Isogeometric Analysis of Fluid–Structure Interaction based on a fully coupled Arbitrary Lagrangian–Eulerian variational formulation

Babak S. Hosseini\textsuperscript{1}, Stefan Turek\textsuperscript{1}, Matthias Möller\textsuperscript{2}

\textsuperscript{1}TU Dortmund University, Institute of Applied Mathematics, LS III
babak.hosseini@math.tu-dortmund.de

\textsuperscript{2}Delft University of Technology, Delft Institute of Applied Mathematics

Seventh International Conference on Isogeometric Analysis (IGA 2019), 18-20 September, 2019, Munich, Germany
Motivation

**Objective:** Develop an Isogeometric Analysis based fully coupled monolithic FSI simulation framework that in the sense of robustness/stability is particularly well suited for biomechanical applications.

We would like to use a method that has the following advantages:

- Stable in biomechanical applications ($\rho^F/\rho^S \approx 1$)
- Fully coupled ALE variational formulation solves the difficulty of common variational description; Facilitates consistent Galerkin discretization of the FSI problem
- High continuity and regularity spaces; Complex geometries $\rightarrow$ Isogeometric Analysis

*Figure*: ©[12]
Compressible solid problem

Elastodynamics (Lagrangian perspective):

\[ J \rho \frac{\partial v}{\partial t} - \nabla_X \cdot P = J \rho b \quad \text{in} \quad \Omega_X \times I, \]

\[ \frac{\partial u}{\partial t} - v = 0 \quad \text{in} \quad \Omega_X \times I, \]

\[ u(\cdot, 0) = \hat{u}, \quad v(\cdot, 0) = \hat{v} \quad \text{in} \quad \Omega_X, \]

\[ u = u_D \quad \text{on} \quad \Gamma_D, \Omega_X \times I, \]

\[ P n_0 = g_0 \quad \text{on} \quad \Gamma_N, \Omega_X \times I. \]

- St. Venant–Kirchhoff material

\[ P := \lambda \text{tr}(E) F + 2\mu F E \]

- Neo–Hookean material

\[ P := \mu(F - F^{-T}) + \lambda \log(\det F) F^{-T} \]

- Green–St. Venant strain tensor

\[ E := \frac{1}{2} \left( \nabla_X u + (\nabla_X u)^T + (\nabla_X u)^T \nabla_X u \right) \]

\[ \varphi : \Omega_X \times [0, T] \rightarrow \Omega_{\omega} \times [0, T] \]

\[ (X, t) \mapsto \varphi(X, t) = (x, t) \]

\[ \varphi(X, t) = X + u(X, t) \]

\[ v(X, t) = \frac{\partial u}{\partial t} \bigg|_X \]

\[ F := F(X, u) := \frac{\partial \varphi}{\partial X} = \nabla_X \varphi(X) = I + \nabla_X u \]

\[ J := \det(F) \]
Incompressible fluid problem

**Incompressible Newtonian flow** (Eulerian perspective):

\[
\rho \left( \frac{\partial v}{\partial t} \right)_x + (v \cdot \nabla)v = \nabla \cdot \sigma + \rho b \quad \text{in } \Omega^F_x(t), t \in I,
\]

\[
\nabla \cdot v = 0 \quad \text{in } \Omega^F_x(t), t \in I,
\]

\[
p(\cdot, 0) = \hat{p}, v(\cdot, 0) = \hat{v} \quad \text{in } \Omega^F_x(0),
\]

\[
v = v_D \quad \text{on } \Gamma^{F,D}_x(t), t \in I,
\]

\[
\sigma \cdot n = g \quad \text{on } \Gamma^{F,N}_x(t), t \in I.
\]

\[
\sigma := -pI + \mu (\nabla v + (\nabla v)^T)
\]

**FSI coupling conditions:**

- Geometric coupling: Fluid- and solid-domain never detach or overlap
- Continuity of velocity:
  \[
  v^F = v^S \quad \text{on } \Gamma^{I}_x(t)
  \]
- Continuity of normal stresses:
  \[
  \sigma^F \cdot n^F = -\sigma^S \cdot n^S \quad \text{on } \Gamma^{I}_x(t)
  \]

Strategy for combination into one conservation equation: **Rewrite fluid equations in a “structure-appropriate” framework (ALE)**
Governing equations

\[ \hat{j} \rho F \left( \frac{\partial v F}{\partial t} \bigg|_\chi + \nabla_\chi v F \left( \hat{F}^{-1} (v F - \partial_t \hat{\mathbf{A}}) \right) \right) \]

\[-\nabla_\chi \cdot \left( \hat{j} \left( -p F I + \mu F \left( \nabla_\chi v F \hat{F}^{-1} + \hat{F}^{-T} (\nabla_\chi v F) T \right) \right) \hat{F}^{-T} \right) = \hat{j} \rho F b F \quad \text{in } \Omega_F^\chi \times (0, T),
\]

\[ \nabla_\chi \cdot \left( \hat{j} \hat{F}^{-1} v F \right) = 0 \quad \text{in } \Omega_F^\chi \times [0, T],
\]

