A monolithic multigrid FEM solver for fluid structure interaction and numerical benchmarking

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Fluid structure interaction

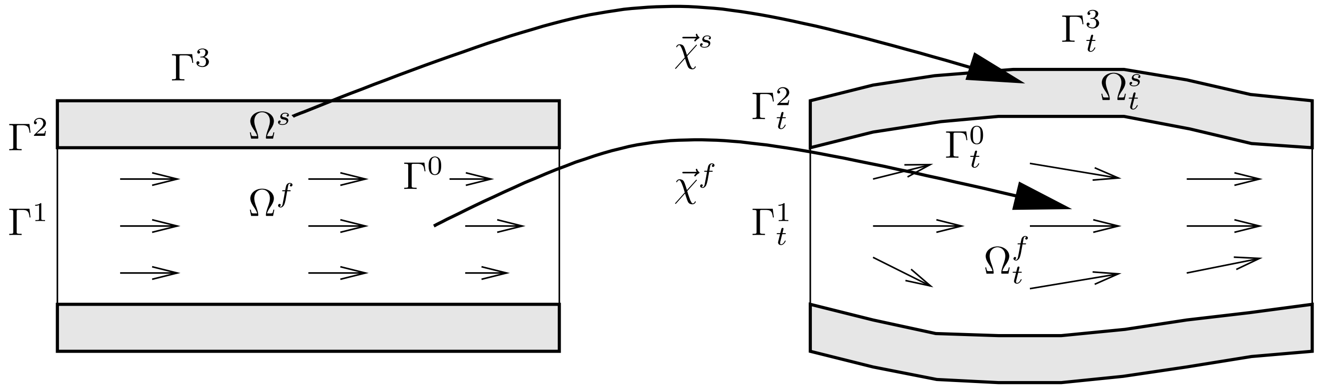
☞ large deformation of a structure in internal/external flow
- aeorelasticity
- bioengineering
- polymer, food, paper processing
- ...

☞ physical models
- viscous fluid flow
- elastic body under large deformations
- interaction between the two parts

☞ numerical tasks involved
- space and time discretization
- nonlinear systems
- large linear systems

☞ testing and validation
- accuracy, efficiency robustnes
- benchmarking

Problem description



reference configuration current configuration

structure part

$$\boldsymbol{\chi}^s : \Omega^s \times [0,T] \mapsto \Omega_t^s$$

$$\boldsymbol{u}^s = \boldsymbol{\chi}^s(\boldsymbol{X},t) - \boldsymbol{X}, \quad \boldsymbol{v}^s = \frac{\partial \boldsymbol{u}^s}{\partial t}$$

$$\mathbf{F} = \mathbf{I} + \mathrm{Grad}\, \boldsymbol{u}^s, \quad J = \det \mathbf{F}$$

fluid part

$$\boldsymbol{\chi}^f : \Omega^f \times [0,T] \mapsto \Omega_t^f$$

$$\boldsymbol{u}^f = \boldsymbol{\chi}^f(\boldsymbol{X},t) - \boldsymbol{X}$$

$$\boldsymbol{v}^f : \Omega_t^f \times [0,T] \mapsto \mathcal{R}^n$$

Governing equations

structure part

$$\frac{\partial \boldsymbol{v}^s}{\partial t} = \operatorname{div}(J\boldsymbol{\sigma}^s \mathbf{F}^{-T}) + \boldsymbol{f} \quad \text{in } \Omega^s$$

$$\det(\mathbf{F}) = 1 \quad \text{in } \Omega^s$$

$$\boldsymbol{u}^s = \mathbf{0} \quad \text{on } \Gamma^2$$

$$\boldsymbol{\sigma}^s \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma^3$$

fluid part

$$\frac{\partial \boldsymbol{v}^f}{\partial t} + (\nabla \boldsymbol{v}^f)\boldsymbol{v}^f = \operatorname{div} \boldsymbol{\sigma}^f + \boldsymbol{f} \quad \text{in } \Omega_t^f$$

