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NATURAL CONVECTION OF INCOMPRESSIBLE VISCOELASTIC FLUID FLOW

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Abstract: We revisit the MIT Benchmark 2001 and introduce a viscoelastic constitutive law into the fluid in motion. Our goal is to study the effect of viscoelasticity into the periodical behavior of the physical quantities of the corresponding benchmark. We use a robust numerical technique in simulating complex fluid flow problems based on higher order Finite Element discretization. While marching in time, an A-stable method of second order is favorable, i.e Crank-Nicolson scheme, to reproduce periodical behaviors. We use a differential form of viscoelastic model, i.e Oldroyd-B type and find out that a small amount of viscoelasticity reduces the oscillatory behavior.

Keywords: Viscoelastic, Oldroyd-B, Natural convection, Non-isothermal.

1. INTRODUCTION

The MIT Benchmark 2001 [1] describes a heat driven cavity flow in a stretched rectangular domain (1:8). This very simple setup leads to a challenging numerical method near a critical Rayleigh number (*Ra*), and introduces already a complex multi-scale phenomena. Thus, the study of transport of temperature in this simple geometry is very important from both numerical and experimental view point. The flow model for the MIT Benchmark can be described as the following:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + 2\eta\,\nabla\cdot\boldsymbol{D} + (1-\gamma\theta)\boldsymbol{j} \\ \nabla\cdot\boldsymbol{u} = 0 \\ \frac{\partial \theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)\theta = \frac{1}{\sqrt{Ra\,Pr}}\Delta\theta \end{cases} \quad (1)$$

with numerical variables are velocity, pressure, temperature $(\boldsymbol{u}, p, \theta)$ and material parameters are thermal expansion and viscosity (γ, η). The values for material parameters can be seen in [1]. The viscosity is expressed in terms of Prandtl number and Rayleigh number ($\eta = \sqrt{\frac{Pr}{Ra}}$). In the above equation (1), the coupling with the temperature is given by Boussinesq approximation using the gravity vector ($\boldsymbol{j}$), in which its direction is visible in Fig. 1. Furthermore, the stress tensors are represented by the symmetric part of gradient velocity, $\boldsymbol{D} = 1/2(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$, and the gradient of hydrostatic pressure.

In the previous study [2] we have shown that our chosen methodology to tackle equation (1) is highly competitive w.r.t. the benchmark work in [1]. This means that all the quantitative data at points of interest (1 and 2 at the corner of the domain) as well as the computed Nusselt number on the side wall have less than 0.02% differences from the references. Thus, we would like to research further in the direction of slightly different material law than the above quasi Newtonian fluid, which is viscoelastic fluid. On this study, we would like to see qualitatively the effect of inserting viscoelasticity and would be done only with one grid.

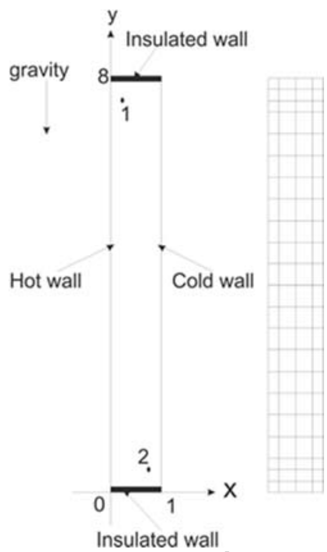

Fig.1. Geometry and set up

2. INSERTING VISCOELASTICITY

We introduce a differential form of viscoelastic material law into equation (1) using an Oldroyd-B fluid [3]. The emerging equation can be written as follows:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + 2\eta_s\,\nabla\cdot\boldsymbol{D} + (1-\gamma\theta)\boldsymbol{j} + \frac{\eta_p}{\Lambda}\nabla\cdot\boldsymbol{\tau} \\ \nabla\cdot\boldsymbol{u} = 0 \\ \frac{\partial \theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)\theta = \frac{1}{\sqrt{Ra\,Pr}}\Delta\theta \\ \frac{\partial \boldsymbol{\tau}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau} - \nabla\boldsymbol{u}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\nabla\boldsymbol{u}^{\boldsymbol{T}} + \frac{1}{\Lambda}(\boldsymbol{\tau} - \boldsymbol{I}) = \boldsymbol{0} \end{cases}$$

(2)