\[ p F (\cdot, 0) = \hat{p} F, \quad u F (\cdot, 0) = \hat{u} F, \quad v F (\cdot, 0) = \hat{v} S
\]

\[ u F = u_D, \quad v F = v_D \quad \text{on } \Gamma_{FD}^\chi \times (0, T), \]

\[ \left( \hat{j} \sigma F \hat{F}^{-T} \right) n_0 F = g_0 F \quad \text{on } \Gamma_{FN}^\chi \times (0, T).\]

\[ \hat{j} \rho S \frac{\partial v S}{\partial t} \bigg|_\chi - \nabla_\chi \cdot \hat{P} S = \hat{j} \rho S b S \quad \text{in } \Omega_S^\chi \times (0, T),
\]

\[ \frac{\partial u S}{\partial t} - v S = 0 \quad \text{in } \Omega_S^\chi \times (0, T),
\]

\[ u S (\cdot, 0) = \hat{u} S, \quad v S (\cdot, 0) = \hat{v} S \quad \text{in } \Omega_S^\chi \times (0, T),\]

\[ u S = u_D \quad \text{on } \Gamma_{SD}^\chi \times (0, T), \]

\[ \hat{P} S n_0 S = g_0 S \quad \text{on } \Gamma_{SN}^S \times (0, T).\]

\[ \nabla_\chi \cdot \left( \alpha_u \hat{j}^{-1} \nabla_\chi u F \right) = 0 \quad \text{in } \Omega_F^\chi \times (0, T),
\]

\[ u F = u S, \quad v F = v S, \left( \hat{j} \sigma F \hat{F}^{-T} \right) n_0 = \hat{P} S n_0 \quad \text{on } \Gamma_T^\chi \times (0, T).\]
Variational formulation

- **Displacement trial and test spaces in the fluid domain:**
  \[ \mathcal{T}_u^F := \{ u^F \in \mathcal{H}^1(\Omega^F_\chi) : u^F = u^S \text{ on } \Gamma^T_\chi, u^F = u^D \text{ on } \Gamma^D_{D,\chi} \} \]
  \[ \mathcal{W}_u^F := \{ \phi^{u,F} \in \mathcal{H}_0^1(\Omega^F_\chi) : \phi^{u,F} = \phi^{u,S} \text{ on } \Gamma^T_\chi \} \]

- **Velocity trial and test spaces in the fluid domain:**
  \[ \mathcal{T}_v^F := \{ v^F \in \mathcal{H}^1(\Omega^F_\chi) : v^F = v^S \text{ on } \Gamma^T_\chi, v^F = v^D \text{ on } \Gamma^D_{D,\chi} \} \]
  \[ \mathcal{W}_v^F := \{ \phi^{v,F} \in \mathcal{H}_0^1(\Omega^F_\chi) : \phi^{v,F} = \phi^{v,S} \text{ on } \Gamma^T_\chi \} \]

- **Pressure trial and test space in the fluid domain:**
  \[ \mathcal{L}^F := \mathcal{L}^2(\Omega^F_\chi)/\mathbb{R} \]

- **Displacement trial and test space in the solid domain:**
  \[ \mathcal{T}_u^S := \{ u^S \in \mathcal{H}^1(\Omega^S_\chi) : u^S = u^D \text{ on } \Gamma^S_{D,\chi} \}, \quad \mathcal{W}_u^S := \mathcal{H}_0^1(\Omega^S_\chi) \]

- **Velocity trial and test space in the solid domain:**
  \[ \mathcal{T}_v^S := \mathcal{H}^1(\Omega^S_\chi), \quad \mathcal{W}_v^S := \mathcal{H}_0^1(\Omega^S_\chi) \]

- **Pressure trial and test spaces in the solid domain:**
  \[ \mathcal{L}^S := \mathcal{L}^2(\Omega^S_\chi)/\mathbb{R} \]
Variational formulation

Let \( \mathcal{T} := \{ \mathcal{T}^F_v \times \mathcal{T}^S_v \times \mathcal{T}^F_u \times \mathcal{T}^S_u \times \mathcal{L}^F \times \mathcal{L}^S \} \), let \( U = \{ v^F, v^S, u^F, u^S, p^F, p^S \} \), and let \( \Phi = \{ \phi^v,F, \phi^v,S, \phi^u,F, \phi^u,S, \phi^p,F, \phi^p,S \} \).

Find \( U \in \mathcal{T} \times I \) such that:

\[
\begin{align*}
\mathcal{F}_1(U; \Phi) &= 0 \quad \forall \phi^v,F \in \mathcal{W}^F_v \\
\mathcal{F}_2(U; \Phi) &= 0 \quad \forall \phi^p,F \in \mathcal{L}^F \\
\mathcal{F}_3(U; \Phi) &= 0 \quad \forall \phi^v,S \in \mathcal{W}^S_v \\
\mathcal{F}_4(U; \Phi) &= 0 \quad \forall \phi^u,S \in \mathcal{W}^S_u \\
\mathcal{F}_5(U; \Phi) &= 0 \quad \forall \phi^p,S \in \mathcal{L}^S \\
\mathcal{F}_6(U; \Phi) &= 0 \quad \forall \phi^u,F \in \mathcal{W}^F_u
\end{align*}
\]