$$\operatorname{div} \boldsymbol{v}^f = 0 \quad \text{in } \Omega_t^f$$

$$\boldsymbol{v}^f = \boldsymbol{v}_0 \quad \text{on } \Gamma_t^1$$

$$\text{or } \boldsymbol{\sigma}^f \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_t^1$$

interface conditions

$$\boldsymbol{v}^f = \boldsymbol{v}^s \quad \text{on } \Gamma_t^0$$

$$\boldsymbol{\sigma}^f \boldsymbol{n} = \boldsymbol{\sigma}^s \boldsymbol{n} \quad \text{on } \Gamma_t^0$$

Arbitrary Lagrangian-Eulerian Formulation

$$\boldsymbol{\chi} : \Omega \times [0,T] \mapsto \Omega_t, \qquad \boldsymbol{v} = \frac{\partial \boldsymbol{\chi}}{\partial t}, \quad \mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{X}}, \quad J = \det \mathbf{F}$$

$$\boldsymbol{\zeta}_{\mathcal{R}} : \mathcal{R} \times [0,T] \mapsto \mathcal{R}_t, \qquad \mathcal{R}_t \subset \Omega_t \quad \forall t \in [0,T], \qquad \boldsymbol{v}_{\mathcal{R}} = \frac{\partial \boldsymbol{\zeta}_{\mathcal{R}}}{\partial t}, \mathbf{F}_{\mathcal{R}} = \frac{\partial \boldsymbol{\zeta}_{\mathcal{R}}}{\partial \boldsymbol{X}}, J_{\mathcal{R}} = \det \mathbf{F}_{\mathcal{R}}$$

$$\frac{\partial}{\partial t} \int_{\mathcal{R}_t} \varrho dv + \int_{\partial \mathcal{R}_t} \varrho (\boldsymbol{v} - \boldsymbol{v}_{\mathcal{R}}) \cdot \boldsymbol{n}_{\mathcal{R}_t} da = 0$$

$$\frac{\partial}{\partial t} (\varrho J_{\mathcal{R}}) + \operatorname{div} \left(\varrho J_{\mathcal{R}} (\boldsymbol{v} - \boldsymbol{v}_{\mathcal{R}}) \mathbf{F}_{\mathcal{R}}^{-T} \right) = 0$$

☞ Lagrangian description:
$\boldsymbol{\zeta}_{\mathcal{R}} = \boldsymbol{\chi} \Rightarrow \mathbf{F}_{\mathcal{R}} = \mathbf{F}, J_{\mathcal{R}} = J, \boldsymbol{v}_{\mathcal{R}} = \boldsymbol{v}$

$$\frac{\partial}{\partial t} (\varrho J) = 0$$

☞ Eulerian description:
$\boldsymbol{\zeta}_{\mathcal{R}} = \mathrm{Id} \Rightarrow \mathbf{F}_{\mathcal{R}} = \mathbf{I}, J_{\mathcal{R}} = 1, \boldsymbol{v}_{\mathcal{R}} = \mathbf{0}$

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0$$

Governing equations

structure part

$$\frac{\partial \boldsymbol{v}^s}{\partial t} = \mathrm{div}(J\boldsymbol{\sigma}^s \mathbf{F}^{-T}) + \boldsymbol{f} \quad \text{in } \Omega^s$$

$$\det(\mathbf{F}) = 1 \quad \text{in } \Omega^s$$

$$\boldsymbol{u}^s = \mathbf{0} \quad \text{on } \Gamma^2$$

$$\boldsymbol{\sigma}^s \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma^3$$

fluid part

$$\frac{\partial \boldsymbol{v}^f}{\partial t} + (\nabla \boldsymbol{v}^f)\mathbf{F}^{-1}\boldsymbol{v}^f = \mathrm{div}\left(J\boldsymbol{\sigma}^f \mathbf{F}^{-T}\right) + \boldsymbol{f} \quad \text{in } \Omega^f$$

$$\mathrm{div}\left(J\boldsymbol{v}^f \mathbf{F}^{-T}\right) = 0 \quad \text{in } \Omega^f$$

$$\boldsymbol{v}^f = \boldsymbol{v}_0 \quad \text{or} \quad \boldsymbol{\sigma}^f \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma^1$$