Different than equation (1), one notices an extra viscoelastic stress equation with a new material parameter which is called relaxation time, Λ. The numerical treatment of viscoelastic fluid governed by the above material law is studied by many authors. Please see in [4] and all the citations therein for further details. Furthermore, one notices that an additional viscoelastic stress tensor, $\boldsymbol{\tau}$, appears on the right hand side, and that the viscosity is now split into two parts, $\eta = \eta_s + \eta_p$, which are the viscous and the viscoelastic part. In this study, we choose a "moderate" number of relaxation time, $\Lambda = 0.1$, to avoid unnecessary numerical effects related to viscoelastic problem, and use a fraction of viscous part, $\beta = \eta_s/\eta$, to give a control on how viscoelastic contribution into the total viscosity may change the flow behavior. So, if we set a value to $\beta = 1$, this corresponds to a full viscous contribution (no viscoelasticity), which would lead to an uncoupling of the additional viscoelastic contribution on the right hand side of the momentum equation. Thus, one would expect the same numerical results as the one from solving equation (1). If we set a value to $\beta < 1$, the viscoelastic effect contributes into the total viscosity, which would be reflected by a different temperature behavior than the one from solving equation (1).

3. NUMERICAL METHOD

The above nonlinear equation (2) is solved using fully coupled strategy, which means that all numerical variables $(\boldsymbol{u}, p, \boldsymbol{\tau}, \theta)$ are iterated simultaneously in each time step and in each nonlinear step. A standard ODE-solver of second order is used, i.e. Crank-Nicolson, for the transient behavior. Inside one time step, Newton iteration is used for solving the nonlinear discrete problem resulting from standard Galerkin Finite Element discretization. Inside one Newton iteration a Jacobi matrix is built by using divided difference approach which uses a machine precision epsilon value. An exemplary form of different Jacobi structures can be found in the previous work of [6], which generally mimics the saddle-point problem,

$$\begin{bmatrix} A & \tilde{B}^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{u}} \\ p \end{bmatrix} = \begin{bmatrix} \mathcal{R}_{\widetilde{\boldsymbol{u}}} \\ \mathcal{R}_p \end{bmatrix} \quad (3)$$

where A is a linearized operator of velocity-stress-temperature, B is divergence operator, $\mathcal{R}$ is the residuals, and $\widetilde{\boldsymbol{u}} = (\boldsymbol{u}, \boldsymbol{\tau}, \theta)$. A conforming second order finite element (with 9 degrees of freedom (dof)) is used for the discretization of $(\boldsymbol{u}, \boldsymbol{\tau}, \theta)$, while a discontinuous linear function is chosen for the pressure. This combination is well-known to satisfy LBB-condition (after Ladyzhenskaya, Babuska and Brezzi) of the mixed problem for velocity-pressure approximation, see for example [5]. This condition is sometimes also called "Babuska-Brezzi" or "Inf-sup".

The grid to produce the initial data is taken by doing two times regular refinement from the coarse grid shown in Fig. 1. This corresponds to 1408 elements or approximately 40000 dofs. More additional grid refinement would be done in the next study. The boundary condition for the two velocity components is non-slip everywhere, for the temperature is "do-nothing" at top and bottom and a prescribed temperature at sides wall ("hot" is set to 0.5 and "cold" is assigned to -0.5). The boundary condition for the viscoelastic stress is "do-nothing" everywhere.

4. RESULTS AND DISCUSSION

We use the same initial solution, at time $t = 0$, for all nonsteady simulations of different β values, which is a steady data solution close to a near critical Rayleigh number. The following Fig. 2 is the solution of temperature oscillation at point 1 by setting full viscous contribution, $\beta = 1$. As expected, this result recovers the MIT Benchmark 2001 initial data, which reflects the onset of transient behavior at critical Rayleigh number. Please notice that the steady oscillation starts to appear approximately at time $t > 2000$.

