A monolithic FEM solver for fluid structure interaction

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Fluid structure interaction

☞ large deformation structure in internal/external flow (bioengineering)

- pulsative flow in large blood vessel with obstacles
- flow through heart flaps
- flow in a heart ventricle
- ...

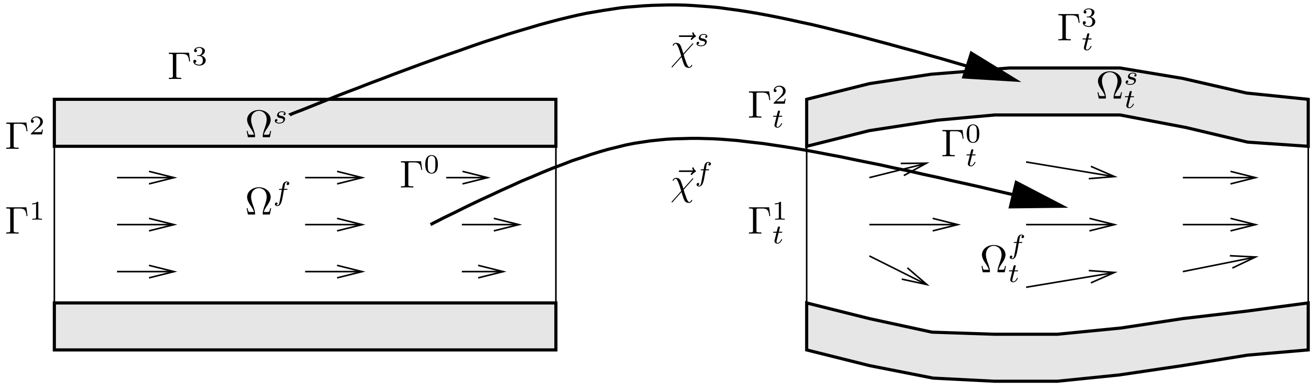
☞ physical model parts to deal with

- viscous fluid flow
- elastic body undergoing large deformations
- the interaction between the two parts

☞ numerical tasks involved

- space and time discretization
- nonlinear system solution
- solution of large linear system

Problem description



reference configuration | current configuration

structure part

$$\boldsymbol{\chi}^s : \Omega^s \times [0,T] \mapsto \Omega_t^s$$

$$\boldsymbol{u}^s = \boldsymbol{\chi}^s(\boldsymbol{X},t) - \boldsymbol{X}, \quad \boldsymbol{v}^s = \frac{\partial \boldsymbol{u}^s}{\partial t}$$

$$\mathbf{F} = \mathbf{I} + \mathrm{Grad}\,\boldsymbol{u}^s, \quad J = \det \mathbf{F}$$

fluid part

$$\boldsymbol{\chi}^f : \Omega^f \times [0,T] \mapsto \Omega_t^f$$

$$\boldsymbol{u}^f = \boldsymbol{\chi}^f(\boldsymbol{X},t) - \boldsymbol{X}$$

$$\boldsymbol{v}^f : \Omega_t^f \times [0,T] \mapsto \mathcal{R}^n$$

Governing equations

structure part

$$\frac{\partial \boldsymbol{v}^s}{\partial t} = \mathrm{div}(J\boldsymbol{\sigma}^s \mathbf{F}^{-T}) + \boldsymbol{f} \quad \text{in } \Omega^s$$

$$\det(\mathbf{I} + \nabla \boldsymbol{u}^s) = 1 \quad \text{in } \Omega^s$$

$$\boldsymbol{u}^s = \mathbf{0} \quad \text{on } \Gamma^2$$

$$\boldsymbol{\sigma}^s \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma^3$$

fluid part

$$\frac{\partial \boldsymbol{v}^f}{\partial t} + (\nabla \boldsymbol{v}^f)\boldsymbol{v}^f = \mathrm{div}\, \boldsymbol{\sigma}^f + \boldsymbol{f} \quad \text{in } \Omega_t^f$$

$$\mathrm{div}\, \boldsymbol{v}^f = 0 \quad \text{in } \Omega_t^f$$

$$\boldsymbol{v}^f = \boldsymbol{v}_0 \quad \text{on } \Gamma_t^1$$

$$\text{or } \boldsymbol{\sigma}^f \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_t^1$$

interface conditions

$$\boldsymbol{v}^f = \boldsymbol{v}^s \quad \text{on } \Gamma_t^0$$

$$\boldsymbol{\sigma}^f \boldsymbol{n} = \boldsymbol{\sigma}^s \boldsymbol{n} \quad \text{on } \Gamma_t^0$$

