A Robust Semi-Implicit Finite Element Scheme for Nonlinear Hyperbolic Systems

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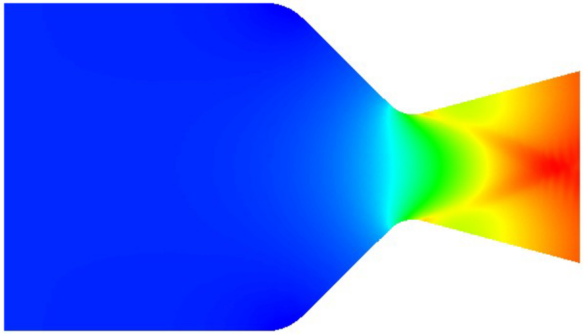
Contents & Aims

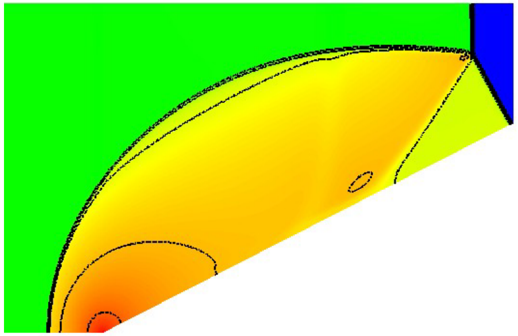
Aims

- Derivation of a robust semi-implicit FE scheme for stationary nonlinear hyperbolic systems
- Parameter-free Newton-like method
- Application to the Euler equations
- Particle-laden gas flows

Euler Equations: JPL Nozzle Flow



Euler Equations: Compression Corner



Contents

- Semi-implicit pseudo time stepping and relation to Newton‘s method
- FEM discretization of the Euler equations
- Weak boundary conditons
- Construction of the preconditioner
- Numerical results

Definition of a Hyperbolic System

Conservative form:

$$\partial_t U + \nabla \cdot F(U) = \partial_t U + \partial_x F^{(x)} + \partial_y F^{(y)} = 0$$

Hyperbolicity:

The PDE-system is called hyperbolic, if the flux jacobians are diagonalizable with real eigenvalues

$$\frac{\partial F^{(x)}(U)}{\partial U} = R^{(x)-1} \Lambda^{(x)} R^{(x)} \qquad \frac{\partial F^{(y)}(U)}{\partial U} = R^{(y)-1} \Lambda^{(y)} R^{(y)}$$

Euler equations:

$$\partial_t \begin{bmatrix} \rho \\ \rho \vec{u} \\ \rho E \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho \vec{u} \\ \rho \vec{u} \otimes \vec{u} + IP \\ \vec{u}(\rho E + P) \end{bmatrix} = 0$$

Implicit Solver & Newton’s Method

Backward Euler scheme:

$$M_L \frac{U^{n+1} - U^n}{\Delta t} = F^{n+1}$$

Taylor Linearization:

$$F^{n+1} = F^n + \left(\frac{\partial F}{\partial U}\right)^n \left(U^{n+1} - U^n\right) + O\left(\left\|U^{n+1} - U^n\right\|^2\right)$$

Semi-implicit scheme:

$$\left[\frac{M_L}{\Delta t} - \left(\frac{\partial F}{\partial U}\right)^n\right]\left(U^{n+1} - U^n\right) = F^n$$

Newton‘s method:

$$-\left(\frac{\partial F}{\partial U}\right)^n \left(U^{n+1} - U^n\right) = F^n$$

- Newton‘s method corresponds to an infinite CFL number
- Second-order convergence if F is differentiable

Discretization of Spatial Derivatives

High-order scheme:

$$M_C \frac{dU}{dt} = KU$$

Low-order scheme:

$$M_L \frac{dU}{dt} = KU + DU = LU$$

High-resolution scheme:

$$M_L \frac{dU}{dt} = KU + DU + F^*U = K^*U$$

Equivalent representation:

$$M_L \frac{dU}{dt} = L^*U$$

High-Order Scheme

Galerkin FEM:

$$\sum_j m_{ij} \frac{dU_j}{dt} = -\sum_j c_{ij} \cdot F_j, \quad \forall i$$

Matrix coefficients:

$$m_{ij} = \int_\Omega \varphi_i \varphi_j dx \qquad c_{ij} = \int_\Omega \varphi_i \nabla \varphi_j dx$$

