A class of Extended one-step methods for solving delay differential equations

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Abstract: We derive a class of extended one-step methods of order m for solving delay-differential equations. This class includes methods of fourth and fifth order of accuracy. Also, the class of these methods depends on two free parameters. A convergence theorem and convergence factor of these methods are given. In addition, we investigate the stability properties of these methods. The results of the article are illustrated by numerical examples.

Keywords: Delay-differential equations, Stability, Convergence, Numerical solutions

1 Introduction

Delay differential equations (DDEs) have a wide range of applications in science and engineering: for examples population dynamics, chemical kinetics, physiological and pharmaceutical kinetics. For example, one may think of modelling the growth of a population where the self-regulatory reaction in case of overcrowding responds after some time lag. More examples are discussed in Driver [7], Gopalsamy[27] and Kuang[28]. First order DDE can be written as

$$\begin{aligned} y'(x) &= f(x, y(x), y(\alpha(x))), & a \le x \le b, \\ y(x) &= g(x), & \nu \le x \le a. \end{aligned} \tag{1}$$

Here f , α and g denote given functions with $\alpha(x) \le x$ for $x \ge a$, the function α is usually called the delay or lag function and y is unknown solution for $x > a$. If the delay is a constant, it is called the constant delay, if it is a function of only time, then it is called the time dependent delay, if it is a function of time and the solution $y(x)$, then it is called the state dependent delay.

Many methods have been proposed for the numerical approximation of problem (1). Oberle and Pesch [18] introduced a class of numerical methods for the treatment of DDEs based on the well-known Runge-Kutta-Fehlberg methods. The retarded argument is approximated by an appropriate Hermite interpolation. The same methods are used by Arndt [2] with a different stepsize control mechanism. Bellen and Zennaro[4] developed a class of numerical methods to approximate solution of DDEs. These methods are based on implicit Runge-Kutta methods. Paul and Baker [19] used explicit Runge-Kutta method for the numerical solution of singular DDEs. Torelli and Vermiglio [20] considered continuous numerical methods for differential equations with several constant delays. These methods are based on continuous quadrature rule. Hayashi [10] used continuous Runge-Kutta methods for the numerical solution of retarded and neutral DDEs. Engelborghs et al. [6] presented collocation methods for the computation of periodic solution of DDEs. Hu and Cahlon [12] considered the numerical solution of initial-value discrete- delay systems.

The most obvious of the above methods for solving problem (1) numerically is that the $s-$ Runge-Kutta methods with $\alpha(x) = x - \tau$ in the form

$$Y^i_{n+1} = y_n + h \sum_j a_{ij} f(x_n + c_j h, Y^j_{n+1}, y(x_n + c_j h - \tau)),$$

$$y_{n+1} = y_n + h \sum_j b_j f(x_n + c_j h, Y^j_{n+1}, y(x_n + c_j h - \tau))$$

$i = 1, 2 \ldots, s$. The b_j are often referred to as the weights of the method, while the c_i are referred to abscissae, they

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belong to $[0,1]$ and satisfy the conditions:

$$c_i = \sum_j^s a_{ij}.$$

There are many concepts of stability of numerical methods for DDEs based on different test equation as well as the delay term. [3] has considered the below scalar equation for $\lambda = 0$ and $\mu \in \mathbb{C}$ and also considered the case, where λ and μ are complex using the linear DDEs

$$\begin{aligned} \acute{y}(x) &= \lambda y(x) + \mu y(x-\tau), \quad x \geq 0 \\ y(x) &= g(x), \quad -\tau \leq x \leq 0 \end{aligned} \tag{2}$$

It is known that from [1] that if $g(x)$ is continuous and if

$$|\mu| < -Re(\lambda), \tag{3}$$

then the soultion $y(x)$ of (2) tends to zero as $x \longrightarrow \infty$.

It is well known that the maximum order of an A-stable linear multistep methods (LMMs) is two. This difficulty has been solved by coupling two LMMS to give an A-stable extended one step method of order three , which had constructed by Usmani and Agarwal [26]. After noting that the maximum order of extended one step methos is three, Kondrat and Jacques [15] gave extended two-step fourth order A-stable methods for solving ordinary differential equations. Later Chawla et al. [24] had constructed a class of extended one-step methods generalizing the method of Usmani and Agrawal [26] and the maximum attainable order for methods of this class is five which are A- and/ or L-stable.

The purpose of this paper is to study an extension the work of Chawla et al. [23,24] for solving DDEs. This class includes methods of fourth and fifth order of accuracy. Furthermore there exists two-parameter sub-family of these methods which are P-stable.

The paper is organized as follows: In the following section 2, we explain the general approach for solving DDEs. The details of the computations for different value of m will be described in Section 3. The Analysis of stability regions for these methods presents in section 4. For three representative examples, section 5 contains a documentation of numerical results illustrating the performance of our methods. Some concluding remarks are given in the final section 6.

2 The general approach

In this section, we extend the work of Chawla et al. (1994,1995) to derive a class of extended one-step methods of order m for solving DDEs. We start with the following discretization for solving problem (1):

$$\begin{aligned} y_{n+1} &= y_n \\ &+ h[\ \alpha_0 f_n + \alpha_1 f_{n+1} + \sum_{j=2}^{m-1} \alpha_j \hat{f}_{n+j}]\ , \quad n = 0,1,\ldots,N-1, \end{aligned} \tag{4}$$

where $\hat{f}_{n+j} = f(x_{n+j}, \hat{y}_{n+j}, y^h(\alpha(x_{n+j})))$ and $\alpha_j,\ j = 0,1,\ldots m-1$ are real coefficients. The function y^h is computed from

$$\begin{cases} y^h(x) = g(x) \quad \text{for} \quad x \leq a \\ y^h(x) = \beta_{j0} y_k + \beta_{j1} y_{k+1} + h[\ \gamma_{j0} f_k \\ \qquad + \gamma_{j1} f_{k+1} + \sum_{i=2}^{j-1} \gamma_{ji} \hat{f}_{k+1}]\ , \\ \qquad\qquad x_k < x \leq x_{k+1} \quad k = 0,1,\ldots \end{cases} \tag{5}$$

where $\beta_{j0},\ \beta_{j1},\ \gamma_{j0},\ \gamma_{j1}$ and γ_{ji} are real coefficients. The function $\hat{y}_{n+j}$ are computed from (5) when $x = x_{n+j}$. In this paper, we will use $\tilde{}$ for the coefficients of $\hat{y}_{n+j}$ as in the following form :

$$\begin{aligned} \hat{y}_{n+j} &= \tilde{\beta}_{j0} y_n + \tilde{\beta}_{j1} \\ &\qquad + h[\ \tilde{\gamma}_{j0} f_n + \tilde{\gamma}_{j0} f_{n+1} + \sum_{i=2}^{j-1} \tilde{\gamma}_{ji} \hat{f}_{n+i}] \end{aligned} \tag{6}$$

We display this class of extended one-step methods in the following Table.