Arbitrary Lagrangian-Eulerian Formulation

$$\boldsymbol{\chi} : \Omega \times [0,T] \mapsto \Omega_t, \qquad \boldsymbol{v} = \frac{\partial \boldsymbol{\chi}}{\partial t}, \quad \mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{X}}, \quad J = \det \mathbf{F}$$

$$\boldsymbol{\zeta}_{\mathcal{R}} : \mathcal{R} \times [0,T] \mapsto \mathcal{R}_t, \qquad \mathcal{R}_t \subset \Omega_t \quad \forall t \in [0,T], \qquad \boldsymbol{v}_{\mathcal{R}} = \frac{\partial \boldsymbol{\zeta}_{\mathcal{R}}}{\partial t}, \mathbf{F}_{\mathcal{R}} = \frac{\partial \boldsymbol{\zeta}_{\mathcal{R}}}{\partial \boldsymbol{X}}, J_{\mathcal{R}} = \det \mathbf{F}_{\mathcal{R}}$$

$$\frac{\partial}{\partial t} \int_{\mathcal{R}_t} \varrho dv + \int_{\partial \mathcal{R}_t} \varrho (\boldsymbol{v} - \boldsymbol{v}_{\mathcal{R}}) \cdot \boldsymbol{n}_{\mathcal{R}_t} da = 0$$

$$\frac{\partial}{\partial t} (\varrho J_{\mathcal{R}}) + \operatorname{div} \left(\varrho J_{\mathcal{R}} (\boldsymbol{v} - \boldsymbol{v}_{\mathcal{R}}) \mathbf{F}_{\mathcal{R}}^{-T} \right) = 0$$

- Lagrangian description:
  $\boldsymbol{\zeta}_{\mathcal{R}} = \boldsymbol{\chi} \Rightarrow \mathbf{F}_{\mathcal{R}} = \mathbf{F}, J_{\mathcal{R}} = J, \boldsymbol{v}_{\mathcal{R}} = \boldsymbol{v}$

  $$\frac{\partial}{\partial t} (\varrho J) = 0$$

- Eulerian description:
  $\boldsymbol{\zeta}_{\mathcal{R}} = \mathsf{Id} \Rightarrow \mathbf{F}_{\mathcal{R}} = \mathbf{I}, J_{\mathcal{R}} = 1, \boldsymbol{v}_{\mathcal{R}} = \mathbf{0}$

  $$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0$$

Governing equations

structure part

$$\frac{\partial \boldsymbol{v}^s}{\partial t} = \mathrm{div}(J\boldsymbol{\sigma}^s\mathbf{F}^{-T}) + \boldsymbol{f} \qquad \text{in } \Omega^s$$

$$\det(\mathbf{I} + \nabla \boldsymbol{u}^s) = 1 \qquad \text{in } \Omega^s$$

$$\boldsymbol{u}^s = \mathbf{0} \qquad \text{on } \Gamma^2$$

$$\boldsymbol{\sigma}^s\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma^3$$

fluid part

$$\frac{\partial \boldsymbol{v}^f}{\partial t} + (\nabla \boldsymbol{v}^f)\mathbf{F}^{-1}\boldsymbol{v}^f = \mathrm{div}\left(J\boldsymbol{\sigma}^f\mathbf{F}^{-T}\right) + \boldsymbol{f} \qquad \text{in } \Omega^f$$

$$\mathrm{div}\left(J\boldsymbol{v}^f\mathbf{F}^{-T}\right) = 0 \qquad \text{in } \Omega^f$$

$$\boldsymbol{v}^f = \boldsymbol{v}_0 \quad \text{or} \quad \boldsymbol{\sigma}^f\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma^1$$