\[
\mathcal{F}_1(U; \Phi) := \\
\int_0^T \int_{\Omega^F_{\chi}} \hat{j}_{\rho,F} \left( \frac{\partial v^F}{\partial t} \bigg|_{\chi} + \nabla_{\chi} v^F \left( \hat{F}^{-1} (v^F - \partial_t \hat{A}) \right) \right) \cdot \phi^v \, d\Omega_{\chi} \, dt \\
+ \int_0^T \int_{\Omega^F_{\chi}} \hat{j} \left( -p^F I + \mu^F \left( \nabla_{\chi} v^F \hat{F}^{-1} + \hat{F}^{-T} (\nabla_{\chi} v^F)^T \right) \right) \hat{F}^{-T} : \nabla_{\chi} \phi^v \, d\Omega_{\chi} \, dt \\
- \int_0^T \int_{\Omega^F_{\chi}} \hat{j}_{\rho,F} f^F \cdot \phi^v \, d\Omega_{\chi} \, dt - \int_0^T \int_{\Gamma^F_{N,\chi}} g^F_0 \cdot \phi^v \, d\Gamma^F_{N,\chi} \, dt.
\]
Variational formulation

\[ F_2(U; \Phi) := \int_0^T \int_{\Omega_\chi^F} \nabla \chi \cdot (\hat{J} \hat{F}^{-1} v^F) \cdot \phi^p \, d\Omega_\chi^F \, dt. \]

\[ F_3(U; \Phi) := \int_0^T \int_{\Omega_\chi^S} \hat{J} \rho^S \frac{\partial v^S}{\partial t} |_{\chi} \cdot \phi^{v,S} \, d\Omega_\chi^S \, dt + \int_0^T \int_{\Omega_\chi^S} \hat{P}^S : \nabla \chi \phi^{v,S} \, d\Omega_\chi^S \, dt \]

\[- \int_0^T \int_{\Omega_\chi^S} \hat{J} \rho^S b^S \cdot \phi^{v,S} \, d\Omega_\chi^S \, dt - \int_0^T \int_{\Gamma_{N,\chi}^S} g_0^S \cdot \phi^{v,S} \, d\Gamma_{N,\chi}^S \, dt. \]

\[ F_4(U; \Phi) := \int_0^T \int_{\Omega_\chi^S} \left( \frac{\partial u^S}{\partial t} |_{\chi} - v^S \right) \cdot \phi^{u,S} \, d\Omega_\chi^S \, dt. \]

\[ F_5(U; \Phi) := \int_0^T \int_{\Omega_\chi^S} p^S \cdot \phi^{p,S} \, d\Omega_\chi^S \, dt. \]

\[ F_6(U; \Phi) := \int_0^T \int_{\Omega_\chi^F} \alpha_u \hat{J}^{-1} \nabla \chi u^F : \nabla \chi \phi^u \, d\Omega_\chi^F \, dt. \]
Discrete Isogeometric approximation spaces

Approximation of velocity and pressure functions with LBB-stable Taylor-Hood like non-uniform rational B-spline space pairs $\hat{V}_{TH}^h / \hat{Q}_{TH}^h$

\[
\hat{V}_{TH}^h \equiv \hat{V}_{TH}^h (p, \alpha) = \mathcal{N}_{\alpha, \alpha}^{p+1,p+1} = \mathcal{N}_{\alpha, \alpha}^{p+1,p+1} \times \mathcal{N}_{\alpha, \alpha}^{p+1,p+1}
\]

\[
\hat{Q}_{TH}^h \equiv \hat{Q}_{TH}^h (p, \alpha) = \mathcal{N}_{\alpha, \alpha}^{p,p}
\]

Corresponding spaces $V_{TH}^h$ and $Q_{TH}^h$ in the physical domain $\Omega$ obtained via component-wise mapping using parametrization $F : \hat{\Omega} \rightarrow \Omega$

\[
V_{TH}^h = \{ v_h = \hat{v}_h \circ F^{-1}, \hat{v}_h \in \hat{V}_{TH}^h \}, \quad Q_{TH}^h = \{ q_h = \hat{q}_h \circ F^{-1}, \hat{q}_h \in \hat{Q}_{TH}^h \}
\]
Solution algorithm and discrete problem

- Discrete spaces:
  \[ \mathcal{T}^h := \{ (\mathcal{T}_v^F \cap V_{hTH}) \times (\mathcal{T}_v^S \cap V_{hTH}) \times (\mathcal{T}_u^F \cap V_{hTH}) \times (\mathcal{T}_u^S \cap V_{hTH}) \times (\mathcal{L}_F \cap Q_{hTH}) \times (\mathcal{L}_S \cap Q_{hTH}) \} \]

  \[ \mathcal{W}^h := \{ (\mathcal{W}_v^F \cap V_{hTH}) \times (\mathcal{W}_v^S \cap V_{hTH}) \times (\mathcal{W}_u^F \cap V_{hTH}) \times (\mathcal{W}_u^S \cap V_{hTH}) \times (\mathcal{L}_F \cap Q_{hTH}) \times (\mathcal{L}_S \cap Q_{hTH}) \} \]