Coupling strategies

- separated, weak coupling

  t^n fluid $\longrightarrow$ t^n solid $\longrightarrow$ t^{n+1} fluid $\longrightarrow$ t^{n+1} solid $\ldots$

- separated, strong coupling

  t^n fluid $\longrightarrow$ t^n solid $\longrightarrow$ t^{n+1} fluid $\longrightarrow$ t^{n+1} solid $\ldots$

- monolithic

  t^n fluid & solid $\longrightarrow$ t^{n+1} fluid & solid $\ldots$

Uniform formulation

$$\Omega = \Omega^f \cup \Omega^s, \qquad \boldsymbol{u} : \Omega \times [0,T] \to \mathcal{R}^3, \qquad \boldsymbol{v} : \Omega \times [0,T] \to \mathcal{R}^3,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = \begin{cases} \boldsymbol{v} & \text{in } \Omega^s \\ \Delta \boldsymbol{u} \quad \text{(“mesh deformation operator”)} & \text{in } \Omega^f \end{cases}$$

$$\beta \frac{\partial \boldsymbol{v}}{\partial t} = \begin{cases} \operatorname{div}\left(J\boldsymbol{\sigma}^s \mathbf{F}^{-T}\right) & \text{in } \Omega^s \\ -\beta(\nabla \boldsymbol{v})\mathbf{F}^{-1}(\boldsymbol{v} - \frac{\partial \boldsymbol{u}}{\partial t}) + \operatorname{div}\left(J\boldsymbol{\sigma}^f \mathbf{F}^{-T}\right) & \text{in } \Omega^f \end{cases}$$

$$0 = \begin{cases} J - 1 & \text{in } \Omega^s \\ \operatorname{div}(J\boldsymbol{v}\mathbf{F}^{-T}) & \text{in } \Omega^f \end{cases}$$

$$\boldsymbol{\sigma}^f \boldsymbol{n} = \boldsymbol{\sigma}^s \boldsymbol{n} \text{ on } \Gamma_t^0$$

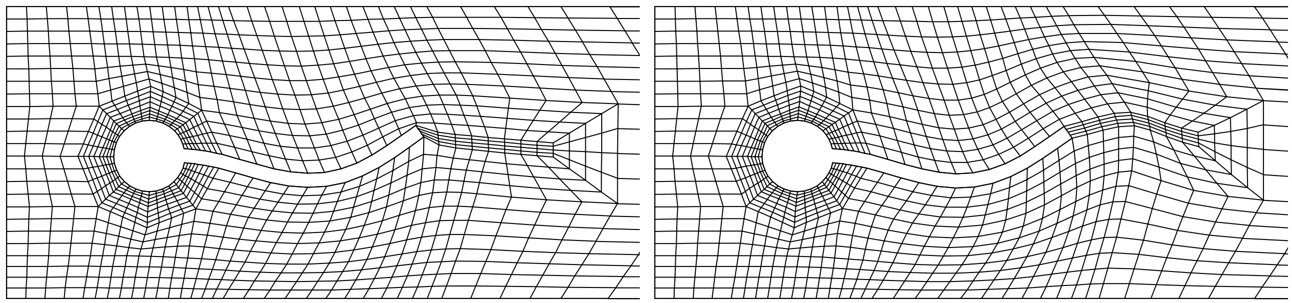
$$\boldsymbol{v} = \boldsymbol{v}_B \quad \text{on } \Gamma_t^1$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_t^2$$

