now yields

$$\|w_1 - w_2\|_{L^\infty} \leq C(d, p, \Omega, A, r, j)\left(\|f_1 - f_2\|_{L^p} + \|c_1 - c_2\|_{L^p}\right).$$

This proves the claim. ◻

4. Strategy for the Differential Sensitivity Analysis. As Theorem 3.1 shows, the solution map $S : L^p_+(\Omega) \times L^p(\Omega) \to H^1_0(\Omega), (c, f) \mapsto w$, $p > \max(d/2, 1)$, is H^1- and L^∞-Lipschitz on bounded sets. This is not only interesting for its own sake (as it implies, for example, that S is even continuous when c is identical zero and the non-differentiable term in (P) degenerates), but also the point of departure for our differential sensitivity analysis: If a tuple $(h, g) \in L^p(\Omega) \times L^p(\Omega)$ satisfying $c + t_0 h \in L^p_+(\Omega)$ for some $t_0 > 0$ is given (with $(c, f) \in L^p_+(\Omega) \times L^p(\Omega)$ arbitrary but fixed), then the local Lipschitz continuity of the solution operator S implies that the difference quotients

$$\delta_t := \frac{S(c + th, f + tg) - S(c, f)}{t}, \quad 0 < t < t_0,$$

remain bounded in $H^1(\Omega)$ and $L^\infty(\Omega)$ as t tends to zero. This yields that for every sequence $t_n \subset (0, t_0)$ satisfying $t_n \to 0$ we can find a subsequence (unrelabeled for simplicity) such that the associated difference quotients δ_{t_n} converge weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$ and pointwise a.e. in Ω to a function $\delta \in H^1_0(\Omega)$. In what follows, the main idea is to show that this weak limit δ is unique, i.e., independent of the choice of (sub)sequences, and that the difference quotients δ_{t_n} converge even strongly in $H^1(\Omega)$. If this is established, then it follows immediately by contradiction that S is directionally differentiable in (c, f) in the direction (h, g) with $S'((c, f); (h, g)) = \delta$ (cf. also with [6]). So let us consider the following situation:

Assumption 4.1.

- $p > \max(d/2, 1)$,
- $(c, f) \in L^p_+(\Omega) \times L^p(\Omega)$ *is arbitrary but fixed (with* $w := S(c, f)$*),*
- $(h, g) \in L^p(\Omega) \times L^p(\Omega)$ *is arbitrary but fixed such that there exists a* $t_0 > 0$ *with* $c + t_0 h \in L^p_+(\Omega)$,
- $0 < t_n < t_0$ *is a sequence tending to zero as* $n \to \infty$,
- *the difference quotients* $\delta_n := \delta_{t_n}$ *associated with* t_n *satisfy*

$$\delta_n \rightharpoonup \delta \text{ in } H^1(\Omega), \quad \delta_n \to \delta \text{ in } L^2(\Omega), \quad \delta_n \to \delta \text{ pointwise a.e. in } \Omega$$

for some $\delta \in H^1_0(\Omega)$.

To prove that the weak limit δ is unique, we first note that the definition of δ_n yields $S(c + t_n h, f + t_n g) = w + t_n \delta_n$. Consequently, for all $v \in H^1_0(\Omega)$ it is true that

$$a(w + t_n\delta_n, v - w - t_n\delta_n) + \int_\Omega (c + t_n h)\, j(v) \mathrm{d}\lambda - \int_\Omega (c + t_n h)\, j(w + t_n\delta_n) \mathrm{d}\lambda \geq \langle f + t_n g, v - w - t_n\delta_n \rangle . \quad (4.1)$$

If we choose functions of the form $v = w + t_n z$, $z \in H^1_0(\Omega)$, in (4.1), then we obtain after some manipulations

$$a(\delta_n, z - \delta_n) + J_n(z) + H_n(z) - J_n(\delta_n) - H_n(\delta_n) \geq \langle g, z - \delta_n \rangle \quad \forall z \in H^1_0(\Omega) \quad (4.2)$$

with

$$H_n(z) := \int_\Omega h\, \frac{j(w + t_n z) - j(w)}{t_n} \mathrm{d}\lambda$$

and

$$J_n(z) := \frac{1}{t_n}\left(a(w,z) + \int_\Omega c\, j'(w;z)\mathrm{d}\lambda - \langle f, z\rangle\right) + \frac{1}{t_n}\left(\int_\Omega c\,\frac{j(w+t_n z) - j(w)}{t_n} - c\, j'(w;z)\mathrm{d}\lambda\right). \tag{4.3}$$

In the following, our aim will be to pass to the limit $n \to \infty$ in (4.2) and to show that the limit δ is itself the solution of an elliptic variational inequality which does not depend on the sequence (t_n) appearing in Assumption 4.1. If this is proved, then δ is clearly unique and the solution map S is indeed directionally differentiable. Note that for the H_n-terms in (4.2) the limit transition $n \to \infty$ is completely unproblematic:

Lemma 4.2. *Let Assumptions 2.1 and 4.1 hold. Then for all $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ it is true that*

$$H_n(z) - H_n(\delta_n) \to \int_\Omega h j'(w;z)\mathrm{d}\lambda - \int_\Omega h j'(w;\delta)\mathrm{d}\lambda.$$

Proof. We know from Theorem 2.4 that $\|w\|_{L^\infty} \le C\|f\|_{L^p}$ and if $z \in L^\infty(\Omega)$, then $\|w + t_n z\|_{L^\infty} \le C\|f\|_{L^p} + t_0\|z\|_{L^\infty}$. Consequently, there exists a constant $C > 0$ such that $\|w\|_{L^\infty}, \|w + t_n z\|_{L^\infty} \in [0, C]$ holds for all n and we may use the local Lipschitz continuity of j in combination with the dominated convergence theorem to obtain

$$H_n(z) = \int_\Omega h\,\frac{j(w+t_n z) - j(w)}{t_n}\mathrm{d}\lambda \to \int_\Omega h j'(w;z)\mathrm{d}\lambda.$$

For $H_n(\delta_n)$, it follows from $w + t_n\delta_n = S(c + t_n h, f + t_n g)$ and Theorem 2.4 that $\|w + t_n\delta_n\|_{L^\infty} \le C(\|f\|_{L^p} + t_0\|g\|_{L^p})$. Further, we have a uniform bound on $\|\delta_n\|_{L^\infty}$ due to Theorem 3.1. Thus, we may again use the dominated convergence theorem and the Hadamard differentiability of j to deduce

$$H_n(\delta_n) = \int_\Omega h\,\frac{j(w+t_n\delta_n) - j(w)}{t_n}\mathrm{d}\lambda \to \int_\Omega h j'(w;\delta)\mathrm{d}\lambda.$$

This proves the claim. □

Unfortunately, passing to the limit $n \to \infty$ with the J_n-terms in (4.2) is much more difficult. Due to the negative powers of t_n in (4.3) and since j is not assumed to be twice continuously differentiable, it is perfectly possible that the sequence $J_n(z)$ diverges and the pointwise behavior of the second order difference quotients

$$\frac{1}{t_n}\left(\frac{j(w+t_n\delta_n) - j(w)}{t_n} - j'(w;\delta_n)\right) \tag{4.4}$$

appearing in $J_n(\delta_n)$ is in general hard to determine. To overcome these problems, we note the following:

Lemma 4.3. *Let Assumptions 2.1 and 4.1 hold. Then it is true that*

$$0 \le \limsup_{n\to\infty}\left(\frac{1}{t_n}\left(a(w,\delta_n) + \int_\Omega c\, j'(w;\delta_n)\mathrm{d}\lambda - \langle f, \delta_n\rangle\right)\right) < \infty$$

and

$$0 \le \limsup_{n\to\infty}\left(\frac{1}{t_n}\left(\int_\Omega c\,\frac{j(w+t_n\delta_n) - j(w)}{t_n} - c\, j'(w;\delta_n)\mathrm{d}\lambda\right)\right) < \infty.$$

Proof. Choosing the test function $z = 0$ in (4.2) yields

$$\begin{aligned}\langle g, \delta_n\rangle - H_n(\delta_n) &\geq \frac{1}{t_n}\left(a(w,\delta_n) + \int_\Omega c\, j'(w;\delta_n)\mathrm{d}\lambda - \langle f, \delta_n\rangle\right) \\ &\quad + \frac{1}{t_n}\left(\int_\Omega c\, \frac{j(w + t_n\delta_n) - j(w)}{t_n} - c\, j'(w;\delta_n)\mathrm{d}\lambda\right) \\ &\geq 0,\end{aligned}$$

where the last inequality holds due to Lemma 2.2 b) and (2.6). The claim now follows immediately from the boundedness of the term $\langle g, \delta_n\rangle - H_n(\delta_n)$. ∎

Lemma 4.3 shows that, although the term $J_n(z)$ might diverge for an arbitrary test function $z \in H_0^1(\Omega)$, the expression $J_n(\delta_n)$ in (4.2) has to remain bounded. In what follows, we will use this boundedness property to obtain additional information about the weak limit δ and to reduce the class of test functions that has to be considered in the variational inequality (4.2). As a first consequence of Lemma 4.3, we obtain:

Lemma 4.4. *Let Assumptions 2.1 and 4.1 hold. Then the weak limit δ of the difference quotients δ_n is an element of the so-called critical cone*

$$T_{crit}(c, f) := \left\{ z \in H_0^1(\Omega) : a(w, z) + \int_\Omega c j'(w; z)\mathrm{d}\lambda = \langle f, z\rangle \right\}.$$

Proof. From the first estimate in Lemma 4.3, it follows

$$\lim_{n\to\infty}\left(a(w,\delta_n) + \int_\Omega c\, j'(w;\delta_n)\mathrm{d}\lambda - \langle f,\delta_n\rangle\right) = 0$$

and the weak convergence $\delta_n \rightharpoonup \delta$ yields $a(w,\delta_n) - \langle f, \delta_n\rangle \to a(w,\delta) - \langle f,\delta\rangle$. Further, we obtain from the uniform L^∞-bound on δ_n, the local Lipschitz continuity of the function j and the dominated convergence theorem

$$\int_\Omega c\, j'(w;\delta_n)\mathrm{d}\lambda = \int_\Omega c\, j'(w;1)\delta_n^+\mathrm{d}\lambda - \int_\Omega c\, j'(w;-1)\delta_n^-\mathrm{d}\lambda \to \int_\Omega c\, j'(w;\delta)\mathrm{d}\lambda.$$

Combining all of the above proves the claim. ∎

Lemma 4.5. *The set $T_{crit}(c, f)$ defined in Lemma 4.4 is a closed convex cone.*

Proof. The cone property and the closedness of $T_{crit}(c, f)$ w.r.t. the H^1-topology are trivial. To see that $T_{crit}(c, f)$ is convex, note that from (2.6) and the convexity of j it follows that for all $z_1, z_2 \in T_{crit}(c, f)$ and all $s \in [0, 1]$ it holds

$$\begin{aligned}0 &\leq a(w, sz_1 + (1-s)z_2) + \int_\Omega c j'(w; sz_1 + (1-s)z_2)\mathrm{d}\lambda - \langle f, sz_1 + (1-s)z_2\rangle \\ &= a(w, sz_1 + (1-s)z_2) - \langle f, sz_1 + (1-s)z_2\rangle \\ &\qquad + \int_\Omega c\left(\lim_{t\to 0^+}\frac{j(sw + (1-s)w + t(sz_1 + (1-s)z_2)) - j(w)}{t}\right)\mathrm{d}\lambda \\ &\leq s\left(a(w,z_1) + \int_\Omega c j'(w;z_1)\mathrm{d}\lambda - \langle f, z_1\rangle\right) \\ &\qquad + (1-s)\left(a(w,z_2) + \int_\Omega c j'(w;z_2)\mathrm{d}\lambda - \langle f, z_2\rangle\right) \\ &= 0.\end{aligned}$$

This proves the claim. ∎

REMARK 4.6. *The critical cone consists precisely of those functions that satisfy the necessary condition* (2.6) *with equality, i.e., those directions for which a linearization of the problem* (P) *does not provide any information about the "optimality" of the solution w (cf. [4]). For an elliptic variational inequality of the first kind whose admissible set is (extended) polyhedric, the critical cone is exactly the admissible set of the projection that characterizes the directional derivatives of the solution map. We refer to [13] and [19] for details on this topic.*

Note that the information $\delta \in T_{crit}(c, f)$ is obtained from the $\mathcal{O}(t_n)$-terms in the perturbed variational inequality (4.1), i.e., Lemma 4.4 is essentially the consequence of a first order perturbation analysis of the problem (P). To be able to pass to the limit in (4.2), it remains to study which "second order information" about δ is encoded in the boundedness property

$$0 \leq \limsup_{n\to\infty} \left(\frac{1}{t_n} \left(\int_\Omega c\, \frac{j(w + t_n\delta_n) - j(w)}{t_n} - c\, j'(w; \delta_n) \mathrm{d}\lambda \right) \right) < \infty. \tag{4.5}$$

Unfortunately, what can be deduced from (4.5) depends heavily on the precise nature of the function j and, at least to the authors' best knowledge, there is currently no general strategy that can be used to study which implications (4.5) has if j is an arbitrary function satisfying the conditions in Assumption 2.1 (although there are, of course, general results about the behavior of second order difference quotients of convex functions, cf. Alexandrov's theorem, [22, Theorem 3.11.2]). As a consequence, in what follows, we have to confine our analysis to a suitable subclass of problems.