- Time discretization: **Shifted Crank-Nicolson** \( (\theta = \frac{1}{2} + O(\Delta t)) \)

**while** \( t \leq T \) **do**

**Solve the nonlinear monolithic FSI problem:**

Find \( U^h \in \mathcal{T}^h \), s.t. \( \forall \Phi^h \in \mathcal{W}^h \) it holds

\[ \mathcal{F}(U^h; \Phi^h) = \sum_i \mathcal{F}_i(U^h; \Phi^h) = 0 \]  

Semilinear form

In each Newton iteration,

Find \( \delta U^h = \{ \delta v^h,F, \delta v^h,S, \delta u^h,F, \delta u^h,S, \delta p^h,F, \delta p^h,S \} \in \mathcal{T}^h \), s.t.

\[ \mathcal{F}'(U^{h,k}; \delta U^h, \Phi^h) = -\mathcal{F}(U^{h,k}; \Phi^h), \quad \forall \Phi^h \in \mathcal{W}^h \]

\[ U^{h,k+1} = U^{h,k} + \omega \delta U^h, \]
2.4 Domain definition

The problem domain, which is based on the 2D version of the well-known CFD benchmark in [7], is illustrated in Figures 1.

The geometry parameters are given as follows (all values in meters):

- The domain has length \( L = 2.5 \) and height \( H = 0.41 \).
- The circle center is positioned at \( C = (0.2, 0.2) \) (measured from the left bottom corner of the channel) and the radius is \( r = 0.05 \).
- The elastic structure bar has length \( l = 0.35 \) and height \( h = 0.02 \), the right bottom corner is positioned at \( (0.6, 0.19) \), and the left end is fully attached to the fixed cylinder.
- The control point is \( A(t) \), attached to the structure and moving in time with \( A(0) = (0.6, 0.2) \).

The setting is intentionally non-symmetric (see [7]) to prevent the dependence of the onset of any possible oscillation on the precision of the computation.

2.5 Boundary conditions

The following boundary conditions are prescribed:

- A parabolic velocity profile is prescribed at the left channel inflow \( \bar{v}_f(0, y) = 1.5 \bar{U}(H - y)(H/2)^2 = 1.5 \bar{U}0.1681y(0.41 - y) \), such that the mean inflow velocity is \( \bar{U} \) and the maximum of the inflow velocity profile is 1.5 \( \bar{U} \).
- The outflow condition can be chosen by the user, for example stress free or do nothing conditions. The outflow condition effectively prescribes some reference value for the pressure variable \( p \). While this value could be arbitrarily set in the

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Unit</th>
<th>FSI 1</th>
<th>FSI 2</th>
<th>FSI 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho^S )</td>
<td>Solid density</td>
<td>( \frac{kg}{m^3} )</td>
<td>1000</td>
<td>10000</td>
<td>1000</td>
</tr>
<tr>
<td>( \nu^S )</td>
<td>Solid Poisson’s ratio</td>
<td></td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>( \mu^S )</td>
<td>Solid Lamé constant</td>
<td>( \frac{kg}{m.s^2} )</td>
<td>0.5 \times 10^6</td>
<td>0.5 \times 10^6</td>
<td>2 \times 10^6</td>
</tr>
<tr>
<td>( \rho^F )</td>
<td>Fluid density</td>
<td>( \frac{kg}{m^3} )</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>( \nu^F )</td>
<td>Fluid kinematic viscosity</td>
<td>( \frac{m^2}{s} )</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>( \bar{U} )</td>
<td>Average inflow velocity</td>
<td>( \frac{m}{s} )</td>
<td>0.2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( \beta = \frac{\rho^S}{\rho^F} )</td>
<td>Fluid-solid density ratio</td>
<td></td>
<td>1</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>( Re = \frac{\bar{U}d}{\nu^F} )</td>
<td>Reynold’s number</td>
<td></td>
<td>20</td>
<td>100</td>
<td>200</td>
</tr>
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</table>
Taylor–Hood NURBS spaces $\hat{V}_h^{TH} = N_{0,0}^{3,3}$, $\hat{Q}_h^{TH} = N_{0,0}^{2,2}$ on multi-patch mesh