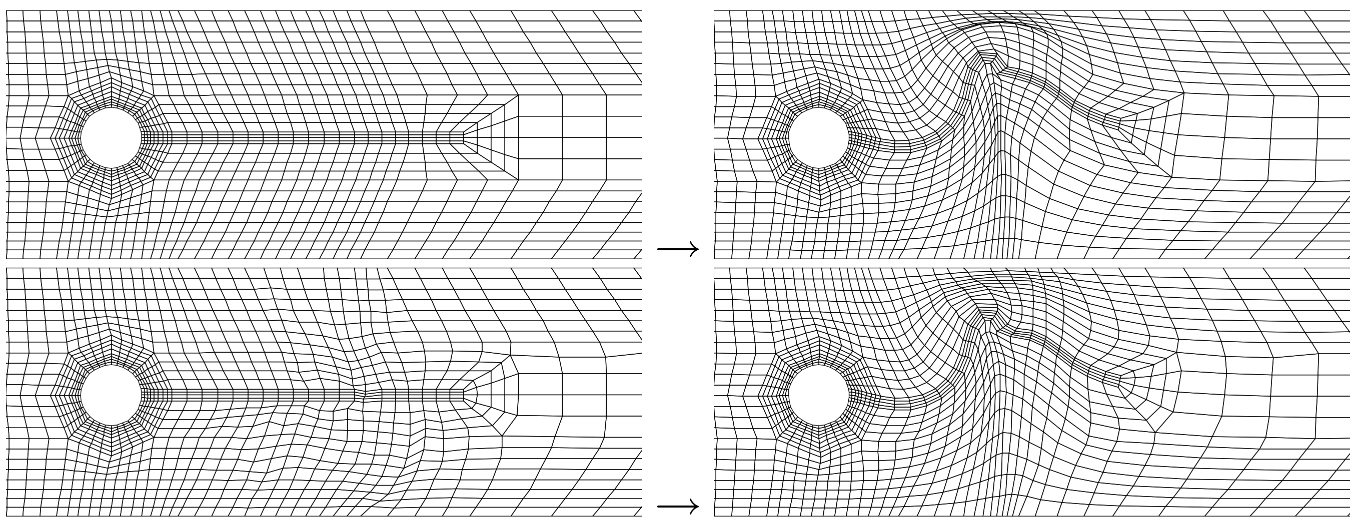
$$\boldsymbol{\sigma}^s \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma_t^3$$

Mesh alignment

- Diferent mesh deformation equations



- Adaptation of the ‘reference mesh’



- Restart after remeshing...

Constitutive equations

- incompressible Newtonian fluid

$$\boldsymbol{\sigma}^f = -p\mathbf{I} + 2\nu\mathbf{D}$$

- hyperelastic material, incompressible

$$\boldsymbol{\sigma}^s = -p\mathbf{I} + 2\mathbf{F}\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{F}^T, \qquad \det F = 1$$

$$\Psi(\mathbf{F}) = \alpha(\mathrm{I}_C - 3) \text{Neo-Hook}$$

$$\Psi(\mathbf{F}) = \alpha_1(\mathrm{I}_C - 3) + \alpha_2(\mathbf{II}_C - 3) + \alpha_3(|\mathbf{F}\boldsymbol{e}| - 1)^2 \text{Mooney-Rivlin + anisotropic}$$

where $\mathbf{C} = \mathbf{F}\mathbf{F}^T$ and $\mathrm{I}_C = \operatorname{tr}\mathbf{C}$, $\mathbf{II}_C = \frac{1}{2}\left(\operatorname{tr}\mathbf{C}^2 - (\operatorname{tr}\mathbf{C})^2\right)$