Coupling strategies

- separated, weak coupling

  t^n fluid $\longrightarrow$ t^n solid $\longrightarrow$ t^{n+1} fluid $\longrightarrow$ t^{n+1} solid $\ldots$

- separated, strong coupling

  t^n fluid $\longrightarrow$ t^n solid $\longrightarrow$ t^{n+1} fluid $\longrightarrow$ t^{n+1} solid $\ldots$

- monolithic

  t^n fluid & solid $\longrightarrow$ t^{n+1} fluid & solid $\ldots$

Uniform formulation

$$\Omega = \Omega^f \cup \Omega^s, \qquad \boldsymbol{u} : \Omega \times [0,T] \to \mathcal{R}^3, \qquad \boldsymbol{v} : \Omega \times [0,T] \to \mathcal{R}^3,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = \begin{cases} \boldsymbol{v} & \text{in } \Omega^s \\ \Delta \boldsymbol{u} \quad (\text{"mesh deformation operator"}) & \text{in } \Omega^f \end{cases}$$

$$\beta \frac{\partial \boldsymbol{v}}{\partial t} = \begin{cases} \operatorname{div}\left(J \boldsymbol{\sigma}^s \mathbf{F}^{-T}\right) & \text{in } \Omega^s \\ -\beta (\nabla \boldsymbol{v}) \mathbf{F}^{-1} (\boldsymbol{v} - \frac{\partial \boldsymbol{u}}{\partial t}) + \operatorname{div}\left(J \boldsymbol{\sigma}^f \mathbf{F}^{-T}\right) & \text{in } \Omega^f \end{cases}$$

$$0 = \begin{cases} J - 1 & \text{in } \Omega^s \\ \operatorname{div}(J \boldsymbol{v} \mathbf{F}^{-T}) & \text{in } \Omega^f \end{cases}$$

$$\begin{aligned} \boldsymbol{\sigma}^f \boldsymbol{n} = \boldsymbol{\sigma}^s \boldsymbol{n} &\text{ on } \Gamma_t^0 \\ \boldsymbol{v} = \boldsymbol{v}_B \quad &\text{ on } \Gamma_t^1 \\ \boldsymbol{u} = \mathbf{0} \quad &\text{ on } \Gamma_t^2 \\ \boldsymbol{\sigma}^s \boldsymbol{n} = \mathbf{0} \quad &\text{ on } \Gamma_t^3 \end{aligned}$$

Constitutive equations

- incompressible Newtonian fluid

$$\boldsymbol{\sigma}^f = -p\mathbf{I} + \nu(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T) \qquad \nu = \nu(\|\mathbf{D}\|), \quad \mathbf{D} = \nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T$$

- hyperelastic material

$$\boldsymbol{\sigma}^s = -p\mathbf{I} + 2\mathbf{F}\frac{\partial \Psi}{\partial \mathbf{F}}\mathbf{F}^T$$

$$\Psi(\mathbf{F}) = \alpha(\mathrm{I}_C - 3)$$

$$\Psi(\mathbf{F}) = \alpha_1(\mathrm{I}_C - 3) + \alpha_2(\mathbf{II}_C - 3) + \alpha_3(|\mathbf{F}\boldsymbol{e}| - 1)^2$$

where $\mathbf{C} = \mathbf{F}\mathbf{F}^T$ and $\mathrm{I}_C = \operatorname{tr}\mathbf{C}$, $\mathbf{II}_C = \frac{1}{2}\left(\operatorname{tr}\mathbf{C}^2 - (\operatorname{tr}\mathbf{C})^2\right)$

neo-Hookean compressible material

$$\boldsymbol{\sigma}^s = -p^s\mathbf{I} + \mu(\mathbf{F}\mathbf{F}^{\mathrm{T}} - \mathbf{I})$$

$$p^s = \lambda(\det \mathbf{F} - \det \mathbf{F}^{-1})$$

neo-Hookean incompressible material

$$\boldsymbol{\sigma}^s = -p^s\mathbf{I} + \mu(\mathbf{F}\mathbf{F}^{\mathrm{T}} - \mathbf{I})$$