5. Variational Inequalities Involving Piecewise Smooth Functions.

Henceforth, we impose the following more restrictive conditions on the function j:

ASSUMPTION 5.1. *It holds $j(x) = j_1(x^+) + j_2(-x^-)$ with non-negative and convex functions $j_1, j_2 \in C^2([0, \infty))$ satisfying $j_1(0) = j_2(0) = 0$ and $j_1'(0), j_2'(0) > 0$.*

REMARK 5.2. *We emphasize that the strategy described in Section 4 and pursued hereafter can also be employed in situations other than that in Assumption 5.1. Our approach is, for example, also applicable if one of the following holds:*

- *j is a C^1-function satisfying the conditions in Assumption 2.1 and j' is locally Lipschitz continuous and directionally differentiable,*
- *$j(x) = |x|^{1+\varepsilon}$ with $\varepsilon > 0$.*

In the above cases, passing to the limit in (4.2) *is even simpler than in the situation of Assumption 5.1 and structural assumptions (as those in Assumption 5.3 below) are not needed to prove the directional differentiability of the solution map S. We remark that it is also possible to extend our analysis to cover the cases where one of the derivatives $j_1'(0)$, $j_2'(0)$ in Assumption 5.1 vanishes and where the function j is non-differentiable at several points. The notational effort, however, increases significantly if this more general setting is considered.*

As we will see in the following, in the situation of Assumption 5.1, the boundedness property (4.5) yields information about the traces of δ on the boundary of the set $\{w \neq 0\}$ (i.e., the boundary of the inactive set). To be able to talk about traces on $\partial\{w \neq 0\}$, we have to make some assumptions:

Assumption 5.3 (Structural Assumptions).

a) *It holds* $f \in L^p(\Omega)$, $p > \max(d/2, 1)$, $0 < c \in C(\overline{\Omega})$ *and* $w \in C^1(\Omega) \cap W^{2,1}(\Omega)$.

b) *The set* $\partial\{w \neq 0\} \subseteq \overline{\Omega}$ *is a* λ^d*-zero set and there exists a set* $\mathcal{C} \subseteq \overline{\Omega}$ *such that the following is true:*
- $\mathcal{C}$ *is closed and has* H^1*-capacity zero (i.e.,* $\operatorname{cap}_2(\mathcal{C}, \mathbb{R}^d) = 0$*),*
- $\partial\{w \neq 0\} \setminus \mathcal{C}$ *is a strong* $(d-1)$*-dimensional Lipschitz submanifold of* $\mathbb{R}^d$*,*
- *the sets*

$$\mathcal{N}^+ := \{\nabla w = 0\} \cap \partial\{w > 0\} \setminus \mathcal{C}$$
$$\mathcal{N}^- := \{\nabla w = 0\} \cap \partial\{w < 0\} \setminus \mathcal{C}$$

are relatively open in $\partial\{w \neq 0\} \setminus \mathcal{C}$.

Here and in what follows, when we use the variable w*, we always mean the* C^1*-representative of the solution* $S(c, f)$.

Some remarks are in order regarding the conditions in Assumption 5.3:

Remark 5.4.

a) *The assumption* $w \in C^1(\Omega) \cap W^{2,1}(\Omega)$ *is automatically fulfilled if* Ω *has a* $C^{1,1}$*-boundary and if* $f \in L^p(\Omega)$ *holds for some* $p > d$ *(see Theorem 2.4 e)).*

b) *Recall that a set* $\mathcal{N} \subset \mathbb{R}^d$ *is called a strong* $(d-1)$*-dimensional Lipschitz submanifold of* $\mathbb{R}^d$ *if the following holds (cf. [25]): For all* $p \in \mathcal{N}$ *there exist an orthogonal transformation* $R \in O(d)$*, an open ball* $B \subset \mathbb{R}^{d-1}$*, an open interval* $J = (a, b)$ *and a Lipschitz continuous map* $h : B \to J$ *such that*

$$p \in R(B \times J) \quad \text{and} \quad \mathcal{N} \cap R(B \times J) = R(\{(x, h(x)) : x \in B\}).$$

Note that here and in what follows, we use the following conventions for the degenerate case $d = 1$*:*
- $\mathbb{R}^0 := \{0\}$ *(with the open ball* $B := \mathbb{R}^0$*),*
- *the set* $\mathbb{R}^0 \times \mathbb{R}$ *is identified with* $\mathbb{R}$*, i.e., if* $J \subseteq \mathbb{R}$ *and* $B = \mathbb{R}^0$*, then* $B \times J := J$ *and* $\overline{B} \times J := J$.

c) *In the situation of Assumption 5.3, the part of* $\partial\{w \neq 0\} \cap \Omega$ *with* $\nabla w \neq 0$*, i.e., the set*

$$\mathcal{M} := \partial\{w \neq 0\} \cap \{\nabla w \neq 0\} = \{w = 0\} \cap \{\nabla w \neq 0\} \subset \Omega,$$

is a $(d-1)$*-dimensional* C^1*-submanifold of* $\mathbb{R}^d$ *(cf. implicit function theorem).*

d) *Since* $\mathcal{N}^+$ *and* $\mathcal{N}^-$ *are relatively open subsets of* $\partial\{w \neq 0\} \setminus \mathcal{C}$*, they are themselves strong* $(d-1)$*-dimensional Lipschitz submanifolds of* $\mathbb{R}^d$.

e) *Since* $\mathcal{M}, \mathcal{N}^+$ *and* $\mathcal{N}^-$ *are strong* $(d-1)$*-dimensional Lipschitz submanifolds, traces on these sets are well-defined (cf. [1, 21]).*

f) *The capacity condition imposed on* $\mathcal{C}$ *in part b) of Assumption 5.3 means the following (cf. [2, Chapter 5.8.2]):*

$$0 = \inf\left\{\|\phi\|_{H^1} : \phi \in C_c(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d),\ 0 \leq \phi \leq 1,\ \phi \equiv 1 \text{ in a nbhd. of } \mathcal{C}\right\}.$$

The above property will ensure that $\mathcal{C}$ *is so "small" that we can neglect it (cf. the proof of Lemma 5.14). In practice,* $\mathcal{C}$ *will contain lower dimensional parts of the boundary* $\partial\{w \neq 0\}$ *that cannot be handled analytically without major problems, e.g., isolated zeros of the function* w *or points where the set* $\partial\{w \neq 0\}$ *has a cusp (cf. Figure 5.1).*

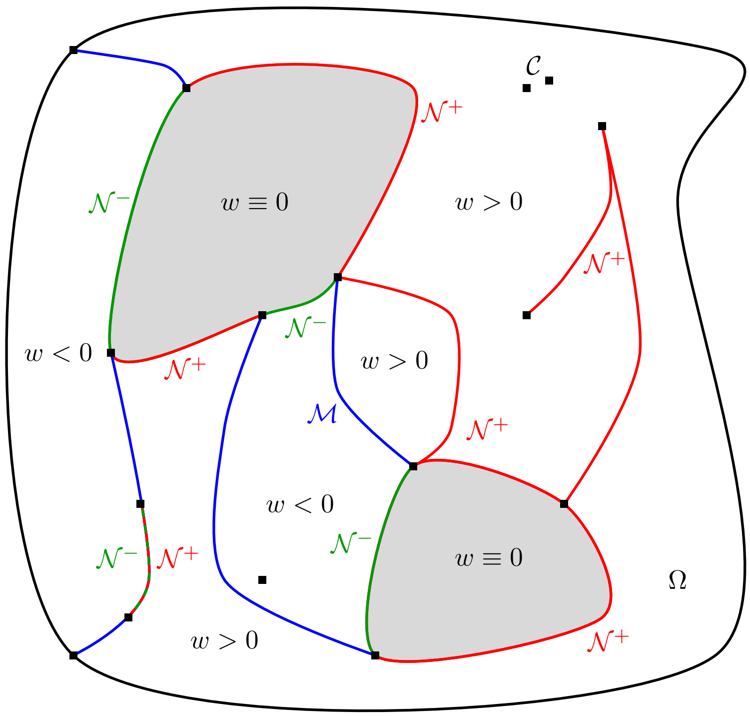

Fig. 5.1. *The geometric situation in Assumption 5.3. All visible lines are part of the boundary $\partial\{w \neq 0\} \subseteq \overline{\Omega}$. In the grey sets, it holds $w \equiv 0$. Points contained in $\mathcal{C}$ are marked by black squares. The sets $\mathcal{M}, \mathcal{N}^-$ and $\mathcal{N}^+$ are depicted in blue, green and red, respectively. We point out that $\mathcal{N}^+$ and $\mathcal{N}^-$ do not necessarily have to be disjoint and that there is always a change of sign along $\mathcal{M}$.*

Note that in the situation of Assumptions 5.1 and 5.3, our first order condition in Lemma 4.4 can be rewritten as follows:

Lemma 5.5. *Suppose that Assumptions 2.1, 5.1 and 5.3 are satisfied. Then it holds $-cj_2'(0) \leq f \leq cj_1'(0)$ λ^d-a.e. in $\{w = 0\}$ and it is true that*

$$\begin{aligned} z \in T_{crit}(c, f) &\iff z^+ \in T_{crit}(c, f) \text{ and } z^- \in T_{crit}(c, f) \\ &\iff z^+ = 0 \ \lambda^d\text{-a.e. in } \{w = 0\} \cap \{-cj_2'(0) \leq f < cj_1'(0)\} \text{ and} \\ &\qquad z^- = 0 \ \lambda^d\text{-a.e. in } \{w = 0\} \cap \{-cj_2'(0) < f \leq cj_1'(0)\}. \end{aligned} \tag{5.1}$$

Proof.

a) Since $w \in W^{2,1}(\Omega)$ and $f \in L^1(\Omega)$, (2.6) implies

$$\int_\Omega (Aw) z \mathrm{d}\lambda + \int_{\{w \neq 0\}} cj'(w) z \mathrm{d}\lambda + \int_{\{w=0\}} c\Big(j_1'(0)z^+ - j_2'(0)z^-\Big)\mathrm{d}\lambda \geq \int_\Omega f z \mathrm{d}\lambda \quad \forall z \in C_c^\infty(\Omega). \tag{5.2}$$

If we choose test functions $z \in C_c^\infty(\{w \neq 0\})$ in the above inequality, we obtain (using that the set $\{w \neq 0\}$ is open)

$$Aw + cj'(w) - f = 0 \text{ a.e. in } \{w \neq 0\}. \tag{5.3}$$

From (5.2) and (5.3), it follows

$$\int_{\{w=0\}} (Aw)z\mathrm{d}\lambda + \int_{\{w=0\}} c\Big(j_1'(0)z^+ - j_2'(0)z^-\Big)\mathrm{d}\lambda \geq \int_{\{w=0\}} fz\mathrm{d}\lambda \qquad (5.4)$$

for all $z \in C_c^\infty(\Omega)$. Since $\partial\{w=0\}$ is a λ^d-zero set, (5.4) yields

$$\int_{\{w=0\}} (cj_1'(0) - f)z^+ - (cj_2'(0) + f)z^-\mathrm{d}\lambda \geq 0 \qquad (5.5)$$

for all $z \in C_c^\infty(\Omega)$ and, by approximation, for all $z \in L^\infty(\Omega)$. Choosing the indicator functions

$$z_1 := \mathbb{1}_{\{f > cj_1'(0)\}} \quad \text{and} \quad z_2 := -\mathbb{1}_{\{f < -cj_2'(0)\}}$$

in (5.5), we readily obtain $-cj_2'(0) \leq f \leq cj_1'(0)$ a.e. in $\{w = 0\}$. This proves the first claim of the lemma.