α_0	α_1	α_2	...		α_{m-1}
β_{20}	β_{21}	γ_{21}			
β_{30}	β_{31}	γ_{31}	γ_{32}		
$\beta_{m-1,0}$	$\beta_{m-1,1}$	$\gamma_{m-1,1}$	$\gamma_{m-1,2}$		$\gamma_{m-1,m-2}$

3 Derivation of some methods for $m = 4,5$

In this section, we describe derivations of some methods for various values of m .

3.1 Case I, $m = 4$

In this case, we describe the derivation of the present methods of fourth order of accuracy. In order to determine the coefficients α_0, α_1 and α_j , we rewrite (4) in the exact form

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + h[\alpha_0 f(x_n, y(x_n), y(\alpha(x_n))) \\ &\quad + \alpha_1 f(x_{n+1}, y(x_{n+1}), y(\alpha(x_{n+1}))) \\ &\quad + \sum_{j=2}^{m-1} \alpha_j f(x_{n+j}, y(x_{n+j}), y(\alpha(x_{n+j})))] + t(x_{n+1}). \end{aligned} \tag{7}$$

We expand the left and right side of (7) in the Taylor series about the point x_{n+1}, equate the coefficients up to the terms $O(h^4)$ and solve the resulting system of equations, we obtain

$$\alpha_0 = \frac{3}{8}, \ \ \alpha_1 = \frac{19}{24}, \ \ \alpha_2 = -\frac{5}{24} \ \ \alpha_3 = \frac{1}{24} \tag{8}$$

and

$$t(x_{n+1}) = \frac{-19}{720} h^5 y^{(5)}(x_{n+1}). \tag{9}$$

By the same way, in order to determine the coefficients $\beta_{j0}, \beta_{j1}, \gamma_{j0}, \gamma_{j1}$ and γ_{ji}, $i = 0, 1, \dots j-1$, we rewrite (5) in the exact form

$$\begin{aligned} y(x) &= \beta_{j0} y(x_k) + \beta_{j1} y(x_{k+1}) + h[\, \gamma_{j0} f(x_k, y(x_k), y(\alpha(x_k))) \\ &\quad + \gamma_{j1} f(x_{k+1}, y(x_{k+1}), y(\alpha(x_{k+1}))) \\ &\quad + \sum_{i=2}^{j-1} \gamma_{ji} f(x_{k+j}, y(x_{k+j}), y(\alpha(x_{k+j})))] \ + t_j(x_{k+1}). \end{aligned} \tag{10}$$

We expand the left and right side of (10) in the Taylor series about the point x_{n+1}, equate the coefficients up to the terms $O(h^3)$ and solve the resulting system of equations, we obtain

$$\begin{cases} \beta_{20} = 2\gamma_{20} + \delta_1^2 \\ \beta_{21} = 1 - \delta_1^2(x) - 2\gamma_{20} \\ \gamma_{21} = \gamma_{20} + \delta_1(x) + \delta_1^2(x) \end{cases} \tag{11}$$

with γ_{20} free, where

$$\begin{aligned} t_2(x_{k+1}) &= \frac{h^3}{6}(\delta_1^3(x) + \delta_1^2(x) - \gamma_{20}) y^{(3)}(x_{k+1}) \\ &\quad + \frac{h^4}{24}(\delta_1^4(x) - \delta_1^2(x) + 2\gamma_{20}) y^{(4)}(x_{k+1}), \ \text{for } j = 2, \end{aligned} \tag{12}$$

here $\delta_1(x) = \frac{1}{h}(x - x_{k+1})$, for $x_k < x \le x_{k+1}$, $x = \alpha(x_{n+2})$, $k = 0, 1, \dots$.
and

$$\begin{cases} \beta_{30} = 2\gamma_{31} + 4\gamma_{32} - 2\delta_2(x) - \delta_2^2(x) \\ \beta_{31} = 1 + 2\delta_2(x) + \delta_2^2(x) - 2\gamma_{31} - 4\gamma_{32} \\ \gamma_{30} = -\delta_2(x) - \delta_2^2(x) + \gamma_{31} + 3\gamma_{32} \end{cases} \tag{13}$$

with γ_{31}, γ_{32} free where

$$\begin{aligned} t_3(x_{k+1}) &= \frac{h^3}{6}(\delta_2^3(x) + 2\delta_2^2(x) + \delta_2(x) - \gamma_{31} \\ &\quad - 8\gamma_{32}) y^{(3)}(x_{k+1}) + \frac{h^4}{24}(\delta_2^4(x) - 3\delta_2^2(x) \\ &\quad - 2\delta_2(x) + 2\gamma_{31} + 4\gamma_{32}) y^{(4)}(x_{k+1}), \ \text{for } j = 3, \end{aligned}$$

$\delta_2(x) = \frac{1}{h}(x - x_{k+1})$, for $x_k < x \le x_{k+1}$, $x = \alpha(x_{n+3})$, $k = 0, 1, \dots$.

The approximations $\hat{y}_{n+2}$ and $\hat{y}_{n+3}$ is determined from (5) and the coefficients in this case take the form

$$\begin{cases} \tilde{\beta}_{20} = 1 + 2\tilde{\gamma}_{20} \\ \tilde{\beta}_{21} = -2\tilde{\gamma}_{20} \\ \tilde{\gamma}_{21} = 2 + \tilde{\gamma}_{20} \end{cases} \tag{14}$$

with γ_{20} free, where

$$\begin{aligned} t_2(x_{n+1}) &= \frac{h^3}{6}(2 - \tilde{\gamma}_{20}) y^{(3)}(x_{n+1}) \\ &\quad + \frac{h^4}{12} \tilde{\gamma}_{20} y^{(4)}(x_{n+1}) \ \ \text{for } j = 2, \end{aligned}$$

and

$$\begin{cases} \tilde{\beta}_{30} = -8 + 2\tilde{\gamma}_{31} + 4\tilde{\gamma}_{32} \\ \tilde{\beta}_{31} = 9 - 2\tilde{\gamma}_{31} - 4\tilde{\gamma}_{32} \\ \tilde{\gamma}_{30} = -6 + \tilde{\gamma}_{31} + 3\tilde{\gamma}_{32} \end{cases} \tag{15}$$

with $\tilde{\gamma}_{31}, \tilde{\gamma}_{32}$ free, where

$$\begin{aligned} t_3(x_{n+1}) &= (3 - \frac{1}{6}\tilde{\gamma}_{31} - \frac{4}{3}\tilde{\gamma}_{32}) h^3 y^{(3)}(x_{n+1}) \\ &\quad + (\frac{1}{12}\tilde{\gamma}_{31} + \frac{1}{6}\tilde{\gamma}_{32}) h^4 y^{(4)}(x_{n+1}) \ \ \text{for } j = 3. \end{aligned}$$

and consider the discretization (4) for $m = 4$ made into one step defined by

$$y_{n+1} = y_n + h[\, \alpha_0 f_n + \alpha_1 f_{n+1} + \alpha_2 \hat{f}_{n+2} + \alpha_3 \hat{f}_{n+3}] + T_{n+1}. \tag{16}$$