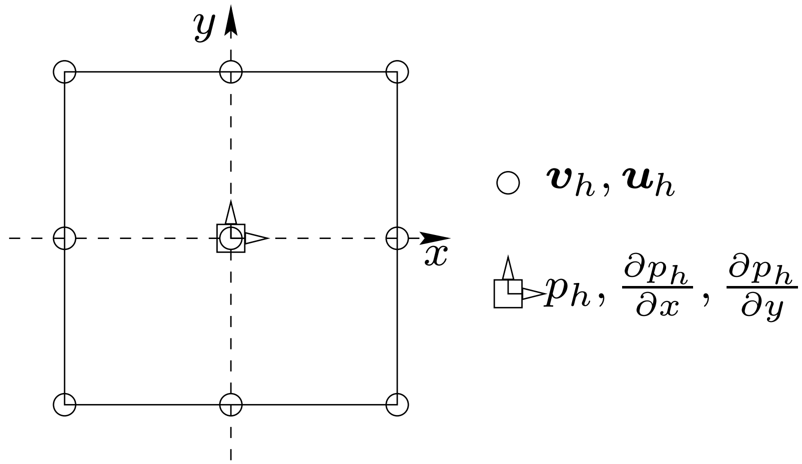
- or **St. Venant–Kirchhoff** material, compressible

$$\boldsymbol{\sigma}^s = \frac{1}{J}\mathbf{F}(\lambda^s(\operatorname{tr}\mathbf{E})\mathbf{I} + \mu^s\mathbf{E})\mathbf{F}^{\mathrm{T}}$$

where $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$

Discretization in space and time

Discretization in space: FEM $Q_2/Q_2/P_1^{disc}$



$$U_h = \{\boldsymbol{u}_h \in [C(\Omega_h)]^2, \boldsymbol{u}_h|_T \in [Q_2(T)]^2 \forall T \in \mathcal{T}_h, \boldsymbol{u}_h = \boldsymbol{0} \text{ on } \Gamma^1\},$$

$$V_h = \{\boldsymbol{v}_h \in [C(\Omega_h)]^2, \boldsymbol{v}_h|_T \in [Q_2(T)]^2 \forall T \in \mathcal{T}_h, \boldsymbol{v}_h = \boldsymbol{0} \text{ on } \Gamma^2\},$$

$$P_h = \{p_h \in L^2(\Omega_h), p_h|_T \in P_1(T) \forall T \in \mathcal{T}_h\}.$$

Discretization in time: Crank-Nicholson scheme with adaptive time-step selection

Remark: high accuracy, edge-based stabilization by $\int_{\text{edges}} [\nabla \phi_i][\nabla \phi_j]$

Discrete nonlinear system

$$\boldsymbol{\mathcal{R}}(\boldsymbol{X}) = \mathbf{0}, \qquad \boldsymbol{X} = (\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) \in U_h \times V_h \times P_h$$

$$\begin{aligned} M u_h - \frac{k}{2}(M^s v_h + L^f u_h) &= \mathrm{rhs}(u_h^n, v_h^n) \\ (M^f + \beta M^s) v_h + \frac{k}{2} N_1(v_h, v_h) + \frac{1}{2} N_2(v_h, u_h) + \frac{k}{2}(S^s(u_h) + S^f(v_h)) - kBp_h &= \mathrm{rhs}(u_h^n, v_h^n, p_h^n) \\ C(u_h) + {B^f}^T v_h &= 1 \end{aligned}$$

$$\Downarrow$$

$$\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \boldsymbol{X}}(\boldsymbol{X}) = \begin{pmatrix} M - \frac{k}{2} L^f & \frac{k}{2} M^s & 0 \\ \frac{1}{2}\frac{\partial N_2}{\partial u_h} + \frac{k}{2}\frac{\partial (N_1 + S^s + S^f)}{\partial u_h} + k\frac{\partial B}{\partial u_h} p_h & M^s + \beta M^f + \frac{1}{2}\frac{\partial N_2}{\partial v_h} + \frac{k}{2}\frac{\partial (N_1 + S_f^2)}{\partial v_h} & kB \\ {B^s}^T + \frac{\partial {B^f}^T}{\partial u_h} v_h & {B^f}^T & 0 \end{pmatrix}$$