b) If $z^+, z^- \in T_{crit}(c, f)$, then it follows from the convexity of the cone $T_{crit}(c, f)$ that $z = z^+ + z^- \in T_{crit}(c, f)$. If, conversely, $z \in T_{crit}(c, f)$, then it holds

$$\begin{aligned}
0 &= a(w, z) + \int_{\{w\neq 0\}} cj'(w)z\mathrm{d}\lambda + \int_{\{w=0\}} c\Big(j_1'(0)z^+ - j_2'(0)z^-\Big)\mathrm{d}\lambda - \langle f, z\rangle \\
&= \left(a(w, z^+) + \int_{\{w\neq 0\}} cj'(w)z^+\mathrm{d}\lambda + \int_{\{w=0\}} cj_1'(0)z^+\mathrm{d}\lambda - \left\langle f, z^+\right\rangle\right) \\
&\quad + \left(a(w, z^-) + \int_{\{w\neq 0\}} cj'(w)z^-\mathrm{d}\lambda - \int_{\{w=0\}} cj_2'(0)z^-\mathrm{d}\lambda - \left\langle f, z^-\right\rangle\right).
\end{aligned}$$

The two bracketed terms on the right-hand side of the last identity are each non-negative due to (2.6). Consequently, they both have to vanish and it follows $z^+, z^- \in T_{crit}(c, f)$ as claimed. To obtain the second equivalence, we note that, due to the regularity of the functions w and f, the condition in the definition of the set $T_{crit}(c, f)$ can also be written as (cf. part a))

$$\int_{\{w=0\}} (cj_1'(0) - f)z^+ - (cj_2'(0) + f)z^-\mathrm{d}\lambda = 0.$$

If we assume that $z^+ \in T_{crit}(c, f)$ holds, then it follows from the above and $-cj_2'(0) \leq f \leq cj_1'(0)$ a.e. in $\{w = 0\}$ that

$$0 = \int_{\{w=0\}} (cj_1'(0) - f)z^+\mathrm{d}\lambda = \int_{\{w=0\}} |(cj_1'(0) - f)z^+|\mathrm{d}\lambda.$$

This yields $z^+ = 0$ a.e. in $\{w = 0\} \cap \{-cj_2'(0) \leq f < cj_1'(0)\}$ as claimed. Completely analogously, we obtain that $z^- \in T_{crit}(c, f)$ implies $z^- = 0$ a.e. in $\{w = 0\} \cap \{-cj_2'(0) < f \leq cj_1'(0)\}$. This proves the implication $\Rightarrow$ in the second equivalence in (5.1). The reverse implication is trivial.

□

Remark 5.6. *In the situation of Lemma 5.5, the critical cone can also be described as follows, using the slack variable q appearing in Theorem 2.4 d):*

$$T_{crit}(c,f) = \left\{ z \in H_0^1(\Omega) : \int_{\{w=0\}} c(j_1'(0) - q)z^+ - (j_2'(0) + q)z^- \mathrm{d}\lambda = 0 \right\}.$$

The above formulation corresponds to that used in [6].

We now turn our attention back to the boundedness condition (4.5), i.e.,

$$0 \leq \limsup_{n\to\infty} \left(\frac{1}{t_n} \left(\int_\Omega c\, \frac{j(w + t_n\delta_n) - j(w)}{t_n} - c\, j'(w;\delta_n)\mathrm{d}\lambda \right) \right) < \infty.$$

To analyze which implications (4.5) has in the situation of Assumptions 5.1 and 5.3, we need the following prototypical result:

Proposition 5.7. *Let $B \subset \mathbb{R}^{d-1}$ be an open ball and let $a > 0$. Suppose that $v, \varphi \in C(\overline{B} \times [0,a])$ are functions satisfying*

$$v = 0 \text{ on } \overline{B} \times \{0\}, \quad v > 0 \text{ in } \overline{B} \times (0,a] \quad \text{and} \quad \varphi \geq 0 \text{ in } \overline{B} \times [0,a].$$

Assume further that $t_n \in (0,\infty)$ and $z_n \in H^1(B \times (0,a))$ are sequences with $t_n \to 0$ and $z_n \rightharpoonup z$ in $H^1(B \times (0,a))$ for some function z. Then the following is true:

a) If it holds $v \in W^{1,\infty}(B \times (0,a))$, $\varphi > 0$ in $\overline{B} \times \{0\}$ and

$$\lim_{t\to 0} \left(\|\nabla v\|_{L^\infty(B\times(0,t))} \right) = 0$$

and if $(\operatorname{tr} z)^-$ is not identical zero on $B \times \{0\}$, then

$$\liminf_{n\to\infty} \left(\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2} \mathrm{d}\lambda \right) = \infty.$$

b) If it holds $v \in C^1(\overline{B} \times [0,a])$ and $\|\nabla v\| \geq \varepsilon > 0$ on $B \times \{0\}$ and if there exists a constant C independent of n with $\|z_n\|_{L^\infty} \leq C$, then

$$\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2} \mathrm{d}\lambda \to \frac{1}{2} \int_{B\times\{0\}} \varphi \frac{(\operatorname{tr} z^-)^2}{(\partial_d v)} \mathrm{d}\mathcal{H}^{d-1}.$$

Here, $\operatorname{tr} z^- \in L^2(B \times \{0\}, \mathcal{H}^{d-1})$ is the trace of the function z^- on $B \times \{0\}$.

Proof. We restrict our attention to the case $d > 1$ (the proof for the one-dimensional case is completely analogous but requires some notational adjustments): Note that we may assume w.l.o.g. $z_n \in C(\overline{B} \times [0,a])$ for all $n \in \mathbb{N}$. If this is not the case, we can simply replace z_n with a sequence $\tilde{z}_n \in C(\overline{B} \times [0,a])$ satisfying $\|z_n - \tilde{z}_n\|_{H^1} \leq t_n^2$ (and $\|\tilde{z}_n\|_{L^\infty} \leq C$ in b)) since this exchange does not alter the limiting behavior of the integral expressions under consideration. In the following, we denote the first $d-1$ coordinates of the Euclidean space with $x \in \mathbb{R}^{d-1}$ and the d-th coordinate with $y \in \mathbb{R}$. Further, we introduce the abbreviations $\mathrm{d}x$ and $\mathrm{d}y$ for $\mathrm{d}\lambda^{d-1}(x)$ and $\mathrm{d}\lambda^1(y)$.

Ad a) If $M>0$ is arbitrary but fixed, then for all n large enough it holds $t_n M < a$ and (since $v \geq 0$ in $B\times(0,a)$)

$$\int_{B\times(0,a)} \varphi \frac{(-v-t_n z_n)^+}{t_n^2}\mathrm{d}\lambda \geq \int_B \int_0^{t_n M} \varphi \frac{(-z_n)^+}{t_n} - \varphi \frac{v}{t_n^2}\mathrm{d}y\mathrm{d}x. \tag{5.6}$$

From $v=0$ on $\overline{B}\times\{0\}$ and $v\in W^{1,\infty}(B\times(0,a))$, we obtain further

$$\begin{aligned}
&\left|\int_B\int_0^{t_n M}\varphi(x,y)\frac{v(x,y)}{t_n^2}\mathrm{d}y\mathrm{d}x\right| \\
&\qquad \leq \|\varphi\|_{L^\infty}\int_B\int_0^{t_n M}\frac{1}{t_n^2}\int_0^y |(\partial_d v)(x,s)|\mathrm{d}s\mathrm{d}y\mathrm{d}x \\
&\qquad \leq \frac{1}{2}\|\varphi\|_{L^\infty}\|\nabla v\|_{L^\infty(B\times(0,t_n M))}\lambda^{d-1}(B)M^2.
\end{aligned} \tag{5.7}$$

Similarly, we may calculate that

$$\begin{aligned}
&\int_B\int_0^{t_n M}\varphi(x,y)\frac{(-z_n(x,y))^+}{t_n}\mathrm{d}y\mathrm{d}x \\
&\qquad = \int_B\int_0^{t_n M}\varphi(x,y)\frac{(-z_n(x,0))^+}{t_n}\mathrm{d}y\mathrm{d}x + R_n \\
&\qquad = \int_B (-z_n(x,0))^+\int_0^M \varphi(x,t_n y)\mathrm{d}y\mathrm{d}x + R_n
\end{aligned} \tag{5.8}$$

with

$$\begin{aligned}
|R_n| &= \left|\int_B\int_0^{t_n M}\varphi(x,y)\frac{(-z_n(x,y))^+ - (-z_n(x,0))^+}{t_n}\mathrm{d}y\mathrm{d}x\right| \\
&\leq \|\varphi\|_{L^\infty}\int_B\int_0^{t_n M}\frac{1}{t_n}\int_0^y |\partial_d z_n(x,s)|\mathrm{d}s\mathrm{d}y\mathrm{d}x \\
&\leq \|\varphi\|_{L^\infty}\int_B\int_0^{t_n M}\frac{1}{t_n}y^{1/2}\left(\int_0^a |\partial_d z_n(x,s)|^2\mathrm{d}s\right)^{1/2}\mathrm{d}y\mathrm{d}x \\
&\leq \frac{2}{3}\|\varphi\|_{L^\infty}\|z_n\|_{H^1}\lambda^{d-1}(B)^{1/2}M^{3/2}t_n^{1/2}.
\end{aligned} \tag{5.9}$$

Using (5.7), (5.8), (5.9), the boundedness of the sequence z_n in $H^1(B\times(0,a))$, our assumptions on the function v and the compactness of the trace operator $\mathrm{tr}: H^1(B\times(0,a)) \to L^2(B,\lambda^{d-1}) \cong L^2(B\times\{0\},\mathcal{H}^{d-1})$ (cf. [21, Chapter 2, Theorem 6.2]), we can pass to the limit $n\to\infty$ in (5.6) to obtain

$$\begin{aligned}
\liminf_{n\to\infty}\left(\int_{B\times(0,a)}\varphi\frac{(-v-t_n z_n)^+}{t_n^2}\mathrm{d}\lambda\right) &\geq M\int_B \varphi(x,0)(-\mathrm{tr}\, z(x))^+\mathrm{d}x \\
&= M\int_{B\times\{0\}}\varphi|(\mathrm{tr}\, z)^-|\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned} \tag{5.10}$$

Since $M>0$ was arbitrarily large and since $\varphi|(\mathrm{tr}\, z)^-|$ is non-negative and not identical zero on $B\times\{0\}$ (cf. our assumptions), it follows that the limes inferior in (5.10) has to be infinite. This proves part a).

Ad b) The claim in b) is obtained similarly to that in a): Due to the C^1-regularity of v, the function

$$r(x,y) := \frac{v(x,y)}{y} = \int_0^1 \partial_d v(x,sy)\mathrm{d}s$$

is continuous and from $v > 0$ in $\overline{B} \times (0,a]$ and $\|\nabla v\| = \partial_d v = r \geq \varepsilon > 0$ on $B \times \{0\}$ it follows that r is positive everywhere in $\overline{B} \times [0,a]$. Thus, there exists an $\tilde{\varepsilon} > 0$ such that $r \geq \tilde{\varepsilon}$ holds everywhere in $\overline{B} \times [0,a]$. On the other hand, the integrand in the integral under consideration can only be non-zero, if it is true that

$$0 \leq -v(x,y) - t_n z_n(x,y) = -yr(x,y) - t_n z_n(x,y),$$

i.e., if it holds

$$0 \leq y \leq t_n \frac{\|z_n\|_{L^\infty}}{\tilde{\varepsilon}} \leq Ct_n$$

with a constant C independent of n. Thus, for large enough n we have

$$\begin{aligned}
&\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2}\mathrm{d}\lambda \\
&\quad= \int_B \int_0^{Ct_n} \varphi(x,y) \frac{(-yr(x,y) - t_n z_n(x,y))^+}{t_n^2}\mathrm{d}y\mathrm{d}x \\
&\quad= \int_B \int_0^{C} \varphi(x,t_n y)\Big(- yr(x,t_n y) - z_n(x,t_n y)\Big)^+\mathrm{d}y\mathrm{d}x \\
&\quad= \int_B \int_0^{C} \varphi(x,t_n y)\Big(- yr(x,0) - z_n(x,0)\Big)^+\mathrm{d}y\mathrm{d}x + R_n
\end{aligned} \tag{5.11}$$

with

$$\begin{aligned}
|R_n| &\leq \|\varphi\|_{L^\infty} \int_B \int_0^C |yr(x,t_n y) - yr(x,0)| + |z_n(x,t_n y) - z_n(x,0)|\mathrm{d}y\mathrm{d}x \\
&\leq \|\varphi\|_{L^\infty} \int_B \int_0^C \int_0^{Ct_n} |\partial_d z_n(x,s)|\mathrm{d}s\mathrm{d}y\mathrm{d}x + o(1) \\
&\leq \|\varphi\|_{L^\infty} \int_B \int_0^C (Ct_n)^{1/2} \left(\int_0^a |\partial_d z_n(x,s)|^2\mathrm{d}s\right)^{1/2}\mathrm{d}y\mathrm{d}x + o(1) \\
&= o(1).
\end{aligned} \tag{5.12}$$

