technische universität dortmund

A Space-Time Multigrid Solver for Optimal Distributed Control of Incompressible Fluid Flow

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Flow Control Model Problem

Distributed Control of the nonstationary Navier-Stokes equation with tracking-type functional for a given z:

$$J(y,u) = \frac{1}{2}||y-z||_Q^2 + \frac{\alpha}{2}||u||_Q^2 + \frac{\gamma}{2}||y(T)-z(T)||_\Omega^2 \quad \rightarrow \quad \text{min!}$$

on $Q = \Omega \times [0,T]$ such that

$$\begin{array}{rcll} y_t - \nu\Delta y + (y\nabla)y + \nabla p & = & u & \text{in } Q \\ -\nabla\cdot y & = & 0 & \text{in } Q \end{array} \qquad + \text{BC}$$

No constraints (for simplicity).

Design goals for optimal control tools

Aim:

- Discretisation with modern CFD techniques
- Solver with optimal complexity:

$$\text{costs for optimisation} = O(N)$$

$$\frac{\text{costs for optimisation}}{\text{costs for simulation}} \le C \approx 10 - 50$$

In the following:

1.) Discretisation
2.) Solver design
3.) Numerical examples

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In the following:

1.) Discretisation
2.) Solver design
3.) Numerical examples

Corresponding KKT-System (unconstrained case):

$$
\begin{aligned}
y_t - \nu\Delta y + (y\nabla)y + \nabla p &= u && \text{in } Q\\
-\nabla \cdot y &= 0 && \text{in } Q\\
\\
-\lambda_t - \nu\Delta\lambda - (y\nabla)\lambda + (\nabla y)^t\lambda + \nabla\xi &= (y - z) && \text{in } Q\\
-\nabla \cdot \lambda &= 0 && \text{in } Q\\
\\
u &= -\tfrac{1}{\alpha}\lambda && \text{in } Q
\end{aligned}
$$

+ boundary conditions
+ initial condition
+ terminal condition $\lambda(T) = \gamma(y(T) - z(T))$ in Ω

'Reduced' KKT-System (unconstrained case):

$$
\begin{aligned}
y_t + N(y)y + \nabla p &= -\tfrac{1}{\alpha}\lambda && \text{in } Q \\
-\lambda_t + N^*(y)\lambda + \nabla \xi &= (y - z) && \text{in } Q
\end{aligned}
$$

+ incompressibility ($-\nabla \cdot y = -\nabla \cdot \lambda = 0$ in Q)
+ initial/boundary conditions
+ terminal condition $\lambda(T) = \gamma(y(T) - z(T))$ in Ω

Properties of the KKT system

Observation:

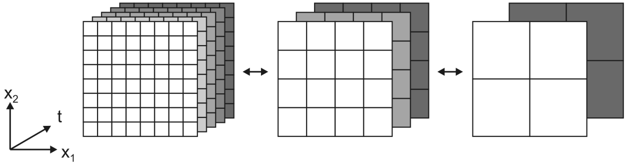
- KKT-system $\rightarrow$ *elliptic* BVP in space/time $(-y_{tt} + \Delta^2 y + \ldots)$

Idea:

- Apply 'optimal' $O(N)$ ingredients from CFD to this BVP!

  $\rightarrow$ unstructured meshes, FEM in space, implicit time-stepping
  $\rightarrow$ monolithic Multigrid + Newton solver techniques

- In particular: Solve on a space-time hierarchy



Space–time discretization

Discretization in time (θ–scheme) leads to a system

$$F(x) = A(x)x = b$$

in the form (here e.g. for 2 timesteps, IE in time):

