A fast and accurate method for grid deformation

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Overview

- Motivation
- Grid deformation: derivation and convergence aspects
- Multilevel deformation
- r-adaptivity
Why Grid Deformation?

main reason: building block for r-adaptivity
Why $r$-Adaptivity?

1. reason: flexibility
Why $r$-Adaptivity?

2. reason: **SPEED**

FEM example:

- tensor product mesh
- $Q_1$ FE, Laplace eq., equidistant grid
- for lexicographical ordering: 9 nonzero bands
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MV-multiplication in **FEATFLOW** (F77-code):

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observation: AFEM: MFlop/s-Rate $\ll$ peak performance
Why $r$-Adaptivity?

Reference machine: AMD Opteron 852

- peak performance: 4.3 GFlop/s
- peak memory bandwidth: 5.96 GB/s

⇒ peak performance, if $\approx 6$ flops per memory access
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arithmetic intensity:

- CSR: $17/28 \ll 1 \Rightarrow \approx 9\%$ of peak performance
- FEAST: $17/19 \leq 1 \Rightarrow \approx 15\%$ of peak performance
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Avoid unstructured meshes!
FEAST-Concept (Grid-Related Part)

global grid: “many” local generalised tensor product meshes (“macros”).
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r-adaptivity
**Derivation**

- domain $Ω$
- triangulation $\mathcal{T}$, quads $T$
- “monitor function” $0 < ε < f \in C^1(\overline{Ω})$: desired area distribution
- “weighting function” $0 < ε < g \in C^1(\overline{Ω})$: current area distribution

**goal:** transformation $\Phi : Ω \to Ω$ with

$$g(x)|J\Phi(x)| = f(\Phi(x)) \quad \forall x \in Ω$$

and

$$\Phi : \partialΩ \to \partialΩ.$$

$$\mathcal{T}^d = \Phi(\mathcal{T}), \quad X := \Phi(x)$$
Derivation

\[ m(\Phi(T)) := \int_{\Phi(T)} 1 \, dx = \int_T |J\Phi(x)| \, dx, \]
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1 × 1 Gauss-formula:

\[ g(x_c) \frac{m(\Phi(T))}{m(T)} = f(\Phi(x_c)) + O(h). \]
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If

\[ g(x) = c(h) \, m(T) + O(h), \, x \in T \]

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Grid Deformation Method

Deformation\((f, \mathcal{T})\)

compute \(\tilde{f} - \tilde{g}, \quad \tilde{f} := c/f, \tilde{g} = C/g, \int \tilde{f} = \int \tilde{g}\)

solve
\[-\text{div}(v(x)) = \tilde{f}(x) - \tilde{g}(x), \quad x \in \Omega, \quad v(x) \cdot n = 0, \quad x \in \partial\Omega\]

DO FORALL \(x \in \mathcal{T}\)

solve
\[
\partial_t \varphi(x, t) = \frac{v(\varphi(x, t), t)}{tf(\varphi(x, t)) + (1-t)\tilde{g}(\varphi(x, t))}, \quad 0 \leq t \leq 1, \varphi(x, 0) = x
\]

\(\Phi(x) := \varphi(x, 1)\)

ENDDO

END Deformation
Theoretical Results

**Theorem** (Moser) Let $0 \geq k \in \mathbb{N}$, $\alpha > 0$. Let $\Omega \subset \mathbb{R}^n$ a domain with $C^{3+k,\alpha}$-smooth boundary. Let $f, g \in C^{k,\alpha}(\overline{\Omega})$ with $\int_{\Omega} f = \int_{\Omega} g$. Then there is a $C^{k+1,\alpha}$-diffeomorphism $\Phi : \bar{\Omega} \to \mathbb{R}^n$ with

$$g(x)|J\Phi(x)| = f(\Phi(x)) \quad \forall x \in \Omega$$

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$$\Phi(x) = x \quad \forall x \in \partial\Omega.$$

**Theorem** Let be $\Omega$ as above. If $\Phi : \Omega \rightarrow \Omega$ exists, it fulfills the aforementioned conditions.
Measuring the Error and Convergence

situation: Let \((T_i)_{i \in I}, N_i < N_{i+1}\) with

\[
h_i := \max_{e \in \mathcal{E}_i} |e| = O\left(N_i^{-0.5}\right) \quad \forall i \in I \quad \text{(edge-length regularity)}
\]

\[
\exists 0 < c, C : c h_i^2 \leq m(T) \leq C h_i^2 \quad \forall T \in \mathcal{T}_i \forall i \in I \quad \text{(size regularity)}
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similarity condition: \( \exists 0 < g_{\text{min}} < g < g_{\text{max}} < \infty \) with

\[ \frac{1}{h_i^2} c_i m(T) = g(x) + O(h_i) \quad \forall x \in T \quad \forall T \in T_i \forall i \in I, \quad c_s \leq c_i \leq C_s. \]
Measuring the Error and Convergence