From the compactness of the trace operator, (5.11) and (5.12), it follows

$$\begin{aligned}
&\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2}\mathrm{d}\lambda \\
&\quad\to \int_B \varphi(x,0) \int_0^C \Big(- y\partial_d v(x,0) - (\operatorname{tr} z)(x)\Big)^+\mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{5.13}$$

Using $\|\nabla v\| = \partial_d v \geq \varepsilon > 0$ on $B \times \{0\}$, we can calculate the inner integral on the right-hand side of (5.13) to obtain

$$\int_{B\times(0,a)} \varphi \frac{(-v - t_n z_n)^+}{t_n^2}\mathrm{d}\lambda \to \frac{1}{2}\int_B \varphi(x,0)\frac{(\operatorname{tr} z^-(x))^2}{(\partial_d v)(x,0)}\mathrm{d}\lambda^{d-1}(x).$$

This proves b). □

Note that for $j(x) = |x|$, it holds

$$\frac{1}{t_n}\left(\int_\Omega c\,\frac{j(w+t_n\delta_n)-j(w)}{t_n} - c\,j'(w;\delta_n)\mathrm{d}\lambda\right) = \int_{\{w>0\}} 2c\,\frac{(-w-t_n\delta_n)^+}{t_n^2}\mathrm{d}\lambda + \int_{\{w<0\}} 2c\,\frac{(w+t_n\delta_n)^+}{t_n^2}\mathrm{d}\lambda,$$

i.e., the parameter integrals studied in Proposition 5.7 are exactly those appearing in the boundedness condition (4.5) when the absolute value function is considered. If j is an arbitrary function satisfying Assumption 5.1, then one can use Taylor expansions and localization arguments to deduce the following from Proposition 5.7:

PROPOSITION 5.8. *Let Assumptions 2.1, 5.1 and 5.3 hold. Then the following is true in the situation of Assumption 4.1:*

$$(\operatorname{tr}\delta)^+ = 0\ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^- \quad \text{and} \quad (\operatorname{tr}\delta)^- = 0\ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^+.$$

Proof. We again restrict our attention to the case $d > 1$. To show that (4.5) implies $(\operatorname{tr}\delta)^- = 0$ $\mathcal{H}^{d-1}$-a.e. on $\mathcal{N}^+$, we use Taylor's formula and Proposition 5.7 a): From (4.5), Lemma 2.2 b) and our assumptions on j, it follows that there exists a constant $C > 0$ independent of n with

$$\begin{aligned} C &\geq \frac{1}{t_n}\left(\int_\Omega c\,\frac{j(w+t_n\delta_n)-j(w)}{t_n} - c\,j'(w;\delta_n)\mathrm{d}\lambda\right) \\ &\geq \frac{1}{t_n}\left(\int_{\{w>0\}} c\,\frac{j(w+t_n\delta_n)-j(w)}{t_n} - c\,j'(w)\delta_n\mathrm{d}\lambda\right) \\ &\geq \frac{1}{t_n}\left(\int_{\{w>0\}} c\,\frac{j((w+t_n\delta_n)^+)-j(w^+)}{t_n} - c\,j'(w^+)\delta_n\mathrm{d}\lambda\right) \\ &= \frac{1}{t_n}\left(\int_{\{w>0\}} c\,\left(j_1'(w^+)\frac{(w+t_n\delta_n)^+-w^+}{t_n}\right) - c\,j_1'(w^+)\delta_n\mathrm{d}\lambda\right) \\ &\quad + \int_{\{w>0\}} c\,\frac{((w+t_n\delta_n)^+-w^+)^2}{t_n^2}\int_0^1(1-s)j_1''((1-s)w^+ + s(w+t_n\delta_n)^+)\mathrm{d}s\mathrm{d}\lambda. \end{aligned} \tag{5.14}$$

Further, we obtain from the boundedness of the sequence δ_n in $L^\infty(\Omega)$ that the integrand of the j_1''-integral in (5.14) satisfies

$$\begin{aligned} &\left|c\,\frac{((w+t_n\delta_n)^+-w^+)^2}{t_n^2}\int_0^1(1-s)j_1''((1-s)w^+ + s(w+t_n\delta_n)^+)\mathrm{d}s\right| \\ &\leq \|c\|_{L^\infty}\|\delta_n\|_{L^\infty}^2 \max\{j_1''(x) : 0 \leq x \leq \|w\|_{L^\infty} + \|\delta_n\|_{L^\infty}\} \\ &\leq C \end{aligned}$$

a.e. in Ω for some constant $C > 0$ independent of n. Thus, the j_1''-term in (5.14) remains bounded as n tends to infinity and it is true that

$$\begin{aligned} C &\geq \int_{\{w>0\}} cj_1'(w^+)\frac{1}{t_n}\left(\frac{(w+t_n\delta_n)^+-w^+}{t_n} - \delta_n\right)\mathrm{d}\lambda \\ &= \int_{\{w>0\}} cj_1'(w^+)\frac{(-w-t_n\delta_n)^+}{t_n^2}\mathrm{d}\lambda. \end{aligned} \tag{5.15}$$

Consider now an arbitrary but fixed point $p \in \mathcal{N}^+ \subseteq \partial\{w \neq 0\} \setminus \mathcal{C}$. Then it follows from our assumptions that (after possibly changing coordinates such that $R = \mathrm{Id}$) we can find an open ball $B \subset \mathbb{R}^{d-1}$, an open interval $J = (a, b)$ and a Lipschitz continuous map $h : B \to J$ with

$$p \in B \times J \quad \text{and} \quad \partial\{w \neq 0\} \setminus \mathcal{C} \cap (B \times J) = \{(x, h(x)) : x \in B\}.$$

Note that, since $\mathcal{C}$ is closed, since $\mathcal{N}^+$ is a subset of Ω and since $\mathcal{N}^+$ is relatively open in $\partial\{w \neq 0\} \setminus \mathcal{C}$, by making the sets J and B smaller, we can always obtain that the following holds true for some $\varepsilon > 0$ (cf. Figure 5.2):

$$\begin{gathered} p \in B \times J, \quad \mathrm{cl}\,(B \times J) \subset \Omega \setminus \mathcal{C}, \\ \mathcal{N}^+ \cap (B \times J) = \{(x, h(x)) : x \in B\}, \\ \{(x, y) : x \in B \text{ and } |y - h(x)| < \varepsilon\}) \subseteq B \times J, \\ \partial\{w \neq 0\} \cap \mathrm{cl}\,(B \times J) = \mathrm{cl}\left(\mathcal{N}^+ \cap (B \times J)\right). \end{gathered}$$

In the above situation, it follows from $\mathcal{N}^+ \subseteq \partial\{w > 0\}$ that w is positive in at least one of the sets

$$D_1 := \{(x, y) \in \mathrm{cl}\,(B \times J) : y > h(x)\}, \quad D_2 := \{(x, y) \in \mathrm{cl}\,(B \times J) : y < h(x)\}.$$

(We, of course, use the unique extension of the function h onto $\overline{B}$ here). Let us assume that this is true for D_1 (the other case is analogous). Then (5.15) and the area formula (cf. [9, Theorem 3.9]) imply

$$C \geq \int_B \int_0^\varepsilon \left(c j_1'(w) \frac{(-w - t_n \delta_n)^+}{t_n^2} \right) \bigg|_{(x, y + h(x))} \mathrm{d}y \mathrm{d}x. \tag{5.16}$$

Defining

$$\begin{gathered} v(x, y) := w(x, y + h(x)), \quad \varphi(x, y) := c(x, y + h(x)) j_1'(w(x, y + h(x))), \\ z_n(x, y) := \delta_n(x, y + h(x)), \end{gathered}$$

the right-hand side of (5.16) takes exactly the form of the integral expression studied in Proposition 5.7. Further, it follows from the definition of $\mathcal{N}^+$ that $w(x, h(x)) = 0$ and $(\nabla w)(x, h(x)) = 0$ holds for all $x \in \overline{B}$. This and the conditions on c and w in Assumption 5.3 yield that in the situation of (5.16) we have

$$\begin{gathered} v, \varphi \in C(\overline{B} \times [0, \varepsilon]), \quad v = 0 \text{ on } \overline{B} \times \{0\}, \quad v > 0 \text{ in } \overline{B} \times (0, \varepsilon], \quad \varphi > 0 \text{ in } \overline{B} \times [0, \varepsilon], \\ v \in W^{1,\infty}(B \times (0, \varepsilon)), \quad \lim_{t \to 0} \left(\|\nabla v\|_{L^\infty(B \times (0, t))} \right) = 0. \end{gathered}$$

From Proposition 5.7 a) and (5.16), it now readily follows by contradiction that $(\mathrm{tr}\,\delta)^- = 0$ $\mathcal{H}^{d-1}$-a.e. on $\mathcal{N}^+ \cap (B \times J)$. This, together with the arbitrariness of the point $p \in \mathcal{N}^+$ proves the claim for $(\mathrm{tr}\,\delta)^-$. The result for $(\mathrm{tr}\,\delta)^+$ is obtained completely analogously. □

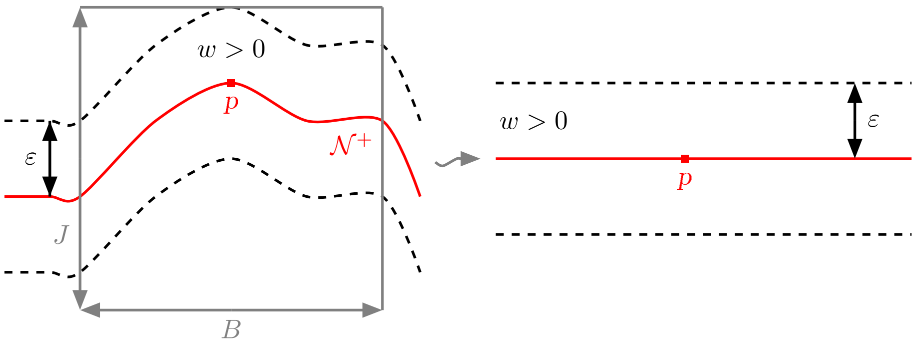

FIG. 5.2. *Local rectification of the set $\mathcal{N}^+$ in two dimensions.*

To study the trace of δ on the set $\mathcal{M} = \{w = 0\} \cap \{\nabla w \neq 0\}$, we need the following corollary of Proposition 5.7 b):

PROPOSITION 5.9. *Let Ω be a bounded Lipschitz domain and suppose that functions j, v, c, ψ_1 and ψ_2 are given such that j satisfies Assumption 5.1 and such that*

$$v \in C^1(\Omega), \quad 0 \le c \in C(\Omega), \quad 0 \le \psi_1 \in C_c(\Omega), \quad 0 \le \psi_2 \in C_c(\Omega),$$
$$\operatorname{supp}(\psi_1) \cap \partial\{v < 0\} \cap \{\nabla v = 0\} = \emptyset$$

and

$$\operatorname{supp}(\psi_2) \cap \partial\{v > 0\} \cap \{\nabla v = 0\} = \emptyset.$$

Assume further that $t_n \in (0, \infty)$ and $z_n \in H^1(\Omega)$ are sequences satisfying

$$t_n \to 0, \quad \|z_n\|_{L^\infty} \le C, \quad z_n \rightharpoonup z \text{ in } H^1(\Omega) \quad \text{and} \quad z_n \to z \text{ pointwise a.e. in } \Omega$$

for some constant C independent of n and some $z \in H^1(\Omega)$. Then it is true that

$$\begin{aligned} &\int_\Omega \psi_1 \frac{c}{t_n} \left(\frac{j(v + t_n z_n^+) - j(v)}{t_n} - j'(v; z_n^+) \right) \mathrm{d}\lambda \\ &\quad \to \frac{1}{2} \int_{\{v \neq 0\}} \psi_1 c j''(v)(z^+)^2 \mathrm{d}\lambda + \frac{1}{2} \int_{\{v=0\}} \psi_1 c j_1''(0)(z^+)^2 \mathrm{d}\lambda \\ &\qquad + \frac{1}{2} \Big(j_1'(0) + j_2'(0) \Big) \int_{\{v=0\} \cap \{\nabla v \neq 0\}} \psi_1 c \frac{(\operatorname{tr} z^+)^2}{\|\nabla v\|} \mathrm{d}\mathcal{H}^{d-1} \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} &\int_\Omega \psi_2 \frac{c}{t_n} \left(\frac{j(v + t_n z_n^-) - j(v)}{t_n} - j'(v; z_n^-) \right) \mathrm{d}\lambda \\ &\quad \to \frac{1}{2} \int_{\{v \neq 0\}} \psi_2 c j''(v)(z^-)^2 \mathrm{d}\lambda + \frac{1}{2} \int_{\{v=0\}} \psi_2 c j_2''(0)(z^-)^2 \mathrm{d}\lambda \\ &\qquad + \frac{1}{2} \Big(j_1'(0) + j_2'(0) \Big) \int_{\{v=0\} \cap \{\nabla v \neq 0\}} \psi_2 c \frac{(\operatorname{tr} z^-)^2}{\|\nabla v\|} \mathrm{d}\mathcal{H}^{d-1}. \end{aligned} \tag{5.18}$$

Proof. We restrict our attention to (5.18). The limit (5.17) is obtained analogously.