Lumped-mass Galerkin scheme:

$$m_i = \sum_j m_{ij} \quad \Rightarrow \quad m_i \frac{dU_i}{dt} = -\sum_j c_{ij} \cdot F_j, \quad \forall i$$

$$F(U) = A(U)U \quad \Rightarrow \quad M_L \frac{dU}{dt} = KU$$

Low-Order Scheme: Edge-Based Decomposition

Characteristic LED criterion:

$$m_i \frac{dU_i}{dt} = \sum_{j \neq i} L_{ij} \left(U_j - U_i \right)$$

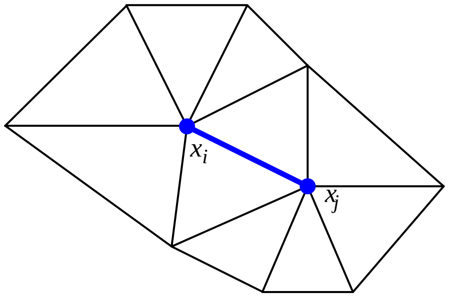
is local extremum diminishing (LED), if L_{ij} are positive semi-definit for $j \neq i$

$$
\begin{aligned}
(KU)_i &= -\sum_j c_{ij} \cdot F_j \\
&= -\sum_{j \neq i} c_{ij} \cdot \left(F_j - F_i \right)
\end{aligned}
$$

$$
\begin{aligned}
(KU)_i &= -\sum_{j \neq i} c_{ij}^{(x)} \left(F_j^{(x)} - F_i^{(x)} \right) + c_{ij}^{(y)} \left(F_j^{(y)} - F_i^{(y)} \right) \\
&= -\sum_{j \neq i} \left(c_{ij}^{(x)} A_{ij}^{(x)} + c_{ij}^{(y)} A_{ij}^{(y)} \right) \left(U_j - U_i \right)
\end{aligned}
$$

Roe linearization

$$F_j^{(x,y)} - F_i^{(x,y)} = A_{ij}^{(x,y)} \left(U_j - U_i \right)$$



Design of the Artificial Diffusion Operator

Requirements:

- Block symmetry, zero row and column sums
- Enough dissipation to enforce the LED criterion

Local diffusion operator:

$$D_{loc} = \begin{pmatrix} -D_{ij} & D_{ij} \\ D_{ij} & -D_{ij} \end{pmatrix}, \qquad D_{ij} = R_{ij} \left| \Lambda_{ij} \right| R_{ij}^{-1}$$

Diffusive fluxes:

$$f_{ij}^{diff} = D_{ij} \left(U_j - U_i \right), \qquad f_{ji}^{diff} = -f_{ij}^{diff}$$

Low-order operator:

$$L_{ij} = K_{ij} + D_{ij} = R_{ij} \Lambda_{ij} R_{ij}^{-1} + R_{ij} \left| \Lambda_{ij} \right| R_{ij}^{-1}$$

Flux Limiting & Characteristic Variables

Characteristic variables:

$$\frac{\partial U}{\partial t} + A^d \frac{\partial U}{\partial x^d} = 0 \quad \xrightarrow[A^d = R \Lambda R^{-1}]{W = R^{-1} U} \quad \frac{\partial W_k}{\partial t} + \lambda_k^d \frac{\partial W_k}{\partial x^d} = 0$$

Compute characteristic variables for each space dimension separately

- Decoupling of the linearized equations along the edge ij
- Characteristic flux limiter of TVD type (Kuzmin 2007)

$$\Delta W_{ij} = R_{ij}^{-1} \left(U_j - U_i \right) \quad \Longrightarrow \quad F_{ij}^* = R_{ij} \Delta \hat{W}_{ij} \qquad F_{ji}^* = -F_{ij}^*$$