To calculate the local truncation error in (16) from (7) and (16) we have

$$\begin{aligned} T_{n+1} &= t(x_{n+1}) + h[\, \alpha_2 (f(x_{n+2}, y(x_{n+2}), y(\alpha(x_{n+2}))) \\ &\quad - \hat{f}_{n+2}) + \alpha_3 (f(x_{n+3}, y(x_{n+3}), y(\alpha(x_{n+3}))) - \hat{f}_{n+3})]. \end{aligned} \tag{17}$$

It can be shown from (17) (we omit details) that

$$\begin{aligned} T_{n+1} &= t(x_{n+1}) \\ &\quad + \frac{h^4}{144}[\, (8 + 5\gamma_{20} - \gamma_{31} - 8\gamma_{32}) g_1(x_{n+1}) + (-5\delta_1^2(x) \\ &\quad - 5\delta_1^3(x) + 5\tilde{\gamma}_{20} + \delta_2^3(x) + \delta_2(x) + 2\delta_2^2(x) - \tilde{\gamma}_{31} \\ &\quad - 8\tilde{\gamma}_{32}) w_1(x_{n+1})]\, y^{(3)}(x_{n+1}) + \Big\{ \frac{h^5}{720}[\, (26 - 5\tilde{\gamma}_{20} - 2\tilde{\gamma}_{31} \\ &\quad - 16\tilde{\gamma}_{32}) g'(x_{n+1}) + (-5\delta_1^2(x) - 5\delta_1^3(x) + 5\gamma_{20} + 2\delta_2^3(x) \\ &\quad + 4\delta_2^2(x) + 2\delta_2(x) - 2\gamma_{31} - 16\gamma_{32}) \acute{w}_1(x_{n+1}) + \tilde{\gamma}_{32}(2 \\ &\quad - \tilde{\gamma}_{20}) g_1^2(x_{n+1}) + \gamma_{32}(2 - \tilde{\gamma}_{20}) g_1(x_{n+1}) w_1(x_{n+1}) + \gamma_{32}(\delta_1^3(x) + \delta_1^2(x) \\ &\quad - \gamma_{20}) w_1^2(x_{n+1})]\, 5y^{(3)}(x_{n+1}) + [\, (-10\tilde{\gamma}_{20} + \gamma_{31} + \tilde{\gamma}_{31}) g_1(x_{n+1}) \\ &\quad + (\delta_2^4 - 3\delta_2^2 - 2\delta_2 - \delta_1^4 + 5\delta_1^2 + 2\gamma_{31} + 4\gamma_{32} - 10\gamma_{20}) w_1(x_{n+1}) \\ &\quad + 4\tilde{\gamma}_{32}(\delta_1^3(x) + \delta_1^2(x) - \gamma_{20}) g_1(x_{n+1}) w_1(x_{n+1})]\, \frac{5}{4} y^{(4)}(x_{n+1}) \,\Big\}, \end{aligned}$$

where we have set

$$g_1(x_{n+1}) = \frac{\partial f(x,y(x),y(\alpha(x)))}{\partial y(x)})|_{x_{n+1}},$$

and

$$w(x_{n+1}) = \frac{\partial f(x,y(x),y(\alpha(x)))}{\partial y(\alpha(x))})|_{x_{n+1}}.$$

In order that T_{n+1} in (**??**) be $O(h^5)$, we must have

$$\begin{aligned}\gamma_{31} &= 8+5\gamma_{20}-8\gamma_{32},\\ \tilde{\gamma}_{31} &= 5\tilde{\gamma}_{20}+\delta_2^3(\alpha(x_{n+3}))+\delta_2(\alpha(x_{n+3}))\\ &\quad -8\tilde{\gamma}_{32}-5\delta_1^2(\alpha(x_{n+2}))-5\delta_1^3(\alpha(x_{n+2})).\end{aligned}$$

By consider $\tilde{\gamma}_{20} = \gamma_{20}$ and $\tilde{\gamma}_{32} = \gamma_{32}$, we have a two-parameter family of extended one-step fourth order methods given, which we will refer it by $PM_4(\gamma_{20}, \gamma_{32})$.

3.2 Case II, $m = 5$

We describe the derivation of a scheme of the fifth order of accuracy. As in case I, we rewrite (4) in the exact form and expand the left and right sides of this equation in the Taylor series about the point x_{n+1}, equate the coefficients up to the terms $O(h^5)$ and solve the resulting system of equations, we obtain

$$\alpha_0 = \frac{251}{720},\ \alpha_1 = \frac{323}{360},\ \alpha_2 = -\frac{-11}{30},\ \alpha_3 = \frac{53}{360},\ \alpha_4 = \frac{-19}{720} \tag{18}$$

and

$$t(x_{n+1}) = \frac{3}{160}h^6 y^{(6)}(x_{n+1}). \tag{19}$$

By the same way, in order to determine the coefficients β_{j0}, β_{j1} and γ_{ji}, $i = 0,1,\dots j-1$, we rewrite (5) in the exact form and expand the left and right sides of this equation in the Taylor series about the point x_{k+1}, equate the coefficients up to the terms $O(h^4)$ and solve the resulting system of equations, we obtain

$$\begin{cases}\beta_{20} = 2\delta_1^3(x)-3\delta_1^2(x)+1\\ \beta_{21} = 3\delta_1^2(x)-2\delta_1^3(x)\\ \gamma_{20} = \delta_1^3(x)-2\delta_1^2(x)+\delta_1(x)\\ \gamma_{21} = \delta_1^3(x)-2\delta_1^2(x)\end{cases} \tag{20}$$

where

$$\begin{aligned}t_2(x_{k+1}) &= \frac{h^4}{24}(\delta_1^4(x)-2\delta_1^3(x)+\delta_1^2(x))y^{(4)}(x_{k+1})\\ &\quad+\frac{h^5}{120}(\delta_1^5(x)-3\delta_1^3(x)+2\delta_1^2(x))y^{(5)}(x_{k+1})\\ &\quad \text{for } j=2,\end{aligned}$$

$\delta_1(x) = \frac{1}{h}(x-x_{k+1})$, for $x = \alpha(x_{n+2});\ k = 0,1,\dots,$
and

$$\begin{cases}\beta_{30} = 2\delta_2^3(x)-3\delta_2^2(x)-12\gamma_{32}+1\\ \beta_{31} = 12\gamma_{32}-2\delta_2^3(x)+3\delta_2^2(x)\\ \gamma_{30} = \delta_2^3(x)-\delta_2^2(x)+\delta_2(x)-5\gamma_{32}\\ \gamma_{31} = \delta_2^3(x)-\delta_2^2(x)-8\gamma_{32}\end{cases} \tag{21}$$

with γ_{32} free, where

$$\begin{aligned}t_3(x_{k+1}) &= \frac{h^4}{24}(\delta_2^4(x)-2\delta_2^3(x)+\delta_2^2(x)-12\gamma_{32})y^{(4)}(x_{k+1})\\ &\quad+\frac{h^5}{120}(\delta_2^5(x)-3\delta_2^3(x)+2\delta_2^2(x)-52\gamma_{32})y^{(5)}(x_{k+1}),\\ &\quad \text{for } j=3,\end{aligned}$$