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>level</th>
<th>ndof</th>
<th>$u_1(A) \times 10^{-3}$ [f]</th>
<th>$u_2(A) \times 10^{-3}$ [f]</th>
<th>$F_D$ [f]</th>
<th>$F_L$ [f]</th>
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<tbody>
<tr>
<td>$1 \times 10^{-2}$</td>
<td>1</td>
<td>25209</td>
<td>$-15.22 \pm 13.34[3.85]$</td>
<td>$1.23 \pm 82.1[1.92]$</td>
<td>211.43 ± 77.41[3.84]</td>
<td>1.1 ± 237.6[1.92]</td>
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<td>2</td>
<td>111573</td>
<td>$-15.14 \pm 13.28[3.85]$</td>
<td>$1.21 \pm 82.1[1.92]$</td>
<td>214.53 ± 78.80[3.84]</td>
<td>1.3 ± 236.0[1.92]</td>
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<td>3</td>
<td>468621</td>
<td>$-15.22 \pm 13.33[3.85]$</td>
<td>$1.27 \pm 82.4[1.92]$</td>
<td>217.48 ± 80.30[3.84]</td>
<td>1.2 ± 236.9[1.93]</td>
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<tr>
<td>$5 \times 10^{-3}$</td>
<td>1</td>
<td>25209</td>
<td>$-15.23 \pm 13.13[3.85]$</td>
<td>$1.23 \pm 82.4[1.92]$</td>
<td>210.70 ± 77.66[3.84]</td>
<td>0.9 ± 243.0[1.93]</td>
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<td>2</td>
<td>111573</td>
<td>$-15.21 \pm 13.10[3.86]$</td>
<td>$1.20 \pm 82.5[1.92]$</td>
<td>213.91 ± 79.13[3.85]</td>
<td>1.2 ± 241.9[1.93]</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>468621</td>
<td>$-15.29 \pm 13.15[3.86]$</td>
<td>$1.26 \pm 82.8[1.92]$</td>
<td>216.80 ± 80.63[3.85]</td>
<td>0.9 ± 242.8[1.93]</td>
</tr>
<tr>
<td>Turek/Hron[1]</td>
<td>$5 \times 10^{-4}$</td>
<td>4 + 0</td>
<td>304128</td>
<td>$-14.85 \pm 12.70[3.86]$</td>
<td>$1.30 \pm 81.6[1.93]$</td>
<td>215.06 ± 77.65[3.86]</td>
</tr>
</tbody>
</table>
$$\Delta t \quad \text{ndof} \quad u_1(A) \times 10^{-3} [f] \quad u_2(A) \times 10^{-3} [f] \quad F_D [f] \quad F_L [f]$$

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</tbody>
</table>

| Present  | 1.0e−3 | 25209 | −3.26 ± 3.08 | 1.48 ± 37.21 | 457.6 ± 31.59 | 1.29 ± 169.74 |
|          | 111573 |       | −2.85 ± 2.69 | 1.38 ± 34.78 | 458.1 ± 27.65 | 2.06 ± 158.95 |
|          | 468621 |       | −2.92 ± 2.76 | 1.45 ± 35.25 | 459.5 ± 28.32 | 2.15 ± 159.57 |

| Present  | 5.0e−4 | 111573 | −2.89 ± 2.72 | 1.49 ± 34.99 | 458.6 ± 27.19 | 2.43 ± 159.59 |

| 1) Schäfer | 1.0e−3 | 941158 | −2.91 ± 2.77 | 1.47 ± 35.26 | 459.9 ± 27.92 | 1.84 ± 157.70 |
| 2b) Rannacher | 5.0e−4 | 72696 | −2.84 ± 2.67 | 1.28 ± 34.61 | 452.4 ± 26.19 | 2.36 ± 152.70 |
| 3) Turek/Hron[4] | 2.5e−4 | 304128 | −2.88 ± 2.72 | 1.47 ± 34.99 | 460.5 ± 27.74 | 2.50 ± 153.91 |
| 4) Münsch/Breuer[7] | 2.0e−5 | 324480 | −4.54 ± 4.34 | 1.50 ± 42.50 | 467.5 ± 39.50 | 16.2 ± 188.70 |
| 5) Krafczyk/Rank | 5.1e−5 | 2480814 | −2.88 ± 2.71 | 1.48 ± 35.10 | 463.0 ± 31.30 | 1.81 ± 154.00 |
| 6) Wall | 5.0e−4 | 27147 | −2.00 ± 1.89 | 1.45 ± 29.00 | 434.0 ± 17.50 | 2.53 ± 88.60 |
| 7) Bletzinger | 5.0e−4 | 271740 | −3.04 ± 2.87 | 1.55 ± 36.63 | 474.9 ± 28.12 | 3.86 ± 165.90 |
| Gallinger[8] |       |       |       |       | 474.9 ± 28.10 | 3.90 ± 165.90 |
| Sandboge[9] |       |       |       |       | 458.5 ± 24.00 | 2.50 ± 147.50 |
| Breuer[10] |       |       |       |       | 464.5 ± 40.50 | 6.00 ± 166.00 |
\( u_1(A) \) vs. L1, L2, L3, Schaefer, Rannacher, Turek/Hron, Muensch/Breuer, Krafczyk/Rank, Wall, Blezinger, Sandboge

\( u_2(A) \) vs. L1, L2, L3, Schaefer, Rannacher, Turek/Hron, Muensch/Breuer, Krafczyk/Rank, Wall, Blezinger, Sandboge
FSI 3

Graphs showing time evolution of variables $u_1$, $u_2$, $F_D$, and $F_L$ for different labels L1, L2, and L3. The graphs display oscillatory behavior with time $t$ ranging from 9 to 9.5.
Conclusions and outlook

Isogeometric Analysis ⊕ fully coupled monolithic ALE-FSI model

- robust numerical method
- successful (benchmarks)

Extension to

- 3D
- Complex geometries from biomechanical contexts
- Local refinement (Hierarchical B-splines, T-splines, etc.)