High-resolution scheme:

$$\left(K^* U \right)_i = \sum_{j \neq i} \left(K_{ij} + D_{ij} \right) \left(U_j - U_i \right) - F_{ij}^*$$

The Interior Flux Jacobian

Finite differences:

$$\frac{\partial F}{\partial U} \approx \frac{F(U+\vec{\varepsilon})-F(U)}{|\vec{\varepsilon}|} \quad \textbf{or} \quad \frac{\partial F}{\partial U} \approx \frac{F(U+\vec{\varepsilon})-F(U-\vec{\varepsilon})}{2|\vec{\varepsilon}|}$$

- A convenient way to ´differentiate´ F
- Convergence behavior depends on the choice of $\vec{\varepsilon}$

Analytic derivation:

- F must be differentiable
- Complicated algebra and programming
- Deterioration of matrix properties

$$F_{ij} \overset{\text{low}}{=} c_{ij} \cdot F_j + |c_{ij}||\mathbf{A}_{ij}|(U_j - U_i)$$

Edge-based approximate Jacobian:

- No free parameter
- No additional fill-in
- Increased robustness due to improved matrix properties
- Extension to a limited version is possible

$$\frac{\partial F_{ij}}{\partial U_j}^{(x,y)} \approx c_{ij}^{(x,y)} \frac{\partial F_{ij}}{\partial U_j}^{(x,y)} + |c_{ij}||\mathbf{A}_{ij}^{(x,y)}|$$

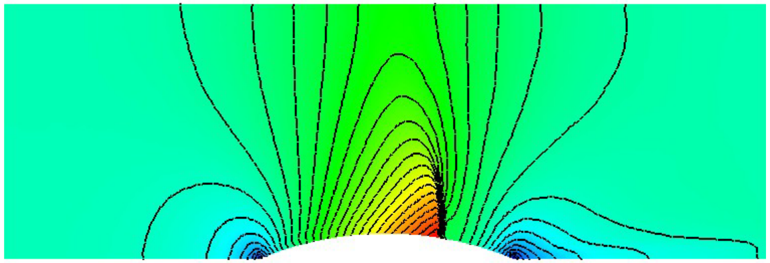
Weak Boundary Conditions

Integration by parts:

$$\sum_j \int_\Omega \varphi_i \varphi_j dx \frac{dU_j}{dt} = \sum_j \int_\Omega \varphi_j \nabla \varphi_i dx \cdot F_j - \sum_j \int_{\partial\Omega} \varphi_i \varphi_j n ds \cdot F_j, \quad \forall i$$

- Improved convergence rates and robustness in steady computations
- Exclusively the boundary integrals are affected by the boundary conditions
- Fully implicit boundary treatment

GAMM Channel



$$\sum_j \int_{\partial\Omega} \varphi_i \varphi_j n ds \cdot F_j$$

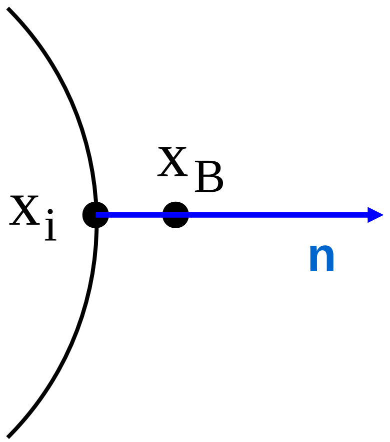
→ Evaluate the boundary flux using the solution of a Riemann problem →

$$\sum_j \int_{\partial\Omega} \varphi_i \varphi_j n ds \cdot \widetilde{F}_j$$

The Boundary Flux

Evaluation of the boundary integral:

- Ghost nodes
- Edgewise evaluation



Roe flux:

$$F_{iB} = \frac{F_i + F_B}{2} - \frac{1}{2}|A_{iB}|(U_B - U_i)$$

- A popular approximate Riemann solver
- U_B is defined in terms of the Riemann invariants
- Boundary values are prescribed for the incoming Riemann invariants