Discrete nonlinear system

$$\mathcal{R}(\boldsymbol{X}) = \mathbf{0}, \qquad \boldsymbol{X} = (\boldsymbol{u}_h, \boldsymbol{v}_h, p_h) \in U_h \times V_h \times P_h$$

$$
\begin{aligned}
Mu_h - \frac{k}{2}(M^s v_h + L^f u_h) &= \mathsf{rhs}(u_h^n, v_h^n) \\
(M^f + \beta M^s) v_h + \frac{k}{2} N_1(v_h, v_h) + \frac{1}{2} N_2(v_h, u_h) + \frac{k}{2}(S^s(u_h) + S^f(v_h)) - kBp_h &= \mathsf{rhs}(u_h^n, v_h^n, p_h^n) \\
C(u_h) + {B^f}^T v_h &= 1
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{bmatrix} S_{\boldsymbol{uu}} & S_{\boldsymbol{uv}} & 0 \\ S_{\boldsymbol{vu}} & S_{\boldsymbol{vv}} & kB \\ c_{\boldsymbol{u}} B_s^T & c_{\boldsymbol{v}} B_f^T & 0 \end{bmatrix}
\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ p \end{bmatrix}
=
\begin{bmatrix} \mathbf{f_u} \\ \mathbf{f_v} \\ f_p \end{bmatrix}
$$

Typical discrete saddle-point problem

Solution of the nonlinear problem

- compute the Jacobian matrix (analytic, automatic differentiation or divided differences)

$$\left[\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \boldsymbol{X}}\right]_{ij}(\boldsymbol{X}^n) \approx \frac{[\boldsymbol{\mathcal{R}}]_i(\boldsymbol{X}^n + \varepsilon \boldsymbol{e}_j) - [\boldsymbol{\mathcal{R}}]_i(\boldsymbol{X}^n - \varepsilon \boldsymbol{e}_j)}{2\varepsilon},$$

- solve for $\delta \boldsymbol{X}$

$$\left[\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \boldsymbol{X}}(\boldsymbol{X}^n)\right] \delta \boldsymbol{X} = \boldsymbol{\mathcal{R}}(\boldsymbol{X}^n)$$

- adaptive line search strategy

$$\boldsymbol{X}^{n+1} = \boldsymbol{X}^n + \omega \delta \boldsymbol{X} \qquad \omega \text{ such that } f(\omega) = \boldsymbol{\mathcal{R}}(\boldsymbol{X} + \omega \delta \boldsymbol{X}) \cdot \boldsymbol{X} \searrow$$

- continuation with respect to time
- MG, BiCGStab or GMRes(m) with ILU(k) preconditioner to solve the linear problems

Jacobian approximation

$$\left[\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \boldsymbol{X}}\right]_{ij}(\boldsymbol{X}^n) \approx \frac{[\boldsymbol{\mathcal{R}}]_i(\boldsymbol{X}^n + \varepsilon \boldsymbol{e}_j) - [\boldsymbol{\mathcal{R}}]_i(\boldsymbol{X}^n - \varepsilon \boldsymbol{e}_j)}{2\varepsilon},$$

ε / TOL	10^{-8}	10^{-4}	10^{-2}	10^{-1}
10^{-8}	7 /107.57 [21.52]	12 /57.08 [26.52]	12 /47.00 [23.75]	17 /33.06 [27.38]
10^{-4}	7 /108.71 [24.57]	**8 /62.75 [17.77]**	**10 /42.20 [18.95]**	18 /31.33 [29.05]
10^{-2}	16 /109.75 [51.65]	20 /47.35 [38.28]	25 /29.80 [38.58]	56 /16.98 [73.83]
10^{-1}	44 /116.11 [141.30]	48 /35.79 [81.72]	49 /17.92 [65.77]	–

nonlinear solver it. / avg. linear solver it. [CPU time] for BiCGStab(ILU(0))