$\delta_2(x) = \frac{1}{h}(x-x_{k+1})$, for $x = \alpha(x_{n+3});\ k = 0,1,\dots$
and

$$\begin{cases}\beta_{40} = 2\delta_3^3(x)-3\delta_3^2(x)-12\gamma_{42}-36\gamma_{43}+1\\ \beta_{41} = 3\delta_3^2(x)-2\delta_3^3(x)+12\gamma_{42}+36\gamma_{43}\\ \gamma_{40} = \delta_3^3(x)-2\delta_3^2(x)+\delta_3(x)-5\gamma_{42}-16\gamma_{43}\\ \gamma_{41} = \delta_3^3(x)-\delta_3^2(x)-8\gamma_{42}-21\gamma_{43}\end{cases} \tag{22}$$

with γ_{42}, γ_{43} free, where

$$\begin{aligned}t_4(x_{k+1}) &= \frac{h^4}{24}(\delta_3^4(x)-2\delta_3^3(x)+\delta_3^2(x)-12\gamma_{42}\\ &\quad -60\gamma_{43})y^{(4)}(x_{k+1})+\frac{h^5}{120}(\delta_3^5(x)+2\delta_3^2(x)-3\delta_3^3(x)\\ &\quad -52\gamma_{42}-336\gamma_{43})y^{(5)}(x_{k+1}),\end{aligned}$$

$\delta_3(x) = \frac{1}{h}(x-x_{k+1})$,for $x = \alpha(x_{n+4});\ k = 0,1,\dots.$
The approximations $\hat{y}_{n+2}$, $\hat{y}_{n+3}$ and $\hat{y}_{n+4}$ determined from (5) and the coefficients in this case take the form

$$\tilde{\beta}_{20} = 5,\ \tilde{\beta}_{21} = -4,\ \tilde{\gamma}_{20} = 2,\ \gamma_{20} = 4,$$

where

$$t_2(x_{n+1}) = \frac{1}{6}h^4 y^{(4)}(x_{n+1}) + \frac{2}{15}h^5 y^{(5)}(x_{n+1}) \text{ for } j=2;$$

$$\begin{cases}\tilde{\beta}_{30} = 28-12\tilde{\gamma}_{32}\\ \tilde{\beta}_{31} = -27+12\tilde{\gamma}_{32}\\ \tilde{\gamma}_{30} = 12-5\tilde{\gamma}_{32}\\ \tilde{\gamma}_{31} = 18-8\tilde{\gamma}_{32}\end{cases} \tag{23}$$

with $\tilde{\gamma}_{32}$ free, where

$$\begin{aligned}t_3(x_{n+1}) &= (\frac{3}{2}-\frac{1}{2}\tilde{\gamma}_{32})h^4 y^{(4)}(x_{n+1}) + (\frac{3}{2}-\frac{13}{30}\tilde{\gamma}_{32})h^5 y^{(5)}(x_{n+1})\\ &\quad +O(h^6) \quad \text{for } j=3;\end{aligned}$$

and

$$\begin{cases}\tilde{\beta}_{40} = 81-12\tilde{\gamma}_{42}-36\tilde{\gamma}_{43}\\ \tilde{\beta}_{41} = -80+12\tilde{\gamma}_{42}+36\tilde{\gamma}_{43}\\ \tilde{\gamma}_{40} = 36-5\tilde{\gamma}_{42}-16\tilde{\gamma}_{43}\\ \tilde{\gamma}_{41} = 48-8\tilde{\gamma}_{42}-21\tilde{\gamma}_{43}\end{cases} \tag{24}$$

with $\tilde{\gamma}_{42}, \tilde{\gamma}_{43}$ free, where

$$t_4(x_{n+1}) = (6 - \frac{1}{2}\tilde{\gamma}_{42} - \frac{5}{2}\tilde{\gamma}_{43})h^4 y^{(4)}(x_{n+1}) + (\frac{36}{5} - \frac{13}{30}\tilde{\gamma}_{42} - \frac{14}{5}\tilde{\gamma}_{43})h^5 y^{(5)}(x_{n+1}) + O(h^6) \quad \text{for } j = 4;$$

By the same way as in case I, we can prove that the global error of fifth order. Thus, with consider $\tilde{\gamma}_{32} = \gamma_{32}$ and $\tilde{\gamma}_{43} = \gamma_{43}$, we have a two-parameter family of extended one-step fifth order methods, which will refer it by $PM_5 = (\gamma_{32}, \gamma_{43})$.

4 Stability definitions and results

The stability investigations are based on the linear equation (4) and the concept of P-stability introduced by Barwell [3]

Definition 1.1. (P-stability region) Given a numerical method for solving (2), the P-stability region of the method is the set S_P of the pairs (X,Y), $X = \lambda h$ and $Y = \mu h$, such that the numerical solution of (2) asymptotically vanishes for step-lengths h satisfying

$$h = \frac{\tau}{m} \tag{25}$$

with m is positive integer.

Definition 1.2. (P-stability) A numerical method for (2) is said to be P-stable if

$$S_P \supseteq R,$$

where

$$R = \{(X,Y) : Y < -X\}.$$

4.1 Case I, $m = 4$

In order to solve the Problem (2), the present methods with $m = 4$ are written as follows

$$\begin{aligned}&\big[24 - 12\lambda h(1+\gamma_{32}) + 2(\lambda h)^2(1+4\gamma_{32} + \gamma_{32}\gamma_{20}) - \gamma_{32}(\lambda h)^3(2+\gamma_{20})\big]y_{n+1} = \\ &\big[24 + 12\lambda h(1-\gamma_{32}) + 2(\lambda h)^2(1-2\gamma_{32}+\gamma_{32}\gamma_{20}) + \gamma_{32}\gamma_{20}(\lambda h)^3\big]y_n + \mu h[(9+\lambda h(2-5\gamma_{32}) \\ &+\gamma_{32}\gamma_{20}(\lambda h)^2)y(x_n-\tau) + (19 - 2\lambda h(1+4\gamma_{32}) + \gamma_{32}(\lambda h)^2(2+\gamma_{20}))y(x_{n+1}-\tau) \\ &-(5-\lambda h\gamma_{32})y(x_{n+2}-\tau) + y(x_{n+3}-\tau),]\end{aligned} \tag{26}$$

with a constant step size h satisfying the constraint (25). The characteristic polynomial associated with (26) takes the form

$$\begin{aligned}W_m(z) = &\big[24 - 12X(1+\gamma_{32}) + 2X^2(1+4\gamma_{32} + \gamma_{32}\gamma_{20}) - X^3\gamma_{32}(2+\gamma_{20})z^{m+1} \\ &- \big[24 + 12X(1-\gamma_{32}) + 2X^2(1-2\gamma_{32}+\gamma_{32}\gamma_{20}) + X^3\gamma_{32}\gamma_{20}\big]z^m - Y\big[9 + X(2-5\gamma_{32}) + X^2\gamma_{32}\gamma_{20} \\ &+ (19 - 2X(1+4\gamma_{32}) + X^2\gamma_{32}(2+\gamma_{20}))z - (5 - X\gamma_{32})z^2 + z^3\big] = 0, \quad m = 1,2,\ldots\end{aligned} \tag{27}$$