Eigenvalues:

$$v_n - c, \quad v_n, \quad v_n, \quad v_n + c$$

The Boundary Flux Jacobian

- The ghost state depends on the imposed boundary condition and on the interior state (subsonic or wall boundary)

Ghost state:

$$U_B = U_B(U_i, BC)$$

Roe flux:

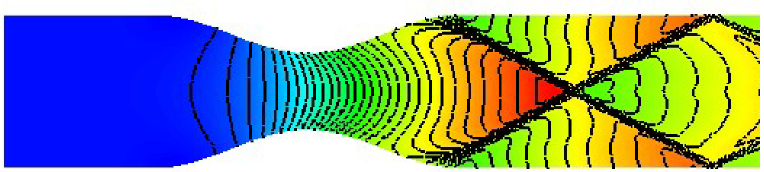
$$F_{iB} = \frac{F_i + F_B}{2} - \frac{1}{2}|A_{iB}|(U_B - U_i)$$

Approximation:

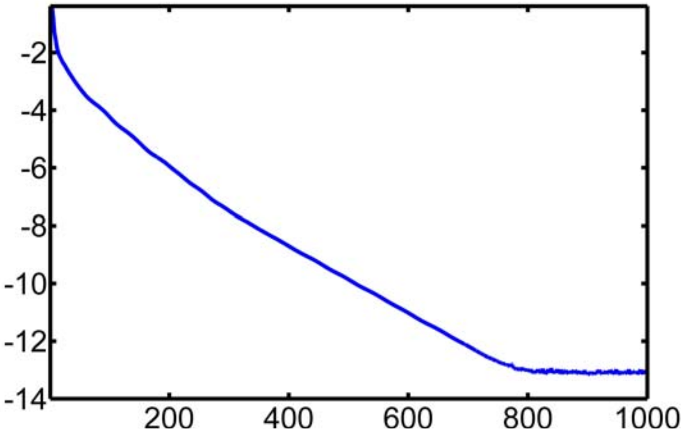
$$\frac{\partial F_{iB}}{\partial U_i} \approx \frac{1}{2}\left[\frac{\partial F(U_i)}{\partial U_i} + |A_{iB}| + \left[\frac{\partial F(U_B)}{\partial U_B} + |A_{iB}|\right]\frac{\partial U_B}{\partial U_i}\right]$$