Work already done:

- ALE “Binary-fluid”–Structure Interaction based on the Cahn–Hilliard phase field model

Figure: ©[5]
References

FeatFlow FSI Benchmarking

J. Hron and S. Turek.
A Monolithic FEM/Multigrid Solver for an ALE Formulation of Fluid–Structure Interaction with Applications in Biomechanics.

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Turek, S. and Hron, J. and Razzaq, M. and Wobker, H. and Schäfer, M.
Numerical Benchmarking of Fluid-Structure Interaction: A comparison of different discretization and solution approaches

Zhang Y. and Bazilevs Y. and Goswami S. and Bajaj C.L., Hughes T.J.R
Patient-Specific Vascular NURBS Modeling for Isogeometric Analysis of Blood Flow

Scovazzi, G. and Hughes, T.J.R.
Lecture Notes on Continuum Mechanics on Arbitrary Moving Domains

Münsch, M. and Breuer, M.

Gallinger, T.G.
Effiziente Algorithmen zur partitionierten Lösung stark gekoppelter Probleme der Fluid-Struktur-Wechselwirkung

Sandboge, R.
Fluid-structure interaction with openfsitm and md nastrantm structural solver

M. Breuer and G. De Nayer and M. Münsch and T. Gallinger and R. Wüchner
Fluid–structure interaction using a partitioned semi-implicit predictor–corrector coupling scheme for the application of large-eddy simulation

Aortic valve
https://en.wikipedia.org/wiki/Aortic_valve

Simulation of flow in the heart valve
### Appendix, FSI 1

<table>
<thead>
<tr>
<th>ndof</th>
<th>$u_1(A) \times 10^{-5}$ (%-Err)</th>
<th>$u_2(A) \times 10^{-4}$ (%-Err)</th>
<th>$F_D$ (%-Err)</th>
<th>$F_L$ (%-Err)</th>
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<tbody>
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<td>Present</td>
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<td>5067</td>
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1) Schäfer  
2b) Rannacher  
3) Turek/Hron[4]  
5) Krafczyk/Rank  
6) Wall  
7) Bletzinger
Appendix, Mesh motion models

- **Harmonic mesh motion model**

  \[-\nabla \chi \cdot (\sigma_{\text{mesh}}) = 0 \quad \text{in } \Omega^F,\]

  \[u^F = u^S \quad \text{on } \Gamma^I,\]

  \[u^F = 0 \quad \text{on } \partial\Omega^F \setminus \Gamma^I,\]

  \[\sigma_{\text{mesh}} = D\nabla \chi u.\]

- **Linear elastic mesh motion model**

  \[-\nabla \chi \cdot (\sigma_{\text{mesh}}) = 0 \quad \text{in } \Omega^F,\]

  \[u^F = u^S \quad \text{on } \Gamma^I,\]

  \[u^F = 0 \quad \text{on } \partial\Omega^F \setminus \Gamma^I,\]

    \[\sigma_{\text{mesh}} = 2\alpha \mu \varepsilon + \alpha \lambda \text{tr } (\varepsilon) I.\]

- **Biharmonic mesh motion model**

  \[\nabla^4 \chi u = \nabla^2 \chi \nabla^2 \chi u = \Delta^2 \chi u = 0 \quad \text{in } \Omega^F,\]

  \[u^F = u^S, \partial_n u^F = \partial_n u^S \quad \text{on } \Gamma^I,\]

  \[u^F = 0, \partial_n u^F = 0 \quad \text{on } \partial\Omega^F \setminus \Gamma^I.\]
Appendix, FSI 3
Appendix, Newton linearization (PDE level)

Linear form (for fixed $U^k$):

$$
\mathcal{F}(U^k; \Phi) = \sum_i \mathcal{F}_i(U^k; \Phi)
$$

$$
\mathcal{F}_1(U^k; \Phi) := \\
\left( \rho^\mathcal{F} \hat{j}^{n,\theta} \left( v^k,\mathcal{F} - v^0,\mathcal{F} \right), \phi v,\mathcal{F} \right)_{\Omega_X^\mathcal{F}} + \left( \Delta t \rho^\mathcal{F} \hat{j} \nabla_X v^k,\mathcal{F} \hat{\mathcal{F}}^{-1} v^k,\mathcal{F}, \phi v,\mathcal{F} \right)_{\Omega_X^\mathcal{F}} \\
+ \left( \Delta t(1 - \theta) \rho^\mathcal{F} \hat{j}^0 \nabla_X v^0,\mathcal{F} \left( \hat{\mathcal{F}}^0 \right)^{-1} v^0,\mathcal{F}, \phi v,\mathcal{F} \right)_{\Omega_X^\mathcal{F}} \\
- \left( \rho^\mathcal{F} \hat{j} \nabla_X v^k,\mathcal{F} \hat{\mathcal{F}}^{-1} \cdot (u^k,\mathcal{F} - u^0,\mathcal{F}), \phi v,\mathcal{F} \right)_{\Omega_X^\mathcal{F}} \\
+ \left( \Delta t \hat{j} \mu^\mathcal{F} \left( \nabla_X v^k,\mathcal{F} \hat{\mathcal{F}}^{-1} + \hat{\mathcal{F}}^{-T} \cdot \left( \nabla_X v^k,\mathcal{F} \right)^T \right) \hat{\mathcal{F}}^{-T}, \nabla_X \phi v,\mathcal{F} \right)_{\Omega_X^\mathcal{F}} \\
+ \left( \Delta t(1 - \theta) \hat{j}^0 \mu^\mathcal{F} \left( \nabla_X v^0,\mathcal{F} \left( \hat{\mathcal{F}}^0 \right)^{-1} + (\hat{\mathcal{F}}^0)^{-T} \cdot \left( \nabla_X v^0,\mathcal{F} \right)^T \right) \left( \hat{\mathcal{F}}^0 \right)^{-T}, \nabla_X \phi v,\mathcal{F} \right)_{\Omega_X^\mathcal{F}} \\
+ \left( \Delta t \hat{j} \left( -p^k,\mathcal{F} I \right) \hat{\mathcal{F}}^{-T}, \nabla_X \phi v,\mathcal{F} \right)_{\Omega_X^\mathcal{F}} - \left( \rho^\mathcal{F} \Delta t \hat{j}^{n,\theta} b^\mathcal{F}, \phi v,\mathcal{F} \right)_{\Omega_X^\mathcal{F}} - \left( \Delta t g_0^\mathcal{F}, \phi v,\mathcal{F} \right)_{\Gamma_{N,\mathcal{F}}^X}
$$
Appendix, Newton linearization (PDE level)