Further, we again focus on the case $d > 1$. Note that the properties of j imply

$$\begin{aligned}
&\int_\Omega \psi_2 \frac{c}{t_n} \left(\frac{j(v + t_n z_n^-) - j(v)}{t_n} - j'(v; z_n^-) \right) \mathrm{d}\lambda \\
&= \int_{\{v \le 0\}} \psi_2 \frac{c}{t_n} \left(\frac{j_2(-v - t_n z_n^-) - j_2(-v)}{t_n} + j_2'(-v) z_n^- \right) \mathrm{d}\lambda \\
&\quad + \int_{\{v > 0\}} \psi_2 \frac{c}{t_n} \left(\frac{j_1((v + t_n z_n^-)^+) - j_1(v)}{t_n} - j_1'(v) z_n^- \right) \mathrm{d}\lambda \\
&\quad + \int_{\{v > 0\}} \psi_2 \frac{c}{t_n} \left(\frac{j_2(-(v + t_n z_n^-)^-)}{t_n} \right) \mathrm{d}\lambda \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

In what follows, we analyze the three integrals I_1, I_2 and I_3 separately:

Ad I_1: Using the dominated convergence theorem, the boundedness of z_n in $L^\infty(\Omega)$ and Taylor's formula, we obtain

$$\begin{aligned}
I_1 &= \int_{\{v \le 0\}} \psi_2 \frac{c}{t_n} \left(\frac{j_2(-v - t_n z_n^-) - j_2(-v)}{t_n} + j_2'(-v) z_n^- \right) \mathrm{d}\lambda \\
&= \int_{\{v \le 0\}} \psi_2 c (z_n^-)^2 \int_0^1 (1 - s) j_2''(-v - s t_n z_n^-) \mathrm{d}s \mathrm{d}\lambda \\
&\to \frac{1}{2} \int_{\{v \le 0\}} \psi_2 c j_2''(-v) (z^-)^2 \mathrm{d}\lambda.
\end{aligned}$$

Ad I_2: It holds (cf. (5.14))

$$\begin{aligned}
I_2 &= \int_{\{v > 0\}} \psi_2 \frac{c}{t_n} \left(\frac{j_1((v + t_n z_n^-)^+) - j_1(v)}{t_n} - j_1'(v) z_n^- \right) \mathrm{d}\lambda \\
&= \frac{1}{t_n} \left(\int_{\{v > 0\}} \psi_2 c j_1'(v) \left(\frac{(v + t_n z_n^-)^+ - v}{t_n} - z_n^- \right) \mathrm{d}\lambda \right) \\
&\quad + \int_{\{v > 0\}} \psi_2 c \, \frac{((v + t_n z_n^-)^+ - v)^2}{t_n^2} \int_0^1 (1 - s) j_1''((1 - s) v + s (v + t_n z_n^-)^+) \mathrm{d}s \mathrm{d}\lambda \\
&= I_{2a} + I_{2b}.
\end{aligned}$$

For I_{2b} we obtain analogously to I_1 that

$$I_{2b} \to \frac{1}{2} \int_{\{v > 0\}} \psi_2 c \, j_1''(v) (z^-)^2 \mathrm{d}\lambda.$$

For I_{2a} a simple distinction of cases shows

$$I_{2a} = \int_{\{v > 0\}} \psi_2 c j_1'(v) \, \frac{(-v - t_n z_n^-)^+}{t_n^2} \mathrm{d}\lambda.$$

To be able to apply Proposition 5.7 to the above integral, we note that our assumption $\operatorname{supp}(\psi_2) \cap \partial\{v > 0\} \cap \{\nabla v = 0\}) = \emptyset$ implies

$$\operatorname{supp}(\psi_2) \cap \partial\{v > 0\} \subset \{\nabla v \neq 0\}. \tag{5.19}$$

Define $K := \operatorname{supp}(\psi_2)$ and (as before) $\mathcal{M} := \{v = 0\} \cap \{\nabla v \neq 0\}$. Then it follows from (5.19) and the compactness of $K \cap \partial\{v > 0\}$ that there exists a constant $m > 0$ with

$$\|\nabla v\| \geq m \quad \text{in} \quad K \cap \partial\{v > 0\}.$$

Further, the implicit function theorem yields that for each point p in the set $K \cap \partial\{v > 0\} \subset \mathcal{M}$ we may find an orthogonal transformation $R_p \in O(d)$, an open ball $B_p \subset \mathbb{R}^{d-1}$, an open interval J_p and a C^1-function $h_p : B_p \to J_p$ such that

$$p \in R_p(B_p \times J_p), \quad R_p(B_p \times J_p) \subset \Omega,$$
$$\mathcal{M} \cap R_p(B_p \times J_p) = \{v = 0\} \cap R_p(B_p \times J_p) = R_p(\{(x, h_p(x)) : x \in B_p\}).$$

Note that by choosing smaller sets B_p and J_p, in the above situation we can always obtain that it holds

$$\operatorname{cl}(R_p(B_p \times J_p)) \subset \Omega,$$
$$\|\nabla v\| \geq m/2 \text{ in } R_p(B_p \times J_p),$$
$$v \neq 0 \text{ in } \operatorname{cl}(R_p(B_p \times J_p)) \setminus \operatorname{cl}(R_p(\{(x, h_p(x)) : x \in B_p\}))$$

and

$$R_p(\{(x, y) : x \in B_p, \, |y - h_p(x)| < \varepsilon_p\}) \subseteq R_p(B_p \times J_p) \tag{5.20}$$

for some $\varepsilon_p > 0$. Let us denote the ε_p-tube on the left-hand side of (5.20) with W_p. Then the collection $\{W_p\}$ defines an open cover of $K \cap \partial\{v > 0\}$ and it follows from the compactness of the set $K \cap \partial\{v > 0\}$ that there exist points $p_1, ..., p_L \in K \cap \partial\{v > 0\}$, $L \in \mathbb{N}$, with

$$K \cap \partial\{v > 0\} \subset \bigcup_{l=1}^{L} W_{p_l} =: U.$$

Further, since U is open, we can find an open set $V \subset \mathbb{R}^d$ such that

$$U \cup V = \mathbb{R}^d \quad \text{and} \quad \overline{V} \cap K \cap \partial\{v > 0\} = \emptyset.$$

Consider now a partition of unity of the Euclidean space $\mathbb{R}^d$ subordinate to the cover $W_{p_1}, ..., W_{p_L}, V$ (cf. [26, Theorem 1.11]), i.e., a collection of smooth functions $\varphi_l : \mathbb{R}^d \to [0, 1]$, $l = 1, ..., L + 1$, satisfying

$$\operatorname{supp}(\varphi_l) \subset W_{p_l}, \quad l = 1, ..., L, \quad \operatorname{supp}(\varphi_{L+1}) \subset V \quad \text{and} \quad \sum_{l=1}^{L+1} \varphi_l \equiv 1.$$

Then we obtain

$$I_{2a} = \sum_{l=1}^{L+1} \int_{K \cap \{v > 0\}} \varphi_l \psi_2 c j_1'(v) \, \frac{(-v - t_n z_n^-)^+}{t_n^2} \mathrm{d}\lambda. \tag{5.21}$$

Note that from $\overline{V} \cap K \cap \partial\{v > 0\} = \emptyset$ it follows

$$\overline{V} \cap K \cap \{v > 0\} = \overline{V} \cap K \cap \operatorname{cl}(\{v > 0\}).$$

Thus, the set $\overline{V} \cap K \cap \{v > 0\}$ is a compact subset of $\{v > 0\}$ and it holds $v \geq \varepsilon > 0$ for some $\varepsilon > 0$ in $\overline{V} \cap K \cap \{v > 0\}$. This implies, together with the uniform L^∞-bound on $\|z_n^-\|_{L^\infty}$, that the integral associated with φ_{L+1} in (5.21) is identical zero for n sufficiently large. It remains to analyze the integrals

$$\begin{aligned}\int_{K\cap\{v>0\}} &\varphi_l \psi_2 c j_1'(v)\, \frac{(-v - t_n z_n^-)^+}{t_n^2} \mathrm{d}\lambda \\ &= \int_{W_{p_l}\cap\{v>0\}} \varphi_l \psi_2 c j_1'(v)\, \frac{(-v - t_n z_n^-)^+}{t_n^2} \mathrm{d}\lambda, \quad l = 1, ..., L, \end{aligned} \tag{5.22}$$

appearing in (5.21), i.e., the contributions to I_{2a} that come from the vicinity of the boundary $\partial\{v > 0\}$. To this end, let us drop the index l and assume w.l.o.g. that $R_p = \mathrm{Id}$. In this prototypical situation, the integral in (5.22) can be rewritten as follows (cf. the proof of Proposition 5.8):

$$\begin{aligned}\int_{W_p\cap\{v>0\}} &\varphi \psi_2 c j_1'(v)\, \frac{(-v - t_n z_n^-)^+}{t_n^2} \mathrm{d}\lambda \\ &= \int_{B_p} \int_{-\varepsilon_p}^{\varepsilon_p} \left[\mathbb{1}_{\{v>0\}} \varphi \psi_2 c j_1'(v)\, \frac{(-v - t_n z_n^-)^+}{t_n^2} \right]\Bigg|_{(x, y+h_p(x))} \mathrm{d}y \mathrm{d}x.\end{aligned}$$

Note that, since $\{v = 0\} \cap W_p = \{(x, h_p(x)) : x \in B_p\} \subset \{v = 0\} \cap \{\nabla v \neq 0\}$ and due to $v \neq 0$ in $\mathrm{cl}(B_p \times J_p) \setminus \mathrm{cl}(\{(x, h_p(x)) : x \in B_p\})$, it has to hold either

$$v(x, y + h_p(x)) > 0 \text{ in } \overline{B_p} \times (0, \varepsilon_p], \quad v(x, y + h_p(x)) < 0 \text{ in } \overline{B_p} \times [-\varepsilon_p, 0)$$

or

$$v(x, y + h_p(x)) < 0 \text{ in } \overline{B_p} \times (0, \varepsilon_p], \quad v(x, y + h_p(x)) > 0 \text{ in } \overline{B_p} \times [-\varepsilon_p, 0).$$

If the first case is true (the second one is analogous), it holds

$$\begin{aligned}\int_{W_p\cap\{v>0\}} &\varphi \psi_2 c j_1'(v)\, \frac{(-v - t_n z_n^-)^+}{t_n^2} \mathrm{d}\lambda \\ &= \int_{B_p} \int_0^{\varepsilon_p} \left[\varphi \psi_2 c j_1'(v)\, \frac{(-v - t_n z_n^-)^+}{t_n^2} \right]\Bigg|_{(x, y+h_p(x))} \mathrm{d}y \mathrm{d}x.\end{aligned}$$

The integral on the right-hand side of the last equation has exactly the form of that studied in Proposition 5.7 b). We may thus deduce

$$\begin{aligned}\int_{W_p\cap\{v>0\}} &\varphi \psi_2 c j_1'(v)\, \frac{(-v - t_n z_n^-)^+}{t_n^2} \mathrm{d}\lambda \\ &\to \frac{1}{2} \int_{B_p} j_1'(0) \left[\varphi \psi_2 c \frac{(\mathrm{tr}\, z^-)^2}{\partial_d v} \right]\Bigg|_{(x, h_p(x))} \mathrm{d}\lambda^{d-1}(x).\end{aligned}$$

Here, with $(\mathrm{tr}\, z^-)(x, h_p(x))$ we, of course, mean the value of the trace

$$\mathrm{tr}\, z^- \in L^2(\mathcal{M}, \mathcal{H}^{d-1})$$

in the point $(x, h_p(x)) \in \mathcal{M}$, $x \in B_p$. Using the identity

$$\|(\nabla v)(x, h_p(x))\| = (\partial_d v)(x, h_p(x))\sqrt{1 + \|\nabla h_p(x)\|^2} \quad \forall x \in B_p$$

and the area formula (cf. [9, Theorem 3.9]), it now follows

$$\begin{aligned}\int_{W_p \cap \{v>0\}} &\varphi \psi_2 c j_1'(v) \frac{(-v - t_n z_n^-)^+}{t_n^2} \mathrm{d}\lambda \\ &\to \frac{1}{2} \int_{\{(x,h_p(x)):x \in B_p\}} j_1'(0) \varphi \psi_2 c \frac{(\operatorname{tr} z^-)^2}{\|\nabla v\|} \mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