It is clear that $(X,Y) \in S_P$ if and only if all roots of the polynomials W_m are inside the unit disc for $m = 1,2,\ldots$. Let

$$\begin{aligned}P(z) := &\big[24 - 2X(1+\gamma_{32}) + 2X^2(1+4\gamma_{32}+\gamma_{32}\gamma_{20}) - X^3\gamma_{32}(2+\gamma_{20})\big]z^{m+1} - [24 + 12X(1-\gamma_{32}) \\ &+ 2X^2(1-2\gamma_{32}+\gamma_{32}\gamma_{20}) + X^3\gamma_{32}\gamma_{20}]z^m, \\ Q(z) := &-Y\big[9 + X(2-5\gamma_{32}) + X^2\gamma_{32}\gamma_{20} + (19 - 2X(1+4\gamma_{32}) + X^2\gamma_{32}(2+\gamma_{20}))z - (5 - X\gamma_{32})z^2 + z^3\big],\end{aligned} \tag{28}$$

and z^* denotes the only nonzero root of $P(z)$. It follows from Rouche's theorem, see Marden [17], that $(X,Y) \in S_P$ if $[z^*] < 1$ and $|P(z)| > |Q(z)|$ on the unit circle. Furthermore, on the unit circle we have

$$\begin{aligned}|P(z)| \geq &||24 - 12X(1+\gamma_{32}) + 2X^2(1+4\gamma_{32}+\gamma_{32}\gamma_{20}) - X^3\gamma_{32}(2+\gamma_{20})| - |24 + 12X(1-\gamma_{32}) \\ &+ 2X^2(1-2\gamma_{32}+\gamma_{32}\gamma_{20}) + X^3\gamma_{32}\gamma_{20}||, \\ |Q(z)| \leq &|Y|(|9 + X(2-5\gamma_{32}) + X^2\gamma_{32}\gamma_{20}| + |19 + 2X(1+4\gamma_{32}) + X^2\gamma_{32}(2+\gamma_{20})| \\ &|-5 + X\gamma_{32}| + 1).\end{aligned} \tag{29}$$

Therefore, $(X,Y) \in S_P$ if the following set of inequalities is satisfied

$$\begin{aligned}&||24 - 12X(1+\gamma_{32}) + 2X^2(1+4\gamma_{32}+\gamma_{32}\gamma_{20}) - X^3\gamma_{32}(2+\gamma_{20})| - |24 + 12X(1-\gamma_{32}) \\ &+ 2X^2(1-2\gamma_{32}+\gamma_{32}\gamma_{20}) + X^3\gamma_{32}\gamma_{20}|| \geq \\ &|Y|(|9 + X(2-5\gamma_{32}) + X^2\gamma_{32}\gamma_{20} + |19 - 2X(1+4\gamma_{32}) + X^2\gamma_{32}(2+\gamma_{20})| + |-5 + X\gamma_{32}| + 1),\end{aligned} \tag{30}$$

and

$$\left|\frac{24 + 12X(1-\gamma_{32}) + 2X^2(1-2\gamma_{32}+\gamma_{32}\gamma_{20}) + X^3\gamma_{32}\gamma_{20}}{24 - 12X(1+\gamma_{32}) + 2X^2(1+4\gamma_{32}+\gamma_{32}\gamma_{20}) - X^3\gamma_{32}(2+\gamma_{20})}\right| < 1. \tag{31}$$

It can be seen that $X \in S_A$ where S_A is the A-stability region of the present methods for solving ordinary differential equation if and only if (31) is satisfied, we refer to Hairer et al. [9] for more details concerning the A-stability concept. It is easy to see that (31) is satisfied if

1. $\gamma_{32} = 0$, with γ_{20} free to choose or
2. $\gamma_{32} > 0$ and $\gamma_{20} \geq -1$

Moreover, the *P*-stability region for various values of free parameters is determined by solving the system of inequalities (30) and (31). Thus we establish the following.

Theorem 1. *For the present methods, the region of P-stability satisfies the relation*

$$S_P \cap R = \{(X,Y) : |Y| < -X \text{ and } |Y| < \phi(X)\}$$

where

$$\phi(X) = \begin{cases} \dfrac{-12X}{17} & \text{for} \quad X \geq \dfrac{-9}{2} \\ \dfrac{-6X}{4-X} & \text{for} \quad X < \dfrac{-9}{2} \end{cases}$$

for $\gamma_{32} = 0$ *and* γ_{20} *free to choose.*

Proof. *The proof follows immediately from inequality (30).*

From among values for the case (2), the choice $\gamma_{20} = 0$ *and* $\gamma_{32} = \frac{1}{2}$ *give the large stability region, so we will present only the theorem of this choice as the following:*

Theorem 2. *For the present methods the region of P-stability satisfies the relation*

$$S_P \cap R = \{(X,Y) : Y < -X \text{and } |Y| < \phi(X)\},$$

where

$$\phi(x) = \begin{cases} \dfrac{-2X^3 - 12X^2 + 48X}{68 - 14X + X^2}, & \text{if} \quad X \geq -4 \\ \dfrac{-2X^3 + 12X^2 - 24X + 96}{68 - 14X + X^2}, & \text{if} \quad X < -4, \end{cases}$$

for $\gamma_{20} = 0$ *and* $\gamma_{32} = \frac{1}{2}$.

Proof. *The proof follows immediately from inequality (27).*
The **Fig. 2** *shows the different regions of the P-stability with respect to different values of* γ_{20} *and* γ_{32}.

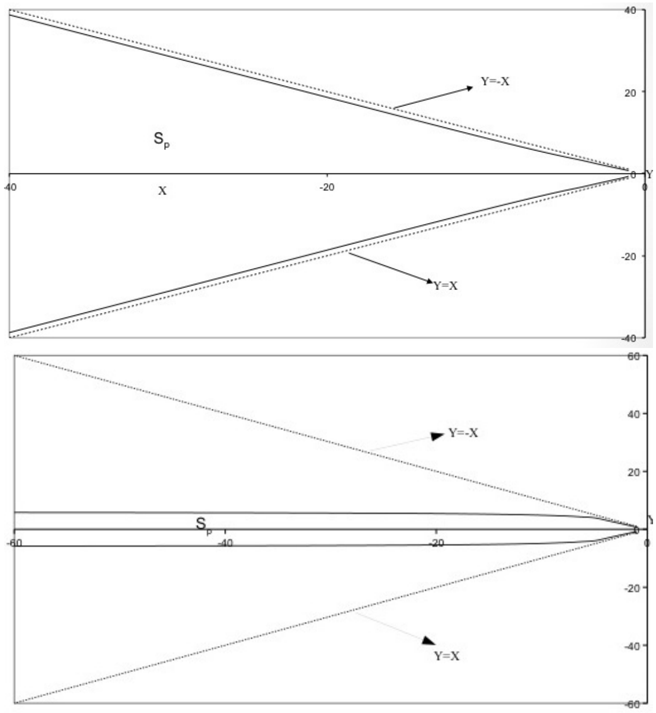

Fig. 1 The *P*-stability region for $PM_4(0, \frac{1}{2})$ and $PM_4(-1,1)$ (Top-Bottom).