\[ F_2(U^k; \Phi) := \left( \hat{J}\hat{F}^{-1} : (\nabla \chi v^k, \hat{F})^T, \phi_p, \mathcal{F} \right)_{\Omega_{\chi}^F} \]

\[ F_3(U^k; \Phi) := \left( \hat{J}_{\rho} \left( v^{k,S} - v^{0,S}, \phi_v, S \right) \right)_{\Omega_{\chi}^S} + \left( \Delta t \hat{P} \left( u^{0,S}, \nabla \chi \phi_v, S \right) \right)_{\Omega_{\chi}^S} - \left( \Delta t^n \hat{J}_{\rho} \left( b^S, \phi_v, S \right) \right)_{\Gamma_{N,\chi}^S} - \left( \Delta t g_{0}^S, \phi_v, S \right)_{\Gamma_{N,\chi}^S} \]

\[ F_4(U^k; \Phi) := \left( u^{k,S} - u^{0,S}, \phi_u, S \right)_{\Omega_{\chi}^S} - \left( \Delta t v^{k,S}, \phi_u, S \right)_{\Omega_{\chi}^S} - \left( \Delta t (1 - \theta) v^{0,S}, \phi_u, S \right)_{\Omega_{\chi}^S} \]

\[ F_5(U^k; \Phi) := \left( p^{k,S}, \phi_p, S \right)_{\Omega_{\chi}^S} \]

\[ F_6(U^k; \Phi) := \left( \alpha_u \hat{J}^{-1} \nabla \chi u^{k}, \hat{F}, \nabla \chi \phi_u \right)_{\Omega_{\chi}^F} \]

\[ \hat{j}^{n,\theta} := \theta \hat{j}^n + (1 - \theta) \hat{j}^{n-1} = \theta \hat{J}(u^k) + (1 - \theta) \hat{J}(u^0) \]
Appendix, Newton linearization (PDE level)

Bilinear form $J = \mathcal{F}'(U^k; \cdot, \cdot)$ from linearization of $\mathcal{F}$ around $U = U^k$:

$$\mathcal{F}'(U^k; \delta U, \Phi) = \sum_i \mathcal{F}'_i(U^k; \delta U, \Phi)$$

$$\mathcal{F}'_1(U^k; \delta U, \Phi) :=$$

$$\int_{\Omega^\mathcal{F}} \left( \rho^\mathcal{F} \theta J \text{tr} \left( \hat{\mathcal{F}}^{-1} \nabla_x \delta u^\mathcal{F} \right) \left( v^k,^\mathcal{F} - v^0,^\mathcal{F} \right) + \rho^\mathcal{F} \hat{j} n,^\mathcal{F} \delta v^\mathcal{F} \right) \cdot \phi^{v,^\mathcal{F}} \, d\Omega_x^\mathcal{F}$$

$$+ \int_{\Omega^\mathcal{F}} \left( \Delta t \rho^\mathcal{F} \hat{j} \left( \nabla_x v^k,^\mathcal{F} \hat{\mathcal{F}}^{-1} \hat{\mathcal{F}} - v^k,^\mathcal{F} \hat{\mathcal{F}}^{-1} \delta v^\mathcal{F} \right) + \hat{j} \nabla_x v^k,^\mathcal{F} \left( -\hat{\mathcal{F}}^{-1} \nabla_x \delta u^\mathcal{F} \hat{\mathcal{F}}^{-1} \right) v^k,^\mathcal{F} \right)$$

$$+ \Delta t \rho^\mathcal{F} \hat{j} \left( \nabla_x \delta v^\mathcal{F} \hat{\mathcal{F}}^{-1} v^k,^\mathcal{F} + \nabla_x v^k,^\mathcal{F} \hat{\mathcal{F}}^{-1} \delta v^\mathcal{F} \right)$$