1. approach: comparison with “reference deformation”:
\[ \| \Phi_h - \Phi \| \to 0 \]
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problems:

- \( \Phi \) unique only by \( \text{curl} v = 0 \)
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2. approach: \( q(x) = \frac{f(x)}{g(x)} - 1 \triangleq 0 \Rightarrow \)

\[
Q_0 := ||q||_{L^2(\Omega)}, \quad Q_{\infty} := ||q||_{L^\infty(\Omega)}
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\[ q(x) = \frac{f(x)}{g(x)} - 1 \approx 0 \Rightarrow \]
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convergence :\( \iff Q_0 \to 0, Q_\infty \to 0, h \to 0 \)
**Convergence Theorem**

Let \((\mathcal{T}_i)_{i \in I}\) be edge-length regular and fulfill the similarity condition, \(0 < \varepsilon < f \in C^1(\overline{\Omega})\). Furthermore, 
\[
||\nabla w - G_h w_h||_\infty = \mathcal{O}(h^{1+\delta}), \quad \delta > 0 \text{ and } ||X_h - \tilde{X}|| = \mathcal{O}(h^{1+\delta}).
\]

Then:

a) \((\tilde{T}_i)_{i \in I}\) is edge-length regular.

b) \((\tilde{T}_i)_{i \in I}\) is size regular.

c) \(\exists c > 0:\)

\[
Q_0 \leq c h^{\min\{1,\delta\}}, \quad Q_\infty \leq c h^{\min\{1,\delta\}}.
\]
**Test Problem**

$\Omega = [0, 1]^2$, tensor product mesh

$$f(x) = \min \left\{ 1, \max \left\{ \frac{|d - 0.25|}{0.25}, \varepsilon \right\} \right\}, \quad d := \sqrt{(x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2}$$
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\[ \varepsilon = 0.1 \]
Convergence of ODE-Solvers

$N E L = 65536$

ODE-error: $\mathcal{O}(\Delta t^2)$
convergence for the Test Problem

**Corollary** Let us assume that

$$\|\nabla w - G_h w_h\|_{L^\infty} = O(h^2), \quad \Delta t = O(h).$$

$$\Rightarrow \quad Q_0 = O(h), \quad Q_\infty = O(h)$$
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Runtime

convergence: time step size $\Delta t = O(h) = O(N^{-1/2})$

complexity: $O(N^{3/2})$
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Multilevel Deformation

**goal:** fixed time step size + convergence

in practical computations: sequence of grids by successive regular refinement
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idea:
• deformation on coarse grid
• regular refinement
• deformation on fine grid (correction step)
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assumption 1:

\[ d_k := \max_{x \in \mathcal{X}_k} \| x - \Phi(x) \| = \mathcal{O}(h^2) \]

assumption 2:

\[ \frac{\| X_h - \tilde{X} \|}{\| x - \Phi(x) \|} \leq c \]
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\[
\frac{||X_h - \tilde{X}||}{||x - \Phi(x)||} \leq c
\]

\[ \Rightarrow ||X_h - \tilde{X}|| = O(h^2) \]
Multilevel Deformation
**Multilevel Deformation (Algorithm)**

\[
\text{MultilevelDef}(f, \mathcal{T}, N_{\text{pre}}) : \mathcal{T}
\]

\[
\mathcal{T}_{i_{\text{min}}} := R(\mathcal{T}, i_{\text{min}})
\]

**DO** \(i = i_{\text{min}}, i_{\text{max}}, i_{\text{incr}}\)

\[
\mathcal{T}_i := \text{PreSmooth}(\mathcal{T}_i, N_{\text{pre}}(i))
\]

\[
\mathcal{T}_i := \text{Deformation}(f, \mathcal{T}_i)
\]

**IF** \((i < i_{\text{max}}) \mathcal{T}_{i+1} := V(\mathcal{T}_i)\)

**ENDDO**

\[
\mathcal{T} := \mathcal{T}_{i_{\text{max}}}
\]

**RETURN** \(\mathcal{T}\)

**END MultilevelDef**
Multilevel Deformation

convergence despite of fixed time step size

\[ i_{\text{min}} = 3, \quad i_{\text{incr}} = 1, \quad N_{\text{Pre}} = 2 \]
Runtime Comparison

almost optimal complexity
Test Problem: L-domain

Poisson equation

\[ \Omega = [-0.5, 0.5]^2/[0, 0.5]^2 \]

\[ u(r, \varphi) = r^{2/3} \sin(2/3\varphi) \]

\[ f(r) = \min \left\{ 1, \max\{c_0 h, \sqrt{2}|r|\} \right\} \]

desired: gradient error
Grid at Reentrant Corner

-0.025 0.025
Discretisation Error

optimal convergence rate by deformed grids
Conclusion

- HPC: locally structured mesh
- Deformation method: derivation and convergence aspects
- Multilevel deformation
- L-domain: r-adaptivity
Thank you for your attention!