4.2 Case II, $m = 5$

By the same way for m=5, we obtain the following characteristic polynomial

$$\begin{aligned} W_m(z) = \big[&720 - 8X(48 + 57\gamma_{43}) + 4X^2(21 + 57\gamma_{43} \\ &+ 57\gamma_{43}\gamma_{32}) - 2X^3(4 + 19\gamma_{43} + 11\gamma_{43}\gamma_{32}) \\ &+ 76X^4\gamma_{43}\gamma_{32}\big]z^{m+1} - [720 + 12X(28 - 38\gamma_{43}) \\ &+ 6X^2(10 - 38\gamma_{43} + 38\gamma_{43}\gamma_{32}) + 2X^3(2 - 19\gamma_{43}) \\ &- 38X^4\gamma_{32}\gamma_{43}]z^m - Y\,[251 + X(50 - 171\gamma_{43}) \\ &+ X^2(4 - 38\gamma_{43} + 95\gamma_{43}\gamma_{32}) + (646 - X(76 + 361\gamma_{43}) \\ &+ 2X^2(4 + 19\gamma_{43} + 76\gamma_{43}\gamma_{32}) - 76X^3\gamma_{43}\gamma_{32})z \\ &- (264 - X(2 + 95\gamma_{43}) + 19X^2\gamma_{43}\gamma_{32})z^2 + (106 \\ &- 19X\gamma_{43})z^3 - 19z^4] = 0, \quad m = 1,2,\ldots \end{aligned} \tag{32}$$

It is clear that $(X,Y) \in S_P$ *if and only if all roots of the polynomials* W_m *are inside the unit disc for* $m = 1,2,\ldots$.

Let

$$\begin{aligned}P(z) :=& \big[720-8X(48+57\gamma_{43})+4X^2(21+57\gamma_{43}\\&+57\gamma_{43}\gamma_{32})-2X^3(4+19\gamma_{43}+11\gamma_{43}\gamma_{32})\\&+76X^4\gamma_{43}\gamma_{32}\big]z^{m+1}-[720+12X(28\\&-38\gamma_{43})+6X^2(10-38\gamma_{43}+38\gamma_{43}\gamma_{32}))\\&+2X^3(2-19\gamma_{43}-38X^4\gamma_{32}\gamma_{43}]z^m\\Q(z) :=& -Y\big[251+X(50-171\gamma_{43})+X^2(4-38\gamma_{43}\\&+95\gamma_{43}\gamma_{32})+(646-X(76+361\gamma_{43})\\&+2X^2(4+19\gamma_{43}+76\gamma_{43}\gamma_{32})\\&-76X^3\gamma_{43}\gamma_{32})z-(264-X(2+95\gamma_{43})\\&+19X^2\gamma_{43}\gamma_{32})z^2+(106-19X\gamma_{43})z^3\\&-19z^4\big]\end{aligned} \quad (33)$$

and z^* denotes the only nonzero root of $P(z)$. It follows from Rouche's theorem, see Marden [17], that $(X,Y) \in S_P$ if $[z^*] < 1$ and $|P(z)| > |Q(z)|$ on the unit circle. Furthermore, on the unit circle we have

$$\begin{aligned}|P(z)| \geq& ||720-8X(48+57\gamma_{43})+4X^2(21+57\gamma_{43}\\&+57\gamma_{43}\gamma_{32})-2X^3(4+19\gamma_{43}+11\gamma_{43}\gamma_{32})\\&+76X^4\gamma_{43}\gamma_{32}-|720+12X(28-38\gamma_{43})\\&+6X^2(10-38\gamma_{43}+38\gamma_{43}\gamma_{32})+2X^3(2\\&-19\gamma_{43})-38X^4\gamma_{32}\gamma_{43}||\\|Q(z)| \leq& |Y|(|251+X(50-171\gamma_{43})+X^2(4-38\gamma_{43}\\&+95\gamma_{43}\gamma_{32})|+|(646-X(76+361\gamma_{43})+2X^2(4\\&+19\gamma_{43}+76\gamma_{43}\gamma_{32})-76X^3\gamma_{43}\gamma_{32})|+|-264\\&+X(2+95\gamma_{43})-19X^2\gamma_{43}\gamma_{32})|+|106\\&-19X\gamma_{43}|+19)\end{aligned} \quad (34)$$

Therefore, $(X,Y)_P$ if the following set of inequalities are satisfied

$$\begin{aligned}&||720-8X(48+57\gamma_{43})+4X^2(21+57\gamma_{43}+57\gamma_{43}\gamma_{32})\\&-2X^3(4+19\gamma_{43}+11\gamma_{43}\gamma_{32})+76X^4\gamma_{43}\gamma_{32}-|720\\&+12X(28-38\gamma_{43})+6X^2(10-38\gamma_{43}+38\gamma_{43}\gamma_{32})\\&+2X^3(2-19\gamma_{43})-38X^4\gamma_{32}\gamma_{43}|| \geq\\&|Y|(|251+X(50-171\gamma_{43})+X^2(4-38\gamma_{43}+95\gamma_{43}\gamma_{32})|\\&+X^2(4-38\gamma_{43}+95\gamma_{43}\gamma_{32})|+|(646-X(76+361\gamma_{43})\\&+2X^2(4+19\gamma_{43}+76\gamma_{43}\gamma_{32})-76X^3\gamma_{43}\gamma_{32})||-264\\&+X(2+95\gamma_{43})-19X^2\gamma_{43}\gamma_{32})|+|106-19X\gamma_{43}|+19)\end{aligned} \quad (35)$$

and

$$|\frac{A_1}{A_2}| < 1 \quad (36)$$

where

$$\begin{aligned}A_1 =& 720+12X(28-38\gamma_{43})+6X^2(10-38\gamma_{43}+38\gamma_{43}\gamma_{32})\\&+2X^3(2-19\gamma_{43})-38X^4\gamma_{32}\gamma_{43}\end{aligned}$$

and

$$\begin{aligned}A_2 =& 720-8X(48+57\gamma_{43})+4X^2(21+57\gamma_{43}+57\gamma_{43}\gamma_{32})\\&-2X^3(4+19\gamma_{43}+11\gamma_{43}\gamma_{32})+76X^4\gamma_{43}\gamma_{32}\end{aligned}$$

It can be seen that $X \in S_A$ where S_A is the A-stability region of the present methods for solving ordinary differential equation if and only if (36) is satisfied, we refer to Hairer et al. [9] for more details concerning the A-stability concept. It is easy to see that (35) is satisfied if

1. *$\gamma_{43} = 0$, with γ_{32} free to choose or*
2. *$\gamma_{32} = 0$ and $\gamma_{43} \geq \frac{-1}{19}$.*

Moreover, the P-stability region for various values of free parameters is determined by solving the system of inequalities(35) and (36). Thus we establish the following.