This implies (using $\sum_{l=1}^{L} \varphi_l \equiv 1$ on $K \cap \partial\{v > 0\} = K \cap \mathcal{M}$)

$$I_{2a} \to \frac{1}{2} j_1'(0) \int_{\mathcal{M}} \psi_2 c \frac{(\operatorname{tr} z^-)^2}{\|\nabla v\|} \mathrm{d}\mathcal{H}^{d-1}.$$

Ad I_3: From Taylor's formula we obtain

$$\begin{aligned}I_3 &= \int_{\{v>0\}} \psi_2 \frac{c}{t_n} \left(\frac{j_2(-(v + t_n z_n^-)^-)}{t_n} \right) \mathrm{d}\lambda \\ &= \int_{\{v>0\}} \psi_2 c \frac{-(v + t_n z_n^-)^-}{t_n^2} \int_0^1 j_2'(-s(v + t_n z_n^-)^-) \mathrm{d}s \mathrm{d}\lambda \\ &= \int_{\{v>0\}} \psi_2 j_2'(0) c \frac{-(v + t_n z_n^-)^-}{t_n^2} \mathrm{d}\lambda \\ &\quad + \int_{\{v>0\}} \psi_2 c \frac{((v + t_n z_n^-)^-)^2}{t_n^2} \left(\int_0^1 \int_0^1 j_2''(-st(v + t_n z_n^-)^-) \mathrm{d}t \, s \mathrm{d}s \right) \mathrm{d}\lambda \\ &= I_{3a} + I_{3b}.\end{aligned}$$

The dominated convergence theorem yields $I_{3b} \to 0$ (analogously to I_{2b}) and since

$$\int_{\{v>0\}} \psi_2 j_2'(0) c \frac{-(v + t_n z_n^-)^-}{t_n^2} \mathrm{d}\lambda = \int_{\{v>0\}} \psi_2 j_2'(0) c \frac{(-v - t_n z_n^-)^+}{t_n^2} \mathrm{d}\lambda$$

the integral I_{3a} behaves exactly like I_{2a}. Thus, using the same argumentation as for I_2, we obtain

$$I_3 \to \frac{1}{2} j_2'(0) \int_{\mathcal{M}} \psi_2 c \frac{(\operatorname{tr} z^-)^2}{\|\nabla v\|} \mathrm{d}\mathcal{H}^{d-1}.$$

Combining all of our results, we readily obtain (5.18) as desired. ∎

We point out that Proposition 5.9 is also interesting for its own sake. It yields, for example, the following second order expansion for the L^1-norm:

Corollary 5.10. *Let Ω be a bounded Lipschitz domain and let $v \in C^1(\Omega)$ be a function with $\{v = 0\} \cap \{\nabla v = 0\} = \emptyset$. Then for all $z \in C_c(\Omega) \cap H^1(\Omega)$ and all $t > 0$ it holds*

$$\int_\Omega |v + tz| \mathrm{d}\lambda = \int_\Omega |v| \mathrm{d}\lambda + t \left(\int_\Omega \operatorname{sgn}(v) z \mathrm{d}\lambda \right) + t^2 \left(\int_{\{v=0\}} \frac{z^2}{\|\nabla v\|} \mathrm{d}\mathcal{H}^{d-1} \right) + o(t^2). \tag{5.23}$$

The Landau symbol $o(t^2)$ appearing here refers to the limit $t \to 0^+$.

Proof. Choose a function $0 \le \psi \in C_c(\Omega)$ with $\psi \equiv 1$ in $\operatorname{supp}(z)$, then it follows from (5.17) and (5.18) with $c \equiv 1$, $j(x) = |x|$, $\psi_1 = \psi_2 = \psi$ and $z_n = z$ that for every sequence $t_n \subset (0, \infty)$ tending to zero it holds

$$\int_\Omega \frac{1}{t_n} \left(\frac{|v + t_n z| - |v|}{t_n} - \operatorname{sgn}(v) z \right) \mathrm{d}\lambda \to \int_{\{v=0\} \cap \{\nabla v \neq 0\}} \frac{(\operatorname{tr} z)^2}{\|\nabla v\|} \mathrm{d}\mathcal{H}^{d-1}.$$

Reformulating the above yields (5.23) as desired. ∎

Remark 5.11. *Expansions involving surface integrals similar to that in (5.23) also appear in the study of highly oscillatory integrals. See [16] for an overview article.*

We are now in the position to pass to the limit in the variational inequality (4.2) for the difference quotients δ_n:

Proposition 5.12. *Let Assumptions 2.1, 5.1 and 5.3 hold. Then in the situation of Assumption 4.1, for all $z \in L^\infty(\Omega) \cap T_{crit}(c, f)$ satisfying $z^+ = 0$ a.e. in a neighborhood of $\partial\Omega \cup \mathcal{N}^- \cup \mathcal{C}$ and $z^- = 0$ a.e. in a neighborhood of the set $\partial\Omega \cup \mathcal{N}^+ \cup \mathcal{C}$ it holds*

$$a(\delta, z) + J(z) - J(\delta) + \int_\Omega h\Big(j'(w; z) - j'(w; \delta)\Big) \mathrm{d}\lambda - \langle g, z - \delta \rangle \ge \limsup_{n\to\infty} a(\delta_n, \delta_n) \tag{5.24}$$

with

$$J(z) := \frac{1}{2} \int_{\{w \neq 0\}} c j''(w) z^2 \mathrm{d}\lambda + \frac{1}{2} \int_{\{w=0\}} c j_1''(0)(z^+)^2 + c j_2''(0)(z^-)^2 \mathrm{d}\lambda + \frac{1}{2} \Big(j_2'(0) + j_1'(0) \Big) \int_{\mathcal{M}} c \frac{(\operatorname{tr} z)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1}. \tag{5.25}$$

In particular, it is true that

$$\int_{\mathcal{M}} c \frac{(\operatorname{tr} \delta)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} < \infty.$$

Here, with $\mathcal{M}$ we again denote the set $\{w = 0\} \cap \{\nabla w \neq 0\}$.

Proof. From (2.6), (4.2) and the definition of the critical cone, it follows that for all $z \in T_{crit}(c, f)$ we have

$$\begin{aligned} &a(\delta_n, z) + H_n(z) - H_n(\delta_n) - \langle g, z - \delta_n \rangle \\ &\quad + \frac{1}{t_n} \left(\int_\Omega c \, \frac{j(w + t_n z) - j(w)}{t_n} - c \, j'(w; z) \mathrm{d}\lambda \right) \\ &\quad \ge a(\delta_n, \delta_n) + \frac{1}{t_n} \left(\int_\Omega c \, \frac{j(w + t_n \delta_n) - j(w)}{t_n} - c \, j'(w; \delta_n) \mathrm{d}\lambda \right). \end{aligned} \tag{5.26}$$

Note that due to the weak H^1-convergence $\delta_n \rightharpoonup \delta$ and Lemma 4.2, the first four terms on the left-hand side of (5.26) satisfy

$$\begin{aligned} &a(\delta_n, z) + H_n(z) - H_n(\delta_n) - \langle g, z - \delta_n \rangle \\ &\quad \to a(\delta, z) + \int_\Omega h j'(w; z) \mathrm{d}\lambda - \int_\Omega h j'(w; \delta) \mathrm{d}\lambda - \langle g, z - \delta \rangle \quad \forall z \in L^\infty(\Omega) \cap H_0^1(\Omega). \end{aligned}$$

Suppose now that $z \in L^\infty(\Omega) \cap T_{crit}(c, f)$ is a function such that there exist open sets $U_1, U_2 \subseteq \mathbb{R}^d$ with $\partial\Omega \cup \mathcal{N}^- \cup \mathcal{C} \subset U_1$, $\partial\Omega \cup \mathcal{N}^+ \cup \mathcal{C} \subset U_2$, $z^+ = 0$ a.e. in $U_1 \cap \Omega$ and $z^- = 0$ a.e. in $U_2 \cap \Omega$. Then we can find functions $\psi_1, \psi_2 \in C_c(\Omega)$ with

$$\begin{gathered} 0 \le \psi_1, \psi_2 \le 1 \text{ in } \Omega, \quad \psi_1 \equiv 1 \text{ in } \Omega \setminus U_1, \quad \psi_2 \equiv 1 \text{ in } \Omega \setminus U_2, \\ \operatorname{dist}(\operatorname{supp}(\psi_1), \mathcal{C} \cup \mathcal{N}^-) > 0 \quad \text{and} \quad \operatorname{dist}(\operatorname{supp}(\psi_2), \mathcal{C} \cup \mathcal{N}^+) > 0. \end{gathered}$$

Further, the definitions of $\mathcal{N}^-$ and $\mathcal{N}^+$ yield $\partial\{w < 0\} \cap \{\nabla w = 0\} \subseteq \mathcal{C} \cup \mathcal{N}^-$ and $\partial\{w > 0\} \cap \{\nabla w = 0\} \subseteq \mathcal{C} \cup \mathcal{N}^+$. Thus, we may employ Proposition 5.9 to obtain

$$\begin{aligned} &\frac{1}{t_n} \left(\int_\Omega c \, \frac{j(w + t_n z) - j(w)}{t_n} - c \, j'(w; z) \mathrm{d}\lambda \right) \\ &\quad = \int_\Omega \psi_1 \frac{c}{t_n} \left(\frac{j(w + t_n z^+) - j(w)}{t_n} - j'(w; z^+) \right) \mathrm{d}\lambda \\ &\qquad + \int_\Omega \psi_2 \frac{c}{t_n} \left(\frac{j(w + t_n z^-) - j(w)}{t_n} - j'(w; z^-) \right) \mathrm{d}\lambda \\ &\quad \to J(z). \end{aligned} \tag{5.27}$$

It remains to study the right-hand side of (5.26). To this end, let $\psi_1^k, \psi_2^k \in C_c(\Omega)$ be sequences of bump functions such that

$$\begin{gathered} 0 \le \psi_1^k, \psi_2^k \le 1 \text{ in } \Omega, \\ \psi_1^k \equiv 1 \text{ in } \{x \in \Omega : \operatorname{dist}(x, \mathcal{C} \cup \mathcal{N}^- \cup \partial\Omega) \ge 1/k\}, \\ \psi_2^k \equiv 1 \text{ in } \{x \in \Omega : \operatorname{dist}(x, \mathcal{C} \cup \mathcal{N}^+ \cup \partial\Omega) \ge 1/k\}, \\ \operatorname{dist}(\operatorname{supp}(\psi_1^k), \mathcal{C} \cup \mathcal{N}^-) > 0, \quad \operatorname{dist}(\operatorname{supp}(\psi_2^k), \mathcal{C} \cup \mathcal{N}^+) > 0 \quad \forall k \in \mathbb{N}, \end{gathered}$$

and note that (5.26), (5.27) and Proposition 5.9 yield that for all $z \in L^\infty(\Omega) \cap T_{crit}(c, f)$ with $z^+ = 0$ a.e. in a neighborhood of $\partial\Omega \cup \mathcal{N}^- \cup \mathcal{C}$ and $z^- = 0$ a.e. in a neighborhood of the set $\partial\Omega \cup \mathcal{N}^+ \cup \mathcal{C}$ it holds

$$\begin{aligned} &a(\delta, z) + J(z) + \int_\Omega h j'(w; z) \mathrm{d}\lambda - \int_\Omega h j'(w; \delta) \mathrm{d}\lambda - \langle g, z - \delta \rangle \\ &\quad \ge \limsup_{n \to \infty} \left[a(\delta_n, \delta_n) + \int_\Omega \psi_1^k \frac{c}{t_n} \left(\frac{j(w + t_n \delta_n^+) - j(w)}{t_n} - j'(w; \delta_n^+) \right) \mathrm{d}\lambda \right. \\ &\qquad\qquad \left. + \int_\Omega \psi_2^k \frac{c}{t_n} \left(\frac{j(w + t_n \delta_n^-) - j(w)}{t_n} - j'(w; \delta_n^-) \right) \mathrm{d}\lambda \right] \\ &\quad = \left(\limsup_{n \to \infty} a(\delta_n, \delta_n) \right) + \frac{1}{2} \left(\int_{\{w \neq 0\}} \psi_1^k c j''(w) (\delta^+)^2 + \psi_2^k c j''(w) (\delta^-)^2 \mathrm{d}\lambda \right) \\ &\qquad + \frac{1}{2} \left(\int_{\{w = 0\}} \psi_1^k c j_1''(0) (\delta^+)^2 + \psi_2^k c j_2''(0) (\delta^-)^2 \mathrm{d}\lambda \right) \\ &\qquad + \frac{1}{2} \Big(j_1'(0) + j_2'(0) \Big) \left(\int_{\mathcal{M}} c \frac{\psi_1^k (\operatorname{tr} \delta^+)^2 + \psi_2^k (\operatorname{tr} \delta^-)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} \right). \end{aligned}$$

Using the lemma of Fatou and our assumption $\lambda^d(\partial\{w \neq 0\}) = 0$, we can pass to the limit $k \to \infty$ on the right-hand side of the last estimate. This yields the claim. □

Let us summarize what we know about the limit δ at this point (cf. Lemma 4.4, Lemma 5.5 and Proposition 5.8):

COROLLARY 5.13. *Let Assumptions 2.1, 5.1 and 5.3 hold. Then in the situation of Assumption 4.1, the weak limit δ is an element of the set*

$$\begin{aligned} T_{crit}^{red}(c,f) := \Big\{ z \in H_0^1(\Omega) : \; & z^+ = 0 \ \lambda^d\text{-a.e. in } \{w=0\} \cap \{-cj_2'(0) \le f < cj_1'(0)\}, \\ & z^- = 0 \ \lambda^d\text{-a.e. in } \{w=0\} \cap \{-cj_2'(0) < f \le cj_1'(0)\}, \\ & \operatorname{tr}(z^+) = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^-, \\ & \operatorname{tr}(z^-) = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^+, \\ & \int_{\mathcal{M}} c \frac{(\operatorname{tr} z)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} < \infty \Big\}. \end{aligned} \tag{5.28}$$

We will refer to $T_{crit}^{red}(c,f)$ as the reduced critical cone.