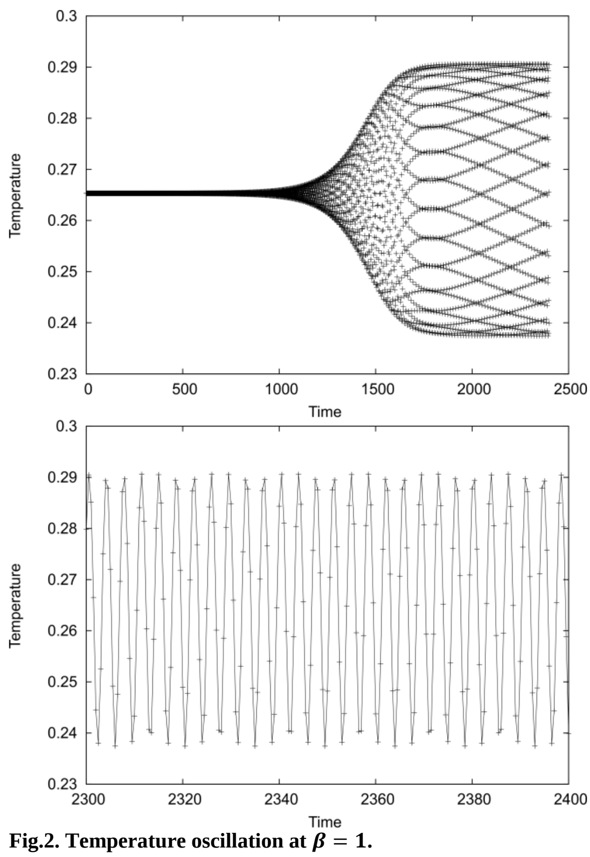

Fig.2. Temperature oscillation at $\boldsymbol{\beta = 1}$.

In the following, we show that a small portion of viscoelasticity into the fluid material reduces the amplitude of the corresponding steady oscillation for $\beta = 0.75, 0.6, 0.5$.

We first introduce viscoelasticity by setting $\beta = 0.75$, see Fig. 3. We obtain in a similar manner as in the case of $\beta = 1$ a steady temperature oscillation at time $t > 2000$ with slightly smaller amplitude than the one from the previous case. At the beginning of time iteration one sees a clear different behavior between the two cases, where multiple temperature oscillations appear in a more chaotic way before it is damped later on in the long time computation.

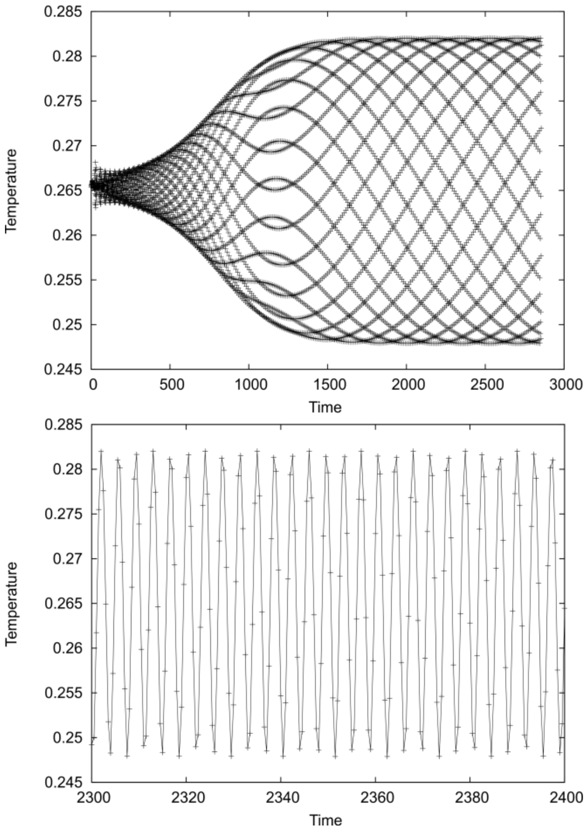

Fig.3. Temperature oscillation at $\boldsymbol{\beta = 0.75}$.

We set now $\beta = 0.65$ and obtain more reduction of the steady amplitude, see Fig. 4. In the same manner as in the previous case, the chaotic oscillation in the beginning of iteration quickly stabilizes in approximately the first 500 time steps, and finally in the long time computation, $t > 3000$, the solution reaches a steady periodical oscillation.

In the case $\beta = 0.5$ one sees a more present damping of the periodical oscillation than the previous cases, see Fig. 5. The same as before, in the first 200 iterations one sees a chaotic multiple oscillations, which are damped in the long time computation to a steady state solution, i.e. the steady periodical behavior of the temperature is completely damped away. The obtained steady data at time $t = 6000$ are as follows: Temperature θ_1=0.2654041 and Nusselt number Nu=4.668066.

The last case provides us information that steady solution data exists starting from a certain value of viscoelastic contribution. And indeed, one can provide this data by using direct steady approach to equation (2), i.e. by canceling time dependency, $\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} = 0$. Table 1 provides these data using the same initial solution used for the above unsteady simulation with only a few nonlinear iterations.