Theorem 3. *For the present methods, the region of P-stability satisfies the relation*

$$S_P \cap R = \{(X,Y) : |Y| < -X \text{ and } |Y| < \phi(X)\}$$

where

$$\phi(X) = \begin{cases} \dfrac{-3X^3+6X^2-720}{3X^2-7X+319} & \text{for} \quad X \geq -6 \\ \dfrac{-X^3+36X^2-12X+360}{3X^2-7X+319} & \text{for} \quad X < -6 \end{cases}$$

for $\gamma_{43} = 0$ and γ_{32} free to choose.

Proof. *The proof follows immediately from inequality (35)).*

The **Fig. 2** *shows the different regions of the P-stability with respect to different values of γ_{43} and γ_{32}.*

In the next part of this section, we state the error estimate for the present methods (4), (5) and (6). Our error estimate is given by the following theorem:

Theorem 4. *Let y_n be obtained by the methods (4), (5) and (6). Then, at each mesh point x_n, we have the following error estimate:*

$$e_n = |y(x_n) - y_n| \leq C_1 h^m, \quad n = 1,2,\ldots \quad (37)$$

where $m = 4,5$ and C_1 is independent of n and h .

Proof. *see(Ibrahim et al.)*

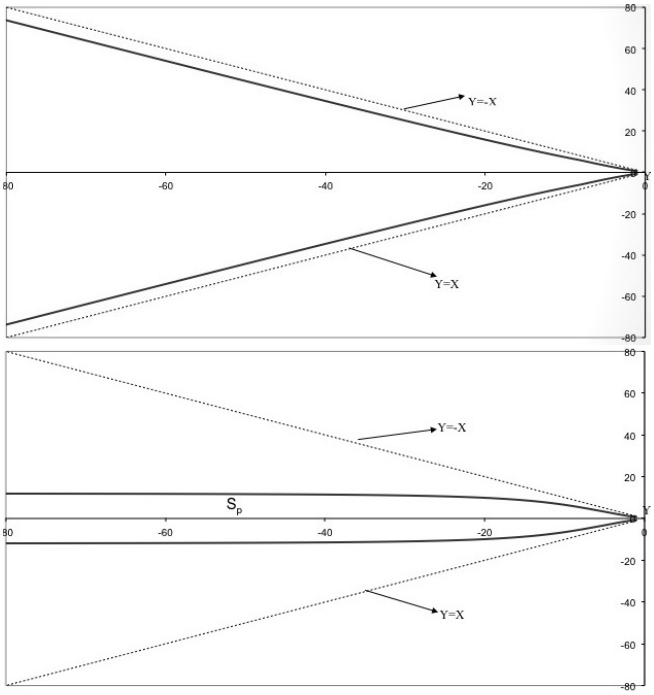

Fig. 2 The P-stability region for $PM_5(0,\frac{2}{19})$ and $PM_5(0,\frac{-1}{19})$ (Top-Bottom).

5 Numerical tests

In this section, we present some numerical results using $PM_4(\gamma_{20},\gamma_{32})$ and $PM_5(\gamma_{32},\gamma_{43})$ with different values of free parameters and also compare the results with Runge-Kutta method. We apply these methods to three examples for each $h=\frac{1}{N}$ where $N=4,8,16,32,64$ and 128 .

Example 1

$$y\prime(x)=\frac{1}{2}e^{\frac{x}{2}}y(\frac{x}{2})+\frac{x}{2}y(x) \qquad 0\le x\le 1$$
$$y(0)=1$$

The exact solution is $y(x)=e^x$.

Example 2

$$y\prime(x)=1-y(\frac{x}{2})^2 \qquad 0\le x\le 1$$
$$y(0)=0$$

The exact solution is $y(x)=\sin(x)$.

***Example 3**: Paul [25]*

$$y_1\prime(x)=y_1(x-1)+y_2(x),\ x\ge 0$$
$$y_2\prime(x)=y_1(x)-y_1(x-1)$$
$$y_1(x)=e^x,\ x\le 0$$
$$y_2(0)=1-e^{-1}$$

The exact solution is
$y1(x)=e^x,\ y_2(x)=e^x-e^{x-1},\ x\ge 0.$

	Runge-Kutta method			
	s=2		s=3	
N	E^N	R^N	E^N	R^N
4	7.80E-02		3.83E-03	
8	2.29E-02	1.77	5.19E-04	2.88
16	6.28E-03	1.87	6.81E-05	2.93
32	1.64E-03	1.93	8.73E-06	2.96
64	4.2E-04	1.97	1.11E-06	2.98
128	1.06E-04	1.98	1.39E-07	2.99
	A class of extended one-step methods			
	$PM_4(0,\frac{1}{5})$		$\mathrm{PM}_5(0,\frac{2}{19})$	
N	E^N	R^N	E^N	R^N
4	1.04E-05		1.39E-06	
8	6.06E-07	4.11	4.05E-08	5.10
16	3.66E-08	4.05	1.23E-09	5.05
32	2.25E-09	4.03	3.77E-11	5.02
84	1.39E-10	4.01	1.17E-12	5.01
128	8.66E-12	4.01	4.06E-14	4.85

Table 1 Comparison of class extended one-step methods with Runge-Kutta method for Example 1.

	Runge-Kutta method			
	s=2		s=3	
N	E^N	R^N	E^N	R^N
4	4.95E-03		1.64E-03	
8	1.21E-03	2.03	1.91E-04	3.10
16	3.38E-04	1.84	2.30E-05	3.05
32	9.17E-05	1.88	2.80E-06	3.04
64	2.35E-05	1.96	3.45E-07	3.02
128	5.94E-06	1.99	4.28E-08	3.01
	A class of extended one-step methods			
	$PM_4(0,\frac{1}{5})$		$\mathrm{PM}_5(0,\frac{2}{19})$	
N	E^N	R^N	E^N	R^N
4	3.46E-06		2.77E-07	
8	2.23E-07	3.96	7.88E-09	5.13
16	1.42E-08	3.98	2.33E-10	5.07
32	8.92E-10	3.99	7.18E-12	5.03
84	5.59E-11	3.99	2.22E-13	5.02
128	3.50E-15	4.00	7.10E-15	4.96

Table 2 Comparison of class extended one-step methods with Runge-Kutta method for Example 2.