$$\det \mathbf{F} = \mathbf{I} \qquad (\lambda \to \infty)$$

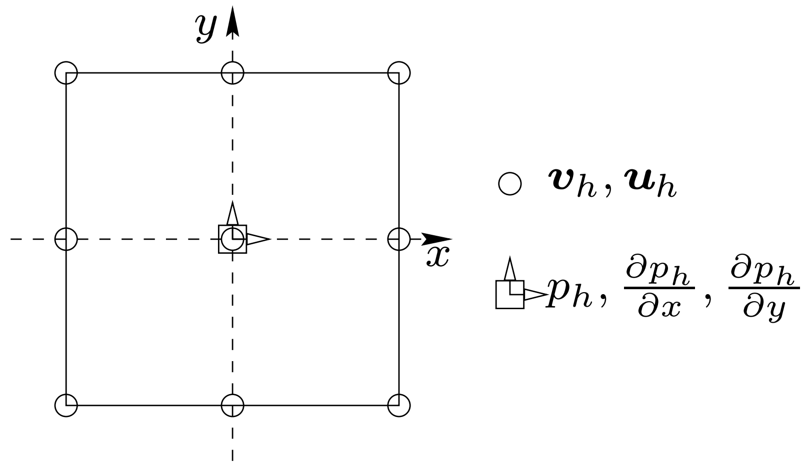
Energy estimate for the system

$$\frac{c}{2}\left\|\boldsymbol{v}(T)\right\|^2_{L^2(\Omega_T)} + \int_0^T \mu \left\|\nabla \boldsymbol{v}\right\|^2_{L^2(\Omega_t^f)} dt + a\left\|\nabla \boldsymbol{u}(T)\right\|^2_{L^2(\Omega^s)}$$
$$\leq \left\|b\right\|_{L^1(\Omega^s)} + \frac{1}{2}\left\|\boldsymbol{v}_0\right\|^2_{L^2(\Omega^f)} + \frac{\beta}{2}\left\|\boldsymbol{v}_0\right\|^2_{L^2(\Omega^s)}$$

$$U = \{\boldsymbol{u} \in L^\infty(I, [W^{1,2}(\Omega)]^3), \boldsymbol{u} = \mathbf{0} \text{ on } \Gamma^2\}$$
$$V = \{\boldsymbol{v} \in L^2(I, [W^{1,2}(\Omega_t)]^3) \cap L^\infty(I, [L^2(\Omega_t)]^3), \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma^1\}$$
$$P = \{p \in L^2(I, L^2(\Omega))\}$$

Discretization in space and time

Discretization in space: FEM $Q_2/Q_2/P_1^{disc}$



$$U_h = \{\boldsymbol{u}_h \in [C(\Omega_h)]^2, \boldsymbol{u}_h|_T \in [Q_2(T)]^2 \forall T \in \mathcal{T}_h, \boldsymbol{u}_h = \mathbf{0} \text{ on } \Gamma^1\},$$

$$V_h = \{\boldsymbol{v}_h \in [C(\Omega_h)]^2, \boldsymbol{v}_h|_T \in [Q_2(T)]^2 \forall T \in \mathcal{T}_h, \boldsymbol{v}_h = \mathbf{0} \text{ on } \Gamma^2\},$$

$$P_h = \{p_h \in L^2(\Omega_h), p_h|_T \in P_1(T) \forall T \in \mathcal{T}_h\}.$$

Discretization in time: Crank-Nicholson scheme with adaptive time-step selection

Discrete nonlinear system

$$\boldsymbol{\mathcal{R}}(\boldsymbol{X}) = \mathbf{0}, \qquad \boldsymbol{X} = (\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) \in U_h \times V_h \times P_h$$

$$
\begin{aligned}
Mu_h - \frac{k}{2}(M^s v_h + L^f u_h) &= \mathsf{rhs}(u_h^n, v_h^n) \\
(M^f + \beta M^s)v_h + \frac{k}{2}N_1(v_h, v_h) + \frac{1}{2}N_2(v_h, u_h) + \frac{k}{2}(S^s(u_h) + S^f(v_h)) - kBp_h &= \mathsf{rhs}(u_h^n, v_h^n, p_h^n) \\
C(u_h) + B^{f\,T} v_h &= 1
\end{aligned}
$$

