Optimal Solvers for Elliptic Optimal Control Problems

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Outline

- Motivating Examples
- Existence and Uniqueness

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Part 1: Motivating Examples

Optimal stationary boundary temperature: Heating of a body $\Omega$by a controlled boundary temperature u to reach the desired temperature y^d

$$\inf_{y,u} \; J(y,u) = \frac{1}{2}||y - y^d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Gamma)} \qquad \text{(P)}$$

subject to state equation

$$-\Delta y = 0 \quad \text{in } \Omega$$

$$y = u \quad \text{on } \Gamma,$$

and control constraints

$$a \leq u \leq b \quad \text{on } \Gamma$$

Optimal stationary heat source: The body Ω is heated e.g. by microwaves. The goal is to find u that minmizes the distance of y and y^d

$$\inf_{y,u} J(y,u) = \frac{1}{2}||y - y^d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)} \qquad \text{(P1)}$$

subject to state equation

$$-\Delta y = u \quad \text{in } \Omega$$

$$y = 0 \qquad \text{on } \Gamma,$$

Notation and data

- J is the objective functional and y^d is the desired state
- $\alpha > 0$ is regularization parameter
- Optimal control problems with linear state equation and quadratic objective functional called linear-quadratic
- (P) is a linear-quadratic elliptic boundary control Problem
- (P1) is a linear-quadratic elliptic distributed control Problem

Part 2: Existence and Uniqueness of optimal control problem

- Problem P1: Defining the control- to- state map $G : u \to y$ from $L^2(\Omega)$ to $H_0^1(\Omega)$, a linear and bounded operator which assigns to a control $u \in L^2(\Omega)$ the unique solution $y = y(u)$ of the state equation,writing the state $y = Gu$
- introduce the reduced objective functional

$$\inf_{y,u} J_{red}(u) := \frac{1}{2}||Gu - y^d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)}$$

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Theorem

The reduced optimal control problem has a unique optimal solution $(y, u) \in H_0^1(\Omega) \times L^2(\Omega)$.

Proof

Direct method of the calculus of variations(minimizing sequence) .

Part 3: Optimize first, then discretize

Theorem

Let $(y, u) \in H_0^1(\Omega) \times L^2(\Omega)$ *be the optimal solution of Problem P1. Then, there exists an adjoint state* $p \in H_0^1(\Omega)$ *as a weak solution of*

$$-\Delta p = \; y - y^d \quad \text{in } \Omega$$

$$p = \; 0 \qquad\qquad \text{on } \Omega$$

and the control equation

$$p + \alpha u = 0 \quad \text{in } \Omega.$$

Finite Element Approximation of Distributed Control Problem

Let $\mathcal{T}_h := \mathcal{T}_h(\Omega)$ be a shape-regular, quadrilateral triangulation and let

$$V_h := \{v_h \in C(\Omega) \mid v_h|_T \in Q_1(T)\ , T \in \mathcal{T}_h(\Omega),\ \mid v_h|_\Gamma = 0\}$$

be the FE space of continuous piecewise linear finite elements. Then a possible FE approximation of (P1) is as follows:

$$\inf J(y_h, u_h) := \frac{1}{2} \int_\Omega |y_h - y^d|^2\ dx\ +\ \frac{\alpha}{2} \int_\Omega |u_h|^2\ dx$$

subject to

$$a(y_h, v_h) = (u_h, v_h), \qquad \forall v_h \in V_h$$

Optimality Conditions for the FE Distributed Control Problem

there exists an adjoint state $p_h \in V_h$ such that the triple (y_h, u_h, p_h) satisfies the following optimality conditions,

$$a(y_h, v_h) = (u_h, v_h), \qquad \forall v_h \in V_h$$

$$a(p_h, v_h) = (y_h - y^d, v_h), \quad \forall v_h \in V_h$$

$$(u_h, v_h) = -\alpha^{-1}(p_h, v_h) \qquad \forall v_h \in V_h$$

The discrete KKT system

$$\begin{pmatrix} A_h & 0 & -M_h \\ -M_h & A_h & 0 \\ 0 & M_h & \alpha M_h \end{pmatrix} \begin{pmatrix} y_h \\ p_h \\ u_h \end{pmatrix} = \begin{pmatrix} 0 \\ -y_h^d \\ 0 \end{pmatrix}$$

A Reduced KKT system

If we substitute $u$in the state equation by means of the control equation according to $u = -\alpha^{-1}\, p_h$, the discrete KKT system can be stated as

$$\begin{pmatrix} A_h & \alpha^{-1} M_h \\ M_h & A_h \end{pmatrix} \begin{pmatrix} y_h \\ p_h \end{pmatrix} = \begin{pmatrix} 0 \\ -y_h^d \end{pmatrix}.$$

There exist also solution methods, which rely on a reduction of the KKT system. In this method, the optimality system is reduced to a single integral equation for the control u

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The integral equation method

We introduce the operator $\mathcal{A} : y \mapsto (-\Delta y)$ and the adjoint operator $\mathcal{A}^* : p \mapsto (-\Delta p)$. The KKT system reads in the following form:

$$\begin{aligned} \mathcal{A} &= u && \text{in } \Omega \\ \mathcal{A}^* &= y - y^d && \text{in } \Omega \\ u &= -\alpha^{-1} p && \text{in } \Omega. \end{aligned}$$