	Runge-Kutta method			
	$y_1(x)$		$y_2(x)$	
N	E^N	R^N	E^N	R^N
4	6.99E-03		5.87E-03	
8	9.63E-04	2.86	8.14E-04	2.85
16	1.26E-04	2.93	1.07E-04	2.93
32	1.62E-05	2.96	1.37E-05	2.96
64	2.05E-06	2.98	1.74E-06	2.98
128	2.57E-07	2.99	2.19E-07	2.99
	Fourth order method $PM_4(0, \frac{1}{2})$			
	$y_1(x)$		$y_2(x)$	
N	E^N	R^N	E^N	R^N
4	4.00E-06		2.30E-05	
8	5.00E-07	4.10	1.20E-06	4.26
16	6.26E-08	4.05	6.82E-08	4.14
32	7.80E-09	4.02	4.05E-09	4.07
64	9.74E-10	4.01	2.47E-10	4.04
128	4.00E-06	4.01	1.53E-11	4.02
	$PM_5(0, \frac{2}{19})$			
	$y_1(x)$		$y_2(x)$	
N	E^N	R^N	E^N	R^N
4	3.19E-05		3.54E-05	
8	8.57E-07	5.22	9.62E-07	5.20
16	2.48E-08	5.11	2.80E-08	5.10
32	7.46E-10	5.05	8.46E-10	5.05
64	2.29E-11	5.03	2.60E-11	5.03
128	7.14E-13	5.00	8.00E-13	5.02

Table 3 Comparison of class extended one-step methods with Runge-Kutta method for Example 3.

6 Conclusion and perspective

we have described a class of numerical methods of order four and five for solving delay differential equation by extending the work of Chawla et all. (1994, 1995). These methods depended on two free parameters, so we can obtain for every method on a family of methods for different value of a free parameters. The region of P-stability for the present methods has been investigated for different values of a free parameters. The large -stability region for the present method of order four occurs at $\gamma_{20} = 0$ and $\gamma_{32} = \frac{1}{2}$, see Fig. 1, further the largeP -stability region for the present method of order five occurs at $\gamma_{32} = 0$ and $\gamma_{43} = \frac{2}{19}$, see Fig. 2. In the last cases, the present methods are L-stable for solving ordinary differential equations. All the obtained numerical results clearly indicate the effectiveness of our methods.

References

[1] *Al-Mutib, A. N. Stability properties of numerical methods for solving delay differential equations,* J. Comput. Appl. Math. **10**, *71-79 (1984)*

[2] *Arndt, H., Numerical solution of retarded initial value problems: local and global error and step size control,* J. Numer. Math. **43**, *71-79 (1984).*

[3] *Barwell, V. K., Special stability problems for functional differential equations,* BIT **15**, *130-135 (1975).*

[4] *Bellen, A. and Zennaro, M., Numerical solution of delay differential equations by uniform corrections to an implicit Runge-Kutta method, Numer. Math.* **47**, *301-316 (1985) .*

[5] *Calvo, M. and Grande, T., On the asymptotic stability of the θ-methods for delay differential equations, Numer. Math.* **54**, *257-269 (1988).*

[6] *Engelborghs , K., Luzyanina, T., In't Hout, K.J., and Roose, D., Collocation methods for the computation of periodic solutions of delay differential equations, SIAM J. Sci. Comput.* **22**, *1593-1609 (2000).*

[7] *Driver R. D.,* Ordinary and Delay Differential equations, *Springer-Verlag, New York, 1977.*

[8] *Guglielmi, N., Delay dependent stability regions of θ-methods for delay differential equations,* IMA J. Numer. Anal. **18**, *399-418 (1998).*

[9] *Hairer, E., NÖresett, S.P. and Wanner, G.,* Solving Ordinary Differential equations I, Non stiff Problems, *Springer-Verlag, New York, 1993.*

[10] *Hayashi, H.,* Numerical Solution of Retarded and Nutral Delay Differential Delay Differential Equations using continuous Runge-Kutta Methods, *PhD Thesis, University of Toronto 1996.*

[11] *Henrici, P.,* Discrete variable methods in ordinary Differential equations, *John Willey, New York, 1962.*

[12] *Hu, G. -D. and Cahlon, B., The numerical solution of discrete-delay system,* Appl. Math Comput. **124**, *403-411(2001).*

[13] *In't Hout, K. J. and Spijker, M. N., Stability analysis of numerical methods for delay differential equations,* Numer. Math. **59**, *807-814 (1991).*

[14] *Jacques, I. B., Extende one-step methods for the numerical solution of ordinary differential equations,* Intern. J. Computer Math. **29**, *247-255 (1989)*

[15] *Kondrat, D. M. and Jacques, I. B., Extended A-stable two-step methods for the numerical solution of ordinary differential equations,* Intern. J. Computer Math. **42**, *117-154 (1985).*

[16] *Liu, M. Z. and Spijker, M. N., The stability of θ-methods in the numerical solution of delay differential equations,* IMA J. Numer. Anal. **10**, *31-48 (1990).*

[17] *Marden, M.,* The Geometry of the zeros of polynomial in a complex variable, *American Math Society, New York1949.*

[18] *Oberle, H. J. and Pesch, H. J., Numerical treatment of delay differential equations by Hermite interpolation,* Numer. Math. **37**, *235-255 (1981).*

[19] *Paul, C. A. and Baker, C. T. H., Explicit Runge-Kutta methods for numerical solutions of singular delay differential equations,* MA Report MCCM report No **212**, *University of Manchester (1992).*

[20] *Torelli, L. and Vermiglio, L., On the stability of continuous quadrature rules for differential equations with several constant delays,* IMA. J. Numer. Anal. **13**, *291-302 (1993)*

[21] *Van Den Heuvel, E. G., New stability for Runge-Kutta methods adapted to delay differential equations,* Appl. Numer. Math. **34**, *293-302 (2000).*

[22] *Van Den Heuvel, E. G., Using resolvent conditions to obtain new stability results for θ-methods for delay differential equation,* IMA J. Numer. Anall. **21**, *421-438 (2001)*

[23] *Chawla, M.M. , Al-Zannaidi, M. A. and Al- Sahhar, M. S., Stabilized fourth order extended methods for the numerical solution of ODEs,* Computers Maths. Intern. J. Computer. Math. **52***, 99-107 (1994)*

[24] *Chawla, M.M. , Al-Zannaidi, M. A. and Al- Sahhar, M. S., A class of stabilized extended one-step methods for the numerical solution of ODEs,* Computers Math. Applic., **29***, 79-84 (1995)*

[25] *C.A.H. Paul, Runge-Kutta Methods for Functional Di. Eqns., Ph.D. thesis, Math. Dept., Manchester Univ. (1992).*

[26] *Usmani, R .A. and Agarwal, R .P., An A-stable extended trapezoidal rule for numerical integration of ordinary equations,* Computers Maths. Applic. **11**, *1183-1191 (1985)*

[27] *Gopalsamy, K., Stability and oscillations in delay differential equations of population dynamics,* Kulwer Academic publishers, London*, (1992)*

[28] *Kuang,Y., Delay differential equations with applications in population dynamics,* Academic Press, London*, (1993)*