$$\Downarrow$$

$$\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \boldsymbol{X}}(\boldsymbol{X}) = \begin{pmatrix} M - \frac{k}{2}L^f & \frac{k}{2}M^s & 0 \\ \frac{1}{2}\frac{\partial N_2}{\partial u_h} + \frac{k}{2}\frac{\partial(N_1 + S^s + S^f)}{\partial u_h} + k\frac{\partial B}{\partial u_h}p_h & M^s + \beta M^f + \frac{1}{2}\frac{\partial N_2}{\partial v_h} + \frac{k}{2}\frac{\partial(N_1 + S_f^2)}{\partial v_h} & kB \\ B^{s\,T} + \frac{\partial B^{f\,T}}{\partial u_h}v_h & B^{f\,T} & 0 \end{pmatrix}$$

Discrete nonlinear system

$$\mathcal{R}(\boldsymbol{X}) = \mathbf{0}, \qquad \boldsymbol{X} = (\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) \in U_h \times V_h \times P_h$$

$$M u_h - \frac{k}{2}(M^s v_h + L^f u_h) = \mathsf{rhs}(u_h^n, v_h^n)$$

$$(M^f + \beta M^s) v_h + \frac{k}{2} N_1(v_h, v_h) + \frac{1}{2} N_2(v_h, u_h) + \frac{k}{2}(S^s(u_h) + S^f(v_h)) - kBp_h = \mathsf{rhs}(u_h^n, v_h^n, p_h^n)$$

$$C(u_h) + {B^f}^T v_h = 1$$

$$\Downarrow$$

$$\begin{bmatrix} S_{\boldsymbol{uu}} & S_{\boldsymbol{uv}} & 0 \\ S_{\boldsymbol{vu}} & S_{\boldsymbol{vv}} & kB \\ c_{\boldsymbol{u}} B_s^T & c_{\boldsymbol{v}} B_f^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f_u} \\ \mathbf{f_v} \\ f_p \end{bmatrix}$$

Typical discrete saddle-point problem

Solution of the nonlinear problem

☞ compute the Jacobian matrix (analytic, automatic differentiation or divided differences)

$$\left[\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \boldsymbol{X}}\right]_{ij}(\boldsymbol{X}^n) \approx \frac{[\boldsymbol{\mathcal{R}}]_i(\boldsymbol{X}^n + \varepsilon \boldsymbol{e}_j) - [\boldsymbol{\mathcal{R}}]_i(\boldsymbol{X}^n - \varepsilon \boldsymbol{e}_j)}{2\varepsilon},$$

☞ solve for $\delta\boldsymbol{X}$

$$\left[\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \boldsymbol{X}}(\boldsymbol{X}^n)\right]\delta\boldsymbol{X} = \boldsymbol{\mathcal{R}}(\boldsymbol{X}^n)$$

☞ adaptive line search strategy

$$\boldsymbol{X}^{n+1} = \boldsymbol{X}^n + \omega\delta\boldsymbol{X} \qquad \omega \text{ such that } f(\omega) = \boldsymbol{\mathcal{R}}(\boldsymbol{X} + \omega\delta\boldsymbol{X}) \cdot \boldsymbol{X} \searrow$$

☞ MG, BiCGStab or GMRes(m) with ILU(k) preconditioner to solve the linear problems

Jacobian approximation

$$\left[\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \boldsymbol{X}}\right]_{ij}(\boldsymbol{X}^n) \approx \frac{[\boldsymbol{\mathcal{R}}]_i(\boldsymbol{X}^n + \varepsilon \boldsymbol{e}_j) - [\boldsymbol{\mathcal{R}}]_i(\boldsymbol{X}^n - \varepsilon \boldsymbol{e}_j)}{2\varepsilon},$$

ε/TOL	10^{-8}	10^{-4}	10^{-2}	10^{-1}
10^{-8}	7 /107.57 [21.52]	12 /57.08 [26.52]	12 /47.00 [23.75]	17 /33.06 [27.38]
10^{-4}	7 /108.71 [24.57]	8 /62.75 [17.77]	10 /42.20 [18.95]	18 /31.33 [29.05]
10^{-2}	16 /109.75 [51.65]	20 /47.35 [38.28]	25 /29.80 [38.58]	56 /16.98 [73.83]
10^{-1}	44 /116.11 [141.30]	48 /35.79 [81.72]	49 /17.92 [65.77]	–

nonlinear solver it. / avg. linear solver it. [CPU time] for BiCGStab(ILU(0))