To obtain a proper elliptic variational inequality for the weak limit δ, it remains to prove that (5.24) holds not only for those functions z that satisfy the conditions of Proposition 5.12, but also for all other elements of the reduced critical cone $T_{crit}^{red}(c,f)$. The following approximation result is useful in this context:

LEMMA 5.14. *Suppose that Assumption 2.1 and Assumption 5.3 hold. Then for every function $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying*

$$\operatorname{tr}(z^+) = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^- \quad \text{and} \quad \operatorname{tr}(z^-) = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{N}^+$$

there exists a sequence $z_l \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that z_l converges to z in $H^1(\Omega)$ and such that for all l it is true that

$$\begin{aligned} z_l^+ &= 0 \text{ a.e. in a neighborhood of } \partial\Omega \cup \mathcal{N}^- \cup \mathcal{C}, \\ z_l^- &= 0 \text{ a.e. in a neighborhood of } \partial\Omega \cup \mathcal{N}^+ \cup \mathcal{C}. \end{aligned}$$

Proof. Let $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be an arbitrary function satisfying the trace conditions in the lemma. Then it follows from our assumptions on $\mathcal{C}$ that there exists a sequence $\phi_m \in C_c(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ with

$$0 \le \phi_m \le 1, \quad \phi_m \equiv 1 \text{ in a nbhd. of } \mathcal{C} \quad \text{and} \quad \|\phi_m\|_{H^1} \to 0 \text{ as } m \to \infty.$$

Define $z_m := (1-\phi_m)z \in H_0^1(\Omega)$. Then each z_m vanishes in a neighborhood of $\mathcal{C}$ and we obtain by direct calculation that $z_m \to z$ holds in $H^1(\Omega)$. Consequently, we may assume w.l.o.g. that the function z that we would like to approximate vanishes almost everywhere in a neighborhood $U \subseteq \mathbb{R}^d$ of $\mathcal{C}$. Further, we have $z = z^+ + z^-$ with $z^+, z^- \in H_0^1(\Omega)$ according to Lemma 2.3. This allows us to approximate the positive and the negative part of the function z separately. Let us focus on the positive part z^+ and consider an arbitrary point $p \in (\partial\Omega \cup \mathcal{N}^-) \setminus U \subset \partial\{w \neq 0\} \setminus \mathcal{C}$. Then it follows from Assumption 5.3 that there exist an orthogonal transformation $R_p \in O(d)$, an open ball $B_p \subset \mathbb{R}^{d-1}$, an open interval J_p and a Lipschitz continuous map $h_p : B_p \to J_p$ such that it holds

$$\begin{aligned} p \in R_p(B_p \times J_p), \quad \operatorname{cl}\left(R_p(B_p \times J_p)\right) \subset \mathbb{R}^d \setminus \mathcal{C}, \\ (\partial\Omega \cup \mathcal{N}^-) \cap R_p(B_p \times J_p) = R_p(\{(x, h(x)) : x \in B_p\}). \end{aligned}$$

Since $\mathrm{cl}(\mathcal{N}^-) \setminus \mathcal{N}^- \subseteq \mathcal{C}$, the set $(\partial\Omega \cup \mathcal{N}^-) \setminus U$ is compact and we can find points $p_l \in (\partial\Omega \cup \mathcal{N}^-) \setminus U$, $l = 1, ..., L$, such that the associated rectification domains $R_{p_l}(B_{p_l} \times J_{p_l})$ cover the set $(\partial\Omega \cup \mathcal{N}^-) \setminus U$. Choose an open set $V \subset \Omega$ such that we have $\mathrm{dist}(\overline{V}, \partial\Omega \cup \mathcal{N}^- \cup \mathcal{C}) > 0$ and

$$\overline{\Omega} \subseteq U \cup V \cup \bigcup_{l=1}^{L} R_{p_l}(B_{p_l} \times J_{p_l}).$$

Then it holds $\mathbb{R}^d = (\mathbb{R}^d \setminus \overline{\Omega}) \cup U \cup V \cup R_{p_1}(B_{p_1} \times J_{p_1}) \cup ... \cup R_{p_L}(B_{p_L} \times J_{p_L})$ and we may find a smooth partition of unity $\varphi_j : \mathbb{R}^d \to [0, 1]$, $j = 1, ..., L + 3$, subordinate to this cover of the Euclidean space. Write

$$z^+ = \sum_{j=1}^{L+3} \varphi_j z^+ \in H_0^1(\Omega).$$

Then the functions $\varphi_j z^+$ associated with the sets $(\mathbb{R}^d \setminus \overline{\Omega})$ and U are identical zero and the function $\varphi_j z^+$ associated with V vanishes almost everywhere in a neighborhood of $\partial\Omega \cup \mathcal{N}^- \cup \mathcal{C}$. Moreover, the functions $\varphi_j z^+$ associated with the sets $R_{p_l}(B_{p_l} \times J_{p_l})$ all vanish outside of their respective rectification domain. This allows us to employ the area formula and the usual modification/mollification arguments for functions in the (half-) plane (as found in [8, Theorem 5.5.-2]) to prove that they can be approximated by continuous functions whose supports have a non-zero distance to $\partial\Omega \cup \mathcal{N}^- \cup \mathcal{C}$. Note that due to Lemma 2.3 we may assume w.l.o.g. that these approximating functions are non-negative everywhere. Combining all of the above, we obtain that z^+ can be approximated by non-negative functions z_l which have the desired properties. Using an analogous argumentation for the negative part z^- and adding the approximating sequences for z^+ and z^-, the claim of the lemma follows immediately. ∎

We now finally arrive at the main result of this section:

THEOREM 5.15. *Let Assumptions 2.1 and 5.1 hold, let* $p > \max(d/2, 1)$, *and let* $(c, f) \in L_+^p(\Omega) \times L^p(\Omega)$ *be a tuple such that* c, f *and the solution* $w := S(c, f)$ *satisfy the conditions in Assumption 5.3. Then the solution operator* $S : L_+^p(\Omega) \times L^p(\Omega) \to H_0^1(\Omega)$ *associated with the problem (P) is Hadamard directionally differentiable in* (c, f) *in all directions* $(h, g) \in \mathbb{R}^+(L_+^p(\Omega) - c) \times L^p(\Omega)$ *and the derivative* $\delta := S'((c, f); (h, g))$ *in a direction* (h, g) *is characterized by the variational inequality*

$$\begin{aligned} \delta \in T_{crit}^{red}(c, f), \quad & a(\delta, z - \delta) + J(z) - J(\delta) \\ & \geq \langle g, z - \delta \rangle - \int_{\{w \neq 0\}} h j'(w)(z - \delta) \mathrm{d}\lambda \\ & \quad - \int_{\{w = 0\}} h j_1'(0)(z^+ - \delta^+) - h j_2'(0)(z^- - \delta^-) \mathrm{d}\lambda \\ & \qquad \forall z \in T_{crit}^{red}(c, f). \end{aligned} \tag{5.29}$$

Here, J *and* $T_{crit}^{red}(c, f)$ *are defined by (5.25) and (5.28), respectively.*

Proof. Consider the situation in Assumption 4.1 and let z be an arbitrary but fixed element of the set $T_{crit}^{red}(c, f)$. Then it follows from the definition of the reduced critical cone and Lemma 5.14 that there exist sequences $z_{k,l} \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $k, l \in \mathbb{N}$, such that $z_{k,l}$ converges to $z_k := \min(z^+, k) + \max(z^-, -k)$ in $H^1(\Omega)$ as $l \to \infty$ for all k and such that for all l it holds

$$z_{k,l}^+ = 0 \text{ a.e. in a neighborhood of } \partial\Omega \cup \mathcal{N}^- \cup \mathcal{C},$$
$$z_{k,l}^- = 0 \text{ a.e. in a neighborhood of } \partial\Omega \cup \mathcal{N}^+ \cup \mathcal{C}.$$

Define

$$\tilde{z}_{k,l} := \min(z_{k,l}^+, \min(z^+, k)) + \max(z_{k,l}^-, \max(z^-, -k)).$$

Then the properties of z, z_k and $z_{k,l}$ yield

$$\begin{gathered}
\tilde{z}_{k,l} \in H_0^1(\Omega) \cap L^\infty(\Omega), \\
|\tilde{z}_{k,l}| \leq |z_k| \leq |z| \ \ \lambda^d\text{-a.e. in } \Omega \ \ \forall k, l, \\
c\frac{(\operatorname{tr} \tilde{z}_{k,l})^2}{\|\nabla w\|} \leq c\frac{(\operatorname{tr} z_k)^2}{\|\nabla w\|} \leq c\frac{(\operatorname{tr} z)^2}{\|\nabla w\|} \in L^1(\mathcal{M}, \mathcal{H}^{d-1}) \ \ \mathcal{H}^{d-1}\text{-a.e. on } \mathcal{M} \ \ \forall k, l, \\
\tilde{z}_{k,l} \to z_k \text{ in } H^1(\Omega) \text{ as } l \to \infty \ \ \forall k, \\
\tilde{z}_{k,l}^+ = 0 \ \lambda^d\text{-a.e. in a neighborhood of } \partial\Omega \cup \mathcal{N}^- \cup \mathcal{C}, \\
\tilde{z}_{k,l}^- = 0 \ \lambda^d\text{-a.e. in a neighborhood of } \partial\Omega \cup \mathcal{N}^+ \cup \mathcal{C}, \\
\tilde{z}_{k,l}^+ = 0 \ \lambda^d\text{-a.e. in } \ \{w = 0\} \cap \{-cj_2'(0) \leq f < cj_1'(0)\}, \\
\tilde{z}_{k,l}^- = 0 \ \lambda^d\text{-a.e. in } \ \{w = 0\} \cap \{-cj_2'(0) < f \leq cj_1'(0)\}.
\end{gathered}$$

Consequently, $\tilde{z}_{k,l}$ is an element of the set $L^\infty(\Omega) \cap T_{crit}^{red}(c, f)$ and it follows from Proposition 5.12 that

$$\begin{aligned}
a(\delta, \tilde{z}_{k,l}) + J(\tilde{z}_{k,l}) - J(\delta) + \int_\Omega h\Big(j'(w; \tilde{z}_{k,l}) - j'(w; \delta)\Big)\mathrm{d}\lambda - \langle g, \tilde{z}_{k,l} - \delta\rangle& \\
\geq \limsup_{n\to\infty} a(\delta_n, \delta_n).& \qquad (5.30)
\end{aligned}$$

Using the dominated convergence theorem and the weak lower semicontinuity of the function $H_0^1(\Omega) \ni z \mapsto a(z, z) \in \mathbb{R}$, we can pass to the limit in (5.30) (first with l then with k) to obtain

$$\begin{aligned}
a(\delta, z) + J(z) - J(\delta) + \int_\Omega h\Big(j'(w; z) - j'(w; \delta)\Big)\mathrm{d}\lambda - \langle g, z - \delta\rangle &\geq \limsup_{n\to\infty} a(\delta_n, \delta_n) \\
&\geq \liminf_{n\to\infty} a(\delta_n, \delta_n) \\
&\geq a(\delta, \delta). \qquad (5.31)
\end{aligned}$$