$$- \rho^\mathcal{F} \hat{j} \left( \nabla_x \delta u^\mathcal{F} \hat{\mathcal{F}}^{-1} \left( u^k,^\mathcal{F} - u^0,^\mathcal{F} \right) \right)$$

$$- \rho^\mathcal{F} \hat{j} \left( \nabla_x v^k,^\mathcal{F} \left( -\hat{\mathcal{F}}^{-1} \nabla_x \delta u^\mathcal{F} \hat{\mathcal{F}}^{-1} \right) \left( u^k,^\mathcal{F} - u^0,^\mathcal{F} \right) + \nabla_x v^k,^\mathcal{F} \hat{\mathcal{F}}^{-1} \delta u^\mathcal{F} \right)$$

$$- \rho^\mathcal{F} \hat{j} \nabla_x \delta v^\mathcal{F} \hat{\mathcal{F}}^{-1} \left( u^k,^\mathcal{F} - u^0,^\mathcal{F} \right) \right) \cdot \phi^{v,^\mathcal{F}} \, d\Omega_x^\mathcal{F}$$
Appendix, Newton linearization (PDE level)

\[ G(\delta u) := \begin{pmatrix} \partial \delta u_2 / \partial y & -\partial \delta u_2 / \partial x \\ -\partial \delta u_1 / \partial y & \partial \delta u_1 / \partial x \end{pmatrix} \]

\[ \mathcal{F}'_2(U^k; \delta U, \Phi) := \int_{\Omega^\mathcal{F}_\chi} \left( \sigma_{uv}^\mathcal{F} G(\delta u) \right. \]
\[ + \mu^\mathcal{F} \left( \nabla \chi v^k,^\mathcal{F} \left( -\hat{F}^{-1} \nabla \chi \delta u^\mathcal{F} \hat{F}^{-1} \right) + \left( -\hat{F}^{-T} \cdot (\nabla \chi \delta u^\mathcal{F})^T \hat{F}^{-T} \right) (\nabla \chi v^k,^\mathcal{F})^T \right) \hat{J} \hat{F}^{-T} \]
\[ + \mu^\mathcal{F} \left( \nabla \chi \delta u^\mathcal{F} \hat{F}^{-1} + \hat{F}^{-T} (\nabla \chi \delta u^\mathcal{F})^T \right) \hat{J} \hat{F}^{-T} \]
\[ - \left( p^\mathcal{F} I \right) G(\delta u) - \left( \delta p^\mathcal{F} I \right) \hat{J} \hat{F}^{-T} - \left( \delta p^\mathcal{F} I \right) \hat{J} \hat{F}^{-T} \right) : \nabla \chi \phi^v,^\mathcal{F} \ d\Omega^\mathcal{F}_\chi \]

\[ \mathcal{F}'_3(U^k; \delta U, \Phi) := \]
\[ \int_{\Omega^\mathcal{F}_\chi} \hat{J} \operatorname{tr} \left( \hat{F}^{-1} \nabla \chi \delta u^\mathcal{F} \right) \operatorname{tr} \left( \nabla \chi v^k,^\mathcal{F} \hat{F}^{-1} \right) \]
\[ + \hat{J} \operatorname{tr} \left( \nabla \chi v^k,^\mathcal{F} \left( -\hat{F}^{-1} \nabla \chi \delta u^\mathcal{F} \hat{F}^{-1} \right) \right) + \hat{J} \operatorname{tr} \left( \nabla \chi \delta u^\mathcal{F} \hat{F}^{-1} \right) \right) : \phi^p,^\mathcal{F} \ d\Omega^\mathcal{F}_\chi \]

\[ \mathcal{F}'_4(U^k; \delta U, \Phi) := \]
\[ \int_{\Omega^\mathcal{F}_\chi} \left( -\alpha_u \hat{J}^{-1} \operatorname{tr} \left( \hat{F}^{-1} \nabla \chi \delta u^\mathcal{F} \right) \nabla \chi u^k,^\mathcal{F} + \alpha_u \hat{J}^{-1} \nabla \chi \delta u^\mathcal{F} \right) \cdot \nabla \chi \phi^u,^\mathcal{F} \ d\Omega^\mathcal{F}_\chi \]
Appendix, Newton linearization (PDE level)

\[ F_5'(U^k; \delta U, \Phi) := \int_{\Omega^S_{\chi}} \rho^S \delta v \cdot \phi^v, S \, d\Omega^S_{\chi} \]

\[ F_6'(U^k; \delta U, \Phi) := \int_{\Omega^S_{\chi}} \Delta t \theta \lambda^S \text{tr} \left( \frac{1}{2} \left( (\nabla_{\chi} \delta u)^T \hat{F} + \hat{F}^T \nabla_{\chi} \delta u \right) \right) \hat{F} : \nabla_{\chi} \phi^v, S \, d\Omega^S_{\chi} \]

\[ F_7'(U^k; \delta U, \Phi) := \int_{\Omega^S_{\chi}} \delta u \cdot \phi^u, S \, d\Omega^S_{\chi} - \int_{\Omega^S_{\chi}} \Delta t \theta \delta v \cdot \phi^u, S \, d\Omega^S_{\chi} \]