Multigrid solver

- standard geometric multigrid approach
- smoother by local MPSC-Ansatz (Vanka-like smoother)

$$\begin{bmatrix} \boldsymbol{u}^{l+1} \\ \boldsymbol{v}^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{l} \\ \boldsymbol{v}^{l} \\ p^{l} \end{bmatrix} - \omega \sum_{\text{Patch } \Omega_i} \begin{bmatrix} S_{\boldsymbol{uu}|\Omega_i} & S_{\boldsymbol{uv}|\Omega_i} & 0 \\ S_{\boldsymbol{vu}|\Omega_i} & S_{\boldsymbol{vv}|\Omega_i} & kB_{|\Omega_i} \\ c_{\boldsymbol{u}} B^T_{s|\Omega_i} & c_{\boldsymbol{v}} B^T_{f|\Omega_i} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{def}^l_{\boldsymbol{u}} \\ \mathbf{def}^l_{\boldsymbol{v}} \\ def^l_p \end{bmatrix}$$

- full inverse of the local problems by standard LAPACK (39×39 systems)
- alternatives: simplified local problems (3×3 systems) or ILU(k)
- combination with GMRES/BiCGStab methods possible
- full Q_2 and P_1^{disc} prolongation **P**, restriction by $\mathbf{R} = \mathbf{P}^T$

Boundary and initial conditions

inflow parabolic velocity profile is prescribed at the left end of the channel

$$v^f(0,y) = 1.5\frac{y(H-y)}{\left(\frac{H}{2}\right)^2} = 1.5\frac{4.0}{0.1681}y(0.41-y),$$

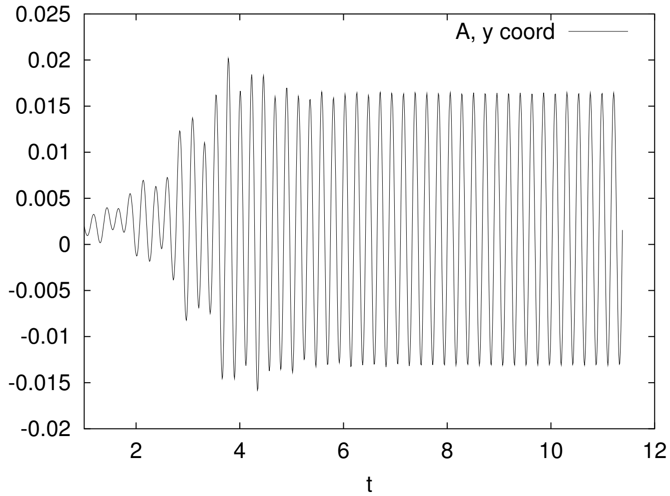
outflow condition can be chosen by the user (*stress free* or *do nothing*)

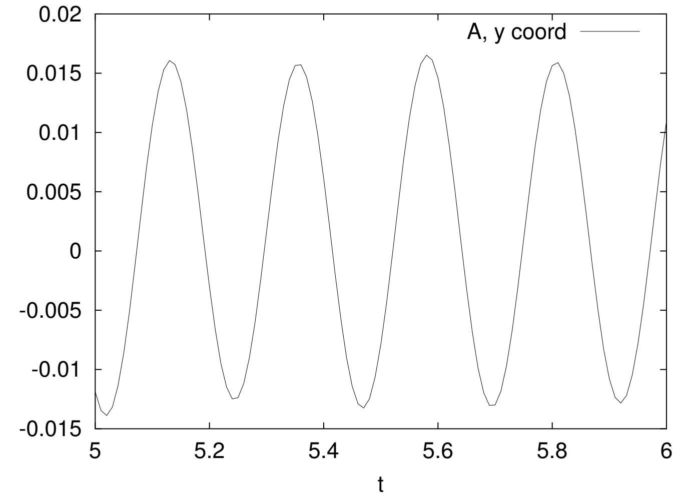
interface condition on Γ_t^0 is $\boldsymbol{v}^f = \boldsymbol{v}^s$ and $\boldsymbol{\sigma}^f \boldsymbol{n} = \boldsymbol{\sigma}^s \boldsymbol{n}$