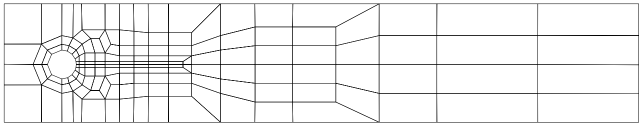
Multigrid solver

- standard geometric multigrid approach
- smoother by local MPSC-Ansatz (Vanka-like smoother)

$$\begin{bmatrix} \boldsymbol{u}^{l+1} \\ \boldsymbol{v}^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{l} \\ \boldsymbol{v}^{l} \\ p^{l} \end{bmatrix} - \omega \sum_{\text{Patch } \Omega_i} \begin{bmatrix} S_{\boldsymbol{uu}|\Omega_i} & S_{\boldsymbol{uv}|\Omega_i} & 0 \\ S_{\boldsymbol{vu}|\Omega_i} & S_{\boldsymbol{vv}|\Omega_i} & kB_{|\Omega_i} \\ c_{\boldsymbol{u}} B^T_{s|\Omega_i} & c_{\boldsymbol{v}} B^T_{f|\Omega_i} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{def}^l_{\boldsymbol{u}} \\ \mathbf{def}^l_{\boldsymbol{v}} \\ def^l_p \end{bmatrix}$$

- full inverse of the local problems by standard LAPACK (39×39 systems)
- alternatives: simplified local problems (3×3 systems) or ILU(k)
- combination with GMRES/BiCGStab methods possible
- full Q_2 and P_1^{disc} prolongation **P**, restriction by $\mathbf{R} = \mathbf{P}^T$

Multigrid solver



- 1 timestep of fully developed solution
- streamline diffusion stabilization
- shown: **number of nonlinear steps/avg. number of linear steps** [**CPU time**]
- timestep 10^{-2}

Level	ndof	MG(2)	MG(4)	BiCGStab(ILU(1))	GMRES(ILU(1),200)
1	12760	2/8 [66]	2/8 [92]	2/51 [32]	2/50 [27]
2	50144	2/8 [190]	2/5 [198]	2/120 [200]	2/117 [151]
3	198784	2/9 [744]	2/6 [852]	2/311 [1646]	2/358 [1432]
4	791552	2/13 [3803]	2/7 [3924]	MEM.	MEM.

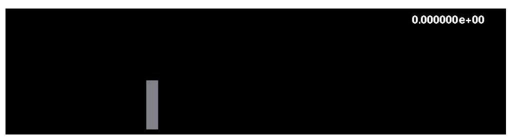
- timestep 10^{0}

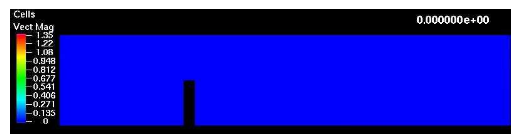
Level	ndof	MG(2)	MG(4)	BiCGStab(ILU(1))	GMRES(ILU(1),200)
1	12760	4/12 [118]	4/11 [177]	20/160 [631]	20/801 [1579]
2	50144	4/12 [466]	4/7 [470]	2/800 [] diverg.	13/801 [] diverg.
3	198784	4/13 [1898]	4/7 [2057]	2/800 [] diverg.	4/801 [] diverg.
4	791552	4/15 [8678]	4/8 [9069]	MEM.	MEM.

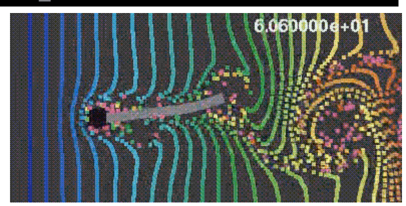
$\Rightarrow$ robust and efficient Newton-MG scheme

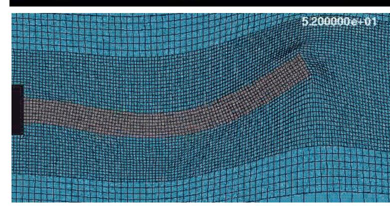
Summary, Examples

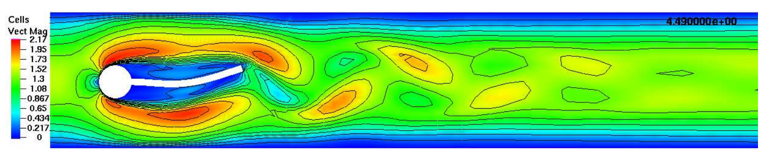
- monolithic, fully coupled FEM (Q_2/P_1) for **viscous incompressible fluid** and **incompressible hyperelastic structure**
- fully implicit 2nd order discretization in time (Crank-Nicholson)
- Newton-like method for the coupled system (Jacobian matrix via divided differences)
- Fully coupled MG-solver
- a priori localy adapted mesh