Transsonic Nozzle & GAMM Channel

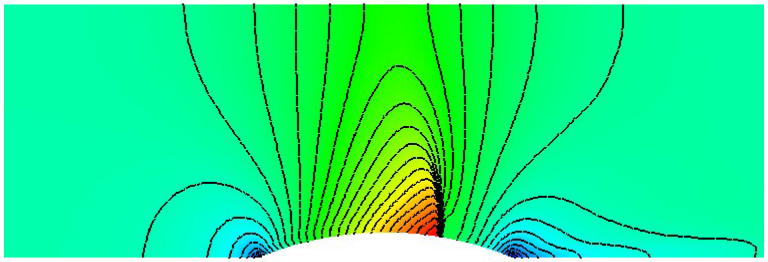
Transsonic Nozzle M=0.8



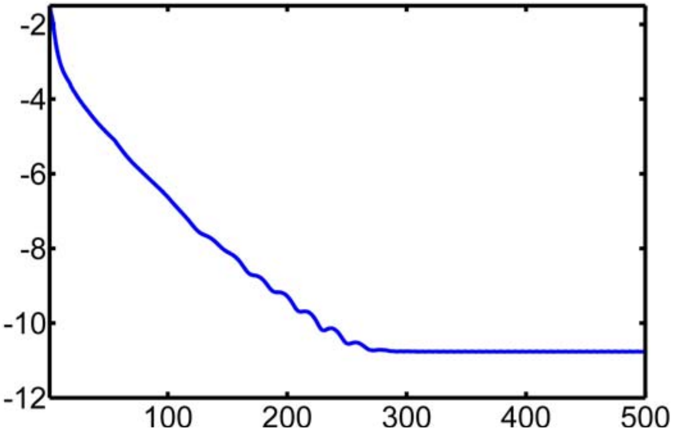
Nonlinear Convergence



GAMM Channel M=0.67

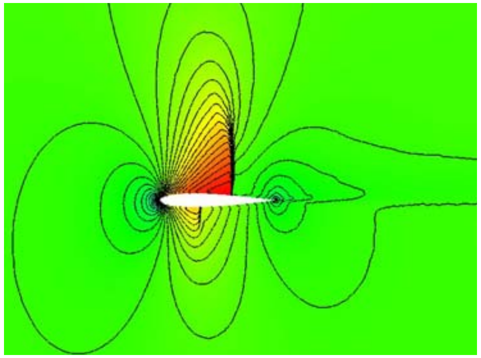


Nonlinear Convergence

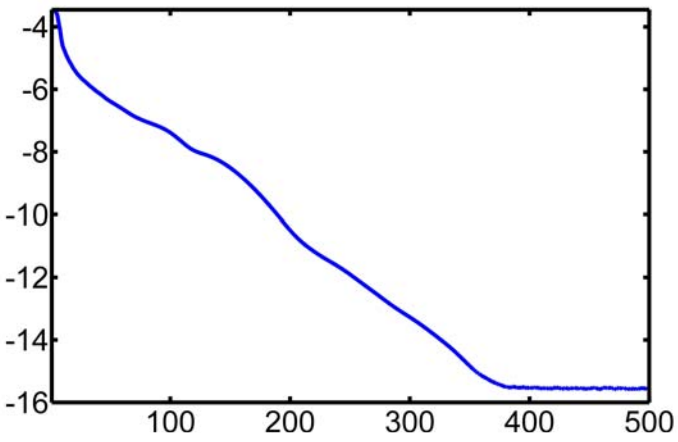


Airfoil & Scramjet

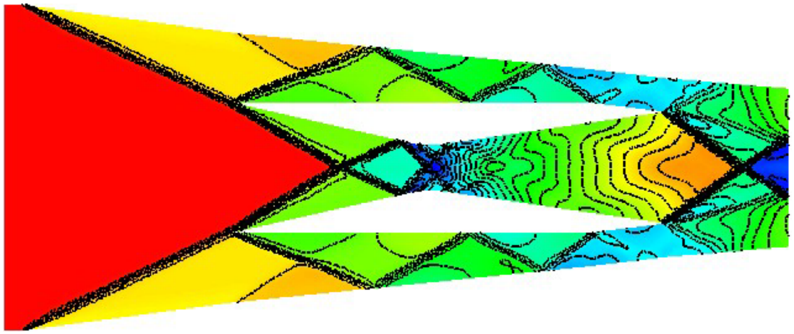
NACA Airfoil M=0.8



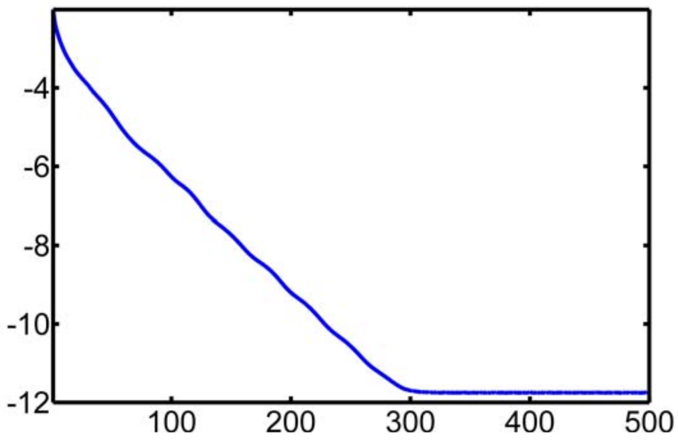
Nonlinear Convergence



Scramjet M=3



Nonlinear Convergence



Conclusions & Outlook

- High resolution, finite element TVD-scheme for the Euler equations
- Semi-implicit time-stepping / Newton – like solver
- No free parameters, information is inferred from the matrix entries
- Generalization to a two – fluid model is feasible and available
- Further work will focus on the development of a genuine Newton‘s method
- Adaptivity is required to reduce the computational cost (Matthias Möller)