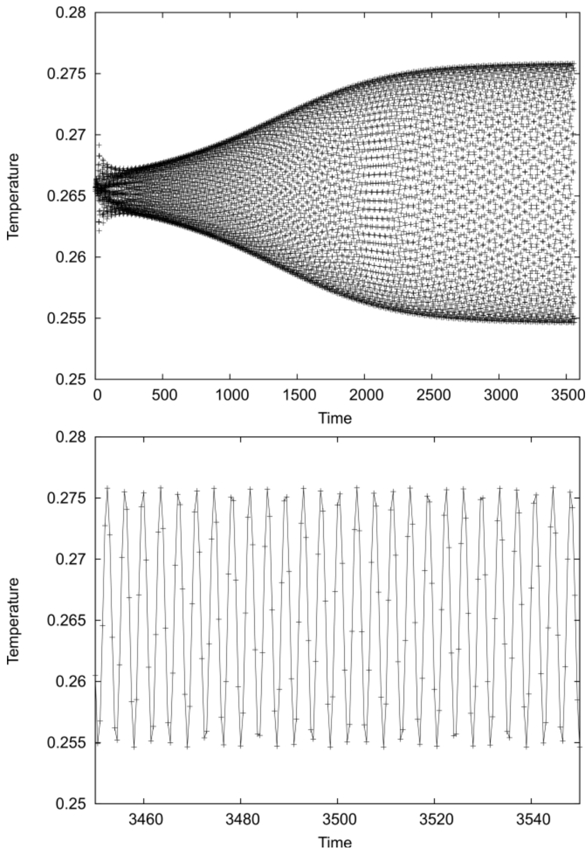

Fig.4. Temperature oscillation at $\boldsymbol{\beta = 0.65}$.

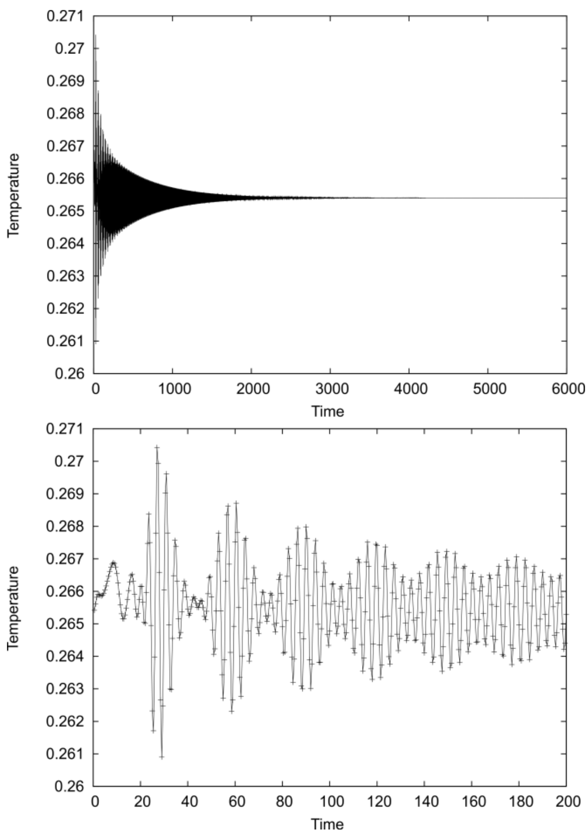

Fig.5. Temperature oscillation at $\boldsymbol{\beta = 0.5}$.

Table 1 Direct steady approach for $\beta \leq 0.5$

★★★

β	Nonlinear steps	θ_1	Nu
0.5	4	0.2654041	4.668066
0.4	4	0.2654271	4.668867
0.3	5	0.2654530	4.669650
0.2	5	0.2654823	4.670387
0.1	6	0.2655137	4.671005
0	5	0.2655297	4.670980

CONCLUSIONS

The MIT Benchmark 2001 is revisited. Natural convection flow clearly behaves differently when a small portion of viscoelastic properties is introduced. Four points can be observed which are: 1). At the beginning of unsteady iteration the oscillations form in a more chaotic way when viscoelasticity is introduced, 2). After a longer time computation, the amplitude of stable periodical oscillation reduces as viscoelastic contribution increases, 3). The time steps needed for the solution to reach steady oscillation increases with increasing viscoelasticity, i.e. decreasing β, 4). There exists a steady solution to equation (2) for a certain amount of viscoelastic contribution, which appears in this work to be $\beta \leq 0.5$.

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