The last equation can be reformulated by the first two equations to the fixed point equation

$$u = (-\alpha^{-1}\mathcal{A}^{-*}\mathcal{A}^{-1})u - (-\alpha^{-1}\mathcal{A}^{-*})y^d$$

$$= \mathcal{K}u + q$$

which we can rewrite it in the compact form as follows:

$$(\mathcal{I} - \mathcal{K})u = q.$$

Based on this compact form, we formulate the following fixed point iteration:

$$u^n := \mathcal{K}u^{n-1} + q, \quad n \in \mathcal{N},$$

Part 4: Numerical results

Example

$$
\begin{aligned}
u &= \sin(3\pi x1)\sin(3\pi x2) \\
y &= \sin(3\pi x1)\sin(3\pi x2) \\
p &= -\alpha\sin(3\pi x1)\sin(3\pi x2)
\end{aligned}
$$

We get the corresponding desired state as

$$
y^d = 18\pi^2\alpha\sin(3\pi x1)\sin(3\pi x2) + \sin(3\pi x1)\sin(3\pi x2).
$$

Table: Comparison of iterations to solve Example 1. with different values of α and different mesh sizes for tolerance of $1E-12$ for Multigrid with different smoother (A Reduced KKT system)

smoother	level \ α	1	1E-02	1E-04	1E-08	1E-12
Jacobi(0.5)	6	7	7	9	div.	div.
	7	6	6	7	10	div.
	8	5	5	5	7	div.
	9	4	4	4	5	div.
Gaus Seidel	6	5	5	6	div.	div.
	7	5	5	6	div.	div.
	8	5	5	5	27	div.
	9	6	6	6	9	div.
SSOR(0.5)	6	7	7	8	div.	div.
	7	6	6	7	div.	div.
	8	6	6	6	10	div.
	9	6	6	6	8	div.

Table: Comparison of iterations to solve Example 1. with different values of α and different mesh sizes for tolerance of $1E-12$ for Multigrid with different smoother and preconditioner (A Reduced KKT system)

smoother	Preconditioner	level \ α	1	1E-02	1E-04	1E-08	1E-12
BICGSTAB	SSOR(1.5)	6	7	8	19	div.	div.
		7	8	10	10	163	div.
		8	8	8	9	28	div.
		9	8	8	9	26	div.
	SSOR(0.5)	6	4	5	7	39*	div.
		7	5	5	6	16	div.
		8	5	5	6	10	41
		9	5	5	5	8	29
GMRES(30)	SSOR(0.5)	6	4	4	7	9*	9*
		7	5	5	6	8	9
		8	5	5	5	7	10
		9	5	5	5	7	10

Table: Comparison of iterations to solve Example 1. with different values of α and different mesh sizes for tolerance of $1E-12$ for Multigrid with different smoother and preconditioner (KKT system)

smoother	Preconditioner	level $\backslash\alpha$	1	1E-02	1E-04	1E-08	1E-12
		6	4	5	9	div	div
	SSOR(0.5)	7	5	6	8	137	div.
		8	5	5	6	13	div.
BICGSTAB		9	5	6	7	9	div.
		6	4	4	7	98^{*}	104^{*}
	SSOR(0.5)	7	5	5	6	35	767
		8	5	5	5	11	1106
GMRES(30)		9	5	5	5	7	23

Table: Comparison of iterations to solve Example 1. with different values of α at different grid level for tolerance of $1E-12$ with different solver (integral equation)

solver	level $\backslash\alpha$	1	1E-02	1E-04	1E-08	1E-12
BICGSTAB	6	2	2	2	576^{*}	-
	7	2	2	2	1281	-
	8	2	2	2	2318	-
	9	2	2	2	2429	-
GMRES	6	2	2	4	3^{*}	3^{*}
	7	2	2	3	7	8
	8	2	2	3	4	4
	9	2	2	4	4	4

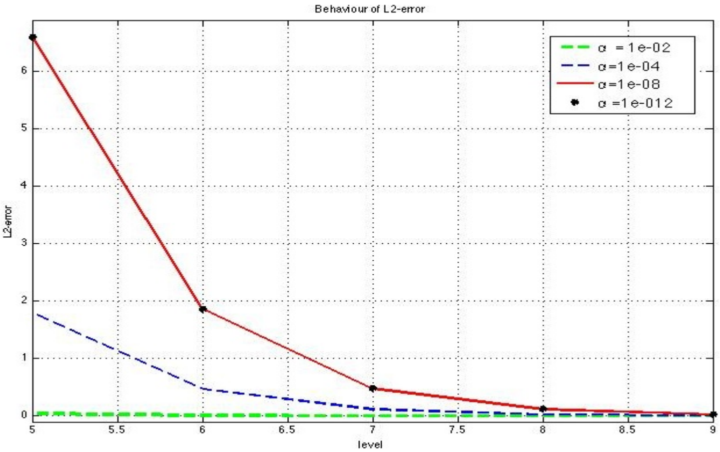

Figure: Example 1: Change in L2-error with level for different values of α

THANK YOU FOR YOUR ATTENTION!