The last estimate has several implications: First of all, it yields that the weak limit δ of the difference quotients δ_n satisfies the variational inequality (5.29). This proves that δ is unique (since (5.29) can only have one solution - just argue by contradiction) and implies that S is weakly directionally differentiable in (c, f) (cf. Section 4). Moreover, if we consider the special choice $z = \delta \in T_{crit}^{red}(c, f)$ in (5.31), then we obtain

$$a(\delta, \delta) \geq \limsup_{n\to\infty} a(\delta_n, \delta_n) \geq \liminf_{n\to\infty} a(\delta_n, \delta_n) \geq a(\delta, \delta)$$

and, consequently,

$$\|\delta - \delta_n\|^2 \leq Ca(\delta - \delta_n, \delta - \delta_n) = C\Big(a(\delta, \delta) - a(\delta, \delta_n) - a(\delta_n, \delta) + a(\delta_n, \delta_n)\Big) \to 0.$$

This shows that the difference quotients converge even strongly in the situation of Assumption 4.1 and that S is strongly directionally differentiable. The Hadamard differentiability now follows immediately from the strong directional differentiability and the Lipschitz continuity of the solution operator S. This completes the proof. ∎

6. Notes Regarding Theorem 5.15 and Concluding Remarks. Several points are noteworthy regarding Theorem 5.15 and the variational inequality (5.29):

First of all, we remark that (5.29) is in general neither a variational inequality of the first nor a variational inequality of the second kind. If we consider, e.g., the special case $j(x) = |x|$, $c \equiv 1$ and $h \equiv -1$, then (5.29) becomes

$$\begin{aligned}
&\delta \in T_{crit}^{red}(c, f),\\
&a(\delta, z - \delta) + \int_{\mathcal{M}} \frac{(\operatorname{tr} z)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} - \int_{\{w=0\}} |z| \mathrm{d}\lambda - \int_{\mathcal{M}} \frac{(\operatorname{tr} \delta)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} + \int_{\{w=0\}} |\delta| \mathrm{d}\lambda\\
&\qquad\qquad \geq \langle g, z - \delta \rangle + \int_{\{w \neq 0\}} \operatorname{sgn}(w)(z - \delta) \mathrm{d}\lambda \quad \forall z \in T_{crit}^{red}(c, f)
\end{aligned}$$

and we end up with a variational inequality which involves an in general non-convex functional of the form

$$z \mapsto \int_{\mathcal{M}} \frac{(\operatorname{tr} z)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} - \int_{\{w=0\}} |z| \mathrm{d}\lambda.$$

Note that, although (5.29) does not fit into the classical setting, the unique solvability of the variational inequality in Theorem 5.15 is still guaranteed: The existence of a solution follows directly from our analysis (since we have proved that the limit of the difference quotients δ_t satisfies (5.29)), and the uniqueness of the solution is a trivial consequence of the ellipticity of the bilinear form a (cf. [12, Theorem 4.1] and the proof of Theorem 5.15).

Secondly, we point out that the natural space for the study of (5.29) is the Hilbert space

$$H := \left\{ z \in H_0^1(\Omega) : \int_{\mathcal{M}} c \frac{(\operatorname{tr} z)^2}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} < \infty \right\}$$

endowed with the scalar product

$$(z_1, z_2)_H := \int_{\Omega} \nabla z_1 \cdot \nabla z_2 \mathrm{d}\lambda + \int_{\mathcal{M}} c \frac{(\operatorname{tr} z_1)(\operatorname{tr} z_2)}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1}.$$

This can be seen, e.g., in the case $j(x) = |x|/2$, $h \equiv 0$ and $A = -\Delta$, where (5.29) can be rewritten as

$$\begin{aligned}
&\delta \in T_{crit}^{red}(c, f),\\
&\int_{\Omega} \nabla \delta \cdot \nabla (z - \delta) \mathrm{d}\lambda + \int_{\mathcal{M}} c \frac{(\operatorname{tr} \delta)(\operatorname{tr} z - \operatorname{tr} \delta)}{\|\nabla w\|} \mathrm{d}\mathcal{H}^{d-1} \geq \langle g, z - \delta \rangle \quad \forall z \in T_{crit}^{red}(c, f)
\end{aligned} \tag{6.1}$$

and the directional derivative $\delta = S'((c, f); (h, g))$ is exactly the $(.,.)_H$-projection of the Riesz representative of g onto the reduced critical cone $T_{crit}^{red}(c, f)$. Note that due to the weight $1/\|\nabla w\|$ in the trace integral of $(.,.)_H$, the space H is usually a proper subspace of $H_0^1(\Omega)$. This has to be taken into account if, e.g., strong stationarity conditions for optimal control problems governed by variational inequalities of the type (P) are considered (cf. the analysis in [6, 20]).

Thirdly, it should be noted that in the situation of Theorem 5.15, the solution operator S associated with (P) is Gâteaux differentiable in the point (c, f) if and only if the map $(h, g) \mapsto \delta$ is linear and continuous. This implies that it sufficies to study the solution behavior of the variational inequality (5.29) to derive criteria for the Gâteaux differentiability of the map S. Consider, for example, the situation in (6.1) where the map $g \mapsto \delta$ can be identified with the H-projection onto the reduced critical cone $T_{crit}^{red}(c, f)$. In this case, the solution operator $g \mapsto \delta$ is certainly linear and continuous whenever the set $T_{crit}^{red}(c, f)$ is a subspace of H. Accordingly (cf. the definition of $T_{crit}^{red}(c, f)$), in the situation in (6.1) a sufficient condition for the Gâteaux differentiability of S is

$$|f| < c/2 \text{ a.e. in } \{w = 0\} \quad \text{and} \quad \mathcal{N}^+ \setminus \operatorname{cl}(\operatorname{int}(\{w = 0\})) = \mathcal{N}^- \setminus \operatorname{cl}(\operatorname{int}(\{w = 0\})).$$

Lastly, we remark that the terms $J(z)$ and $J(\delta)$ appearing in (5.29) are closely related to the pullback $w * j''$ of the second distributional derivative of j by w (in the sense of Hörmander [15, Chapter VI]). To be more precise, we have the formal identities

$$J(z) = \frac{1}{2}\left\langle c(w * j''), z^2 \right\rangle \quad \text{and} \quad J(\delta) = \frac{1}{2}\left\langle c(w * j''), \delta^2 \right\rangle,$$

where the brackets $\langle .,. \rangle$ denote the distributional pairing (cf. [15, Example 6.1.5] and [27, Section V.13]). Recall that in the classical theory, the pullback of a distribution by a function $v \in C^1(\Omega)$ is defined by extending the composition map

$$C_c^\infty(\mathbb{R}) \ni \varphi \mapsto \varphi(v) \in C^1(\Omega)$$

continuously to the space $\mathcal{D}'(\mathbb{R})$ and that this extension is only possible if the gradient of the function v under consideration vanishes nowhere in Ω (cf. [15, Theorem 6.1.2]). What can be observed in the situation of Theorem 5.15 is that in the variational inequality (5.29) for the directional derivative $S'((c, f); (h, g))$ the terms known from the classical pullback $w * j''$ appear "everywhere where they make sense" and that on the set $\partial\{w \neq 0\} \cap \{\nabla w = 0\}$, where the classical construction fails, the pullback terms are replaced with the trace conditions $\operatorname{tr}(z^+) = 0$ $\mathcal{H}^{d-1}$-a.e. on $\mathcal{N}^-$ and $\operatorname{tr}(z^-) = 0$ $\mathcal{H}^{d-1}$-a.e. on $\mathcal{N}^+$. That the quantity $w * j''$ emerges in the above way when the convergence of second order difference quotients of the type (4.4) is studied (or the Mosco epi-convergence to be more precise since this is what we have, in fact, considered in Section 5, cf. [18]) is remarkable and has, at least to the authors' best knowledge, not been studied systematically so far.

It should be noted that the analysis of the difference quotients in (4.4) becomes even more complicated when the bilinear form a in (P) is assumed to be H^k-elliptic for some $k > 1$. In this situation, also $(d-2)$-, $(d-3)$- etc. dimensional "features" of the level sets of w are relevant for the sensitivity analysis and traces of derivatives have to be taken into account, too. Finding a systematic approach towards the study of such H^k-elliptic problems seems to be difficult and is subject to further research. The same holds true for the extension of our differentiability results to variational inequalities that do not fit into the setting of Theorem 5.15, e.g., inequalities which involve terms of the form $j(\|\nabla v\|)$.

REFERENCES

[1] Adams, R.A.: *Sobolev Spaces*, Academic Press, New York (1975)

[2] Attouch, H., G. Buttazzo and G. Michaille: *Variational Analysis in Sobolev and BV Spaces*, SIAM, Philadelphia (2005)

[3] Betz, T. and C. Meyer: *Second-order sufficient optimality conditions for optimal control of static elastoplasticity with hardening*, ESAIM: Control, Optimisation and Calculus of Variations, 21, 271-300 (2015)

[4] Bonnans, J.F., and A. Shapiro: *Perturbation analysis of optimization problems*, Springer Series in Operations Research, Springer-Verlag, New York (2000)

[5] De los Reyes, J.C.: *Optimal control of a class of variational inequalities of the second kind*, SIAM Journal on Control and Optimization, 49, 1629–1658 (2011)

[6] De los Reyes, J.C., and C. Meyer: *Strong Stationarity Conditions for a Class of Optimization Problems Governed by Variational Inequalities of the Second Kind*, Journal of Optimization Theory and Applications, 168, 2, 375-409 (2016)

[7] Ekeland, I., and R. Temam: *Convex Analysis and Variational Problems*, North Holland, Amsterdam (1976)

[8] Evans, L.C.: *Partial Differential Equations*, 2. Edition, AMS, Providence, RI (2010)

[9] Evans, L.C., and R.F. Gariepy: *Measure Theory and Fine Properties of Functions*, Revised Edition, CRC Press, Boca Raton (2015)

[10] Fitzpatrick, S., and R.R. Phelps: *Differentiability of the metric projection in Hilbert space*, Transactions of the American Mathematical Society, 270, 483-501 (1982)

[11] Gilbarg, D., and N.S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 Edition, Springer-Verlag, Berlin/Heidelberg/New York (2001)

[12] Glowinski, R.: *Lectures on Numerical Methods for Non-Linear Variational Problems*, Tata Institute of Fundamental Research, Bombay (1980)

[13] Haraux, A.: *How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities*, Journal of the Mathematical Society of Japan, 29, 615–631 (1977)

[14] Hintermüller, M. and T. Surowiec: *First-order optimality conditions for elliptic mathematical programs with equilibrium constraints via variational analysis*, SIAM Journal on Optimization, 21, 1561–1593 (2012)

[15] Hörmander, L.: *The Analysis of Linear Partial Differential Operators I*, Second Edition, Springer, Berlin/Heidelberg/New York (1990)

[16] Iserles, A., S.P. Nørsett and S. Olver: *Highly oscillatory quadrature: The story so far*, Proceedings of ENuMath, Springer-Verlag, Berlin, 97–118 (2006)

[17] Kinderlehrer, D., and G. Stampacchia: *An Introduction to Variational Inequalities and Their Applications*, SIAM, Philadelphia (2000)

[18] Levy, A.B.: *Sensitivity of solutions to variational inequalities on Banach spaces*, SIAM Journal on Control and Optimization, 38, 50–60 (1999)

[19] Mignot, F.: *Controle dans les inéquations variationelles elliptiques*, Journal of Functional Analysis, 22, 130–185 (1976)

[20] Mignot, F., and J.P. Puel: *Optimal control in some variational inequalities*, SIAM Journal on Control and Optimization, 22, 466-476 (1984)

[21] Nečas, J.: *Direct Methods in the Theory of Elliptic Equations*, Springer, Berlin/Heidelberg (2012)

[22] Niculescu, C.P., and L.E. Persson: *Convex Functions and their Applications - A Contemporary Approach*, CMS Books in Mathematics, 23, Springer-Verlag, New York (2006)

[23] Shapiro, A.S.: *Existence and differentiability of metric projections in Hilbert spaces*, SIAM Journal on Optimization, 4, 130–141 (1994)

[24] Sokolowski, J.: *Sensitivity analysis of contact problems with prescribed friction*, Applied Mathematics and Optimization, 18, 99-117 (1988)

[25] Walter, R.: *Some analytical properties of geodesically convex sets*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, 45, 263-282 (1976)

[26] Warner, F.W.: *Foundations of Differentiable Manifolds and Lie Groups*, Springer, Berlin/Heidelberg/New York (1983)

[27] Wong, R.: *Asymptotic Approximations of Integrals*, SIAM, Philadelphia (2001)