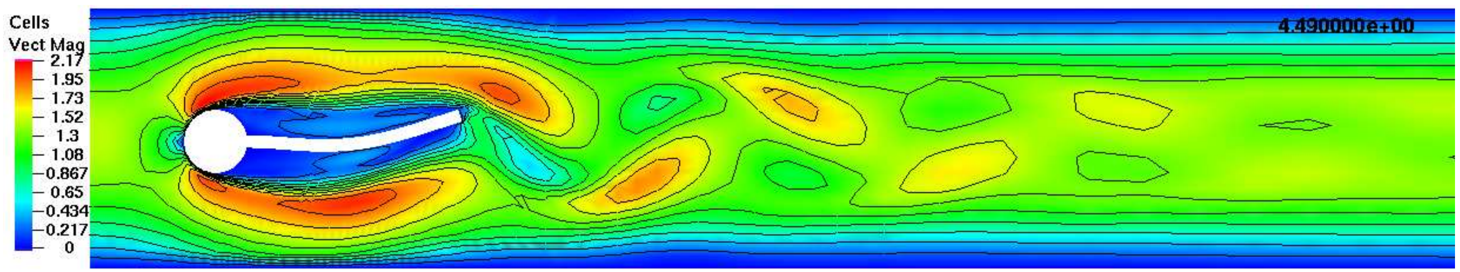
otherwise the *no-slip* condition is prescribed for the fluid on the other boundary parts. i.e. top and bottom wall and cylinder

Examples

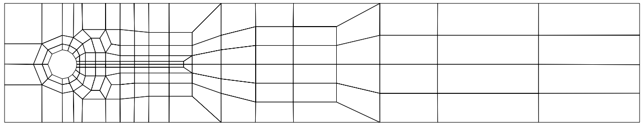
$$\beta = 1 \qquad \alpha = 1 \times 10^3 \qquad \nu_P = 0.5 \qquad \nu = 5 \times 10^{-4} \quad (\text{Re} = 200)$$







Multigrid solver



- 1 timestep started with fully developed solution
- standard streamline diffusion, CN time step, each linear step solved to relative prec. 10^{-4}
- shown: **number of nonlinear steps/avg. number of linear steps** [**CPU time**]
- timestep 10^{-2}

Level	ndof	MG(2)	MG(4)	MG(8)	BiCGStab(ILU(1))	GMRES(ILU(1),200)
1	12760	2/8 [66]	2/8 [92]	2/7 [112]	2/51 [32]	2/50 [27]
2	50144	2/8 [190]	2/5 [198]	2/4 [302]	2/120 [200]	2/117 [151]
3	198784	2/9 [744]	2/6 [852]	2/4 [1185]	2/311 [1646]	2/358 [1432]
4	791552	2/13 [3803]	2/7 [3924]	2/6 [6241]	MEM.	MEM.

- timestep 10^{0}

Level	ndof	MG(2)	MG(4)	MG(8)	BiCGStab(ILU(1))	GMRES(ILU(1),200)
1	12760	4/12 [118]	4/11 [177]	4/10 [262]	20/160 [631]	20/801 [1579]
2	50144	4/12 [466]	4/7 [470]	4/5 [681]	2/800 [] diverg.	13/801 [] diverg.
3	198784	4/13 [1898]	4/7 [2057]	4/5 [2874]	2/800 [] diverg.	4/801 [] diverg.
4	791552	4/15 [8678]	4/8 [9069]	4/6 [13808]	MEM.	MEM.

$\Rightarrow$ robust and efficient Newton-MG scheme

Summary

- monolithic, fully coupled FEM (Q_2/P_1) for **viscous incompressible fluid** and **incompressible hyperelastic structure**
- fully implicit 2nd order discretization in time (Crank-Nicholson)
- Newton-like method for the coupled system (Jacobian matrix via divided differences)
- preconditioned Krylov space linear solver (ILU(k)/GMRES(m))
- adaptive time step control
- a priori space-adapted mesh