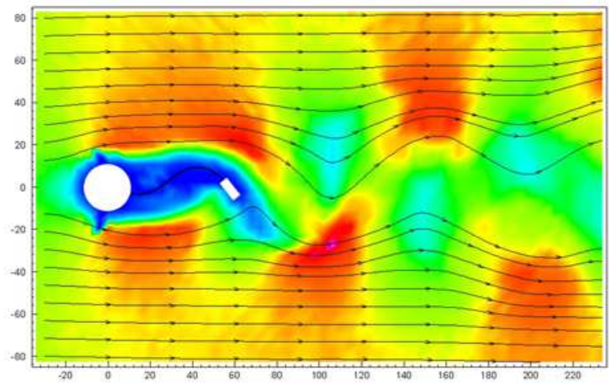




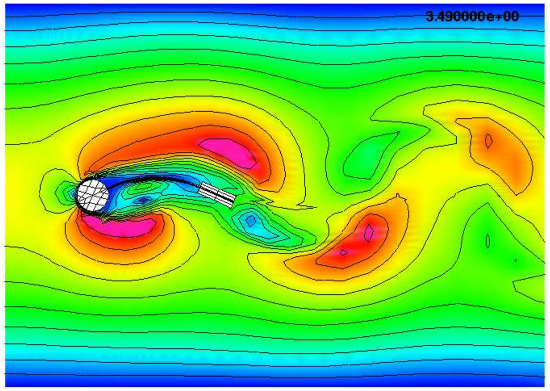


Benchmarking of the experimental data

- Flustruc experiment, Erlangen

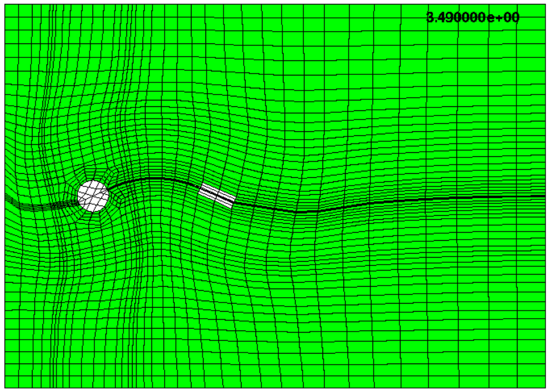


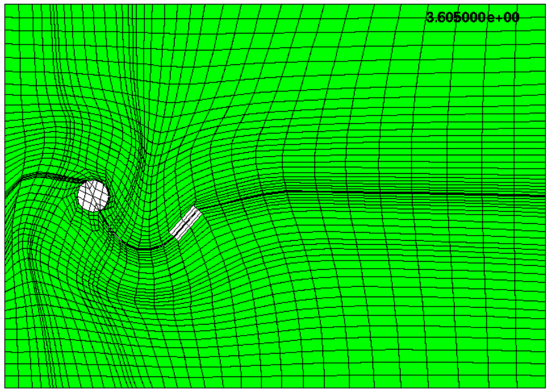
experiment



computation

- First computational results and tasks





Outlook

- Discretization
  - adaptive time step control
  - edge-oriented stabilization of convective terms for fluid and structure for Q_2
  - dynamic mesh alignment (spatial r-adaptivity), updated ALE method for fluid and solid
- Solvers
  - more robust smothers with respect to anisotropy
  - decoupled solvers of "Discrete Projection" type: algebraic decoupling of "large" velocity + deformation part (=nonlinear, but well conditioned for small Δt) and "small" pressure part (=linear, but ill-conditioned)
- Numerical benchmarks (tests, comparisons, 3D)