$$A(x)x =$$

$$\left(\begin{array}{c|c|c}
\begin{array}{cc|cc}
\frac{I}{\Delta t}+N & & \nabla & \\
-I & \frac{I}{\Delta t}+N^* & & \nabla \\
\hline
-\nabla\cdot & & 0 & \\
 & -\nabla\cdot & & 0
\end{array} & -\frac{1}{\Delta t} & \\
\hline
-\frac{I}{\Delta t} & \begin{array}{cc|cc}
\frac{I}{\Delta t}+N & \frac{I}{\alpha} & \nabla & \\
-I & \frac{I}{\Delta t}+N^* & & \nabla \\
\hline
-\nabla\cdot & & 0 & \\
 & -\nabla\cdot & & 0
\end{array} & -\frac{I}{\Delta t} \\
\hline
 & -\frac{I}{\Delta t} & \begin{array}{cc|cc}
\frac{I}{\Delta t}+N & \frac{I}{\alpha} & \nabla & \\
-c(\gamma,\Delta t)I & \frac{I}{\Delta t}+N^* & & \nabla \\
\hline
-\nabla\cdot & & 0 & \\
 & -\nabla\cdot & & 0
\end{array}
\end{array}\right)
\left(\begin{array}{c}
y_0 \\ \lambda_0 \\ p_0 \\ \xi_0 \\ \hline y_1 \\ \lambda_1 \\ p_1 \\ \xi_1 \\ \hline y_2 \\ \lambda_2 \\ p_2 \\ \xi_2
\end{array}\right)$$

$\rightarrow$ **Sparse, (block) tridiagonal system**

Design of an efficient optimal control solver

- Nonlinearity: Newton method for quadratic convergence

$$x^{i+1} = x^i + F'^{-1}(x^i)(b - A(x^i)x^i)$$

- Linear subproblems: space-time MG solver (complexity $O(N)$)

  → using Block-Jacobi/Block-SOR smoothing techniques

- Linear subproblems in space: Monolithic Multigrid solver

  → using 'local Pressure-Schur complement' techniques in each timestep for the coupled Navier–Stokes subproblems

Space–time multigrid ingredients

Essential multigrid components:

- Mesh hierarchy for a space-time cylinder $Q = \Omega \times [0, T]$:
  Choose arbitrary space-time coarse mesh and refine!
- Prolongation/Restriction in space + time.
  Combination of FE in space & FD in time.
- An efficient smoother! E.g. with $\tilde{A} := A(x^i)$:

$$\tilde{A}v = d \qquad \Rightarrow \qquad v^{j+1} = v^j + \omega P^{-1}(d - \tilde{A}v^j)$$

What to use as preconditioner P?

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What to use as preconditioner P?

Space–time discretisation and preconditioner

$\tilde{A}$ in compressed form (omitting ∇ and $\nabla\cdot$ here):

$$
\left(\begin{array}{cc|cc|cc|c}
I+N & & & & & & \\
-I & \frac{I}{\Delta t}+N^* & & -\frac{I}{\Delta t} & & & \\
\hline
-\frac{I}{\Delta t} & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & & & \\
 & & -I & \frac{I}{\Delta t}+N^* & & -\frac{I}{\Delta t} & \\
\hline
 & & -\frac{I}{\Delta t} & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & \\
 & & & & -I & \frac{I}{\Delta t}+N^* & -\frac{I}{\Delta t} \\
\hline
 & & & & & \dots & \dots
\end{array}\right)
$$

$\Rightarrow$ Block-Jacobi preconditioner:

$$
P := P_{Jac} := \left(\begin{array}{cc|cc|cc|c}
I+N & & & & & & \\
-I & \frac{I}{\Delta t}+N^* & & & & & \\
\hline
 & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & & & \\
 & & -I & \frac{I}{\Delta t}+N^* & & & \\
\hline
 & & & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & \\
 & & & & -I & \frac{I}{\Delta t}+N^* & \\
\hline
 & & & & & & \dots
\end{array}\right)
$$

Space–time discretisation and preconditioner

$\tilde{A}$ in compressed form (omitting ∇ and $\nabla\cdot$ here):

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\left(\begin{array}{cc|cc|cc|c}
I+N & & & & & & \\
-I & \frac{I}{\Delta t}+N^* & & -\frac{I}{\Delta t} & & & \\
\hline
-\frac{I}{\Delta t} & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & & & \\
 & & -I & \frac{I}{\Delta t}+N^* & & -\frac{I}{\Delta t} & \\
\hline
 & & -\frac{I}{\Delta t} & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & \\
 & & & & -I & \frac{I}{\Delta t}+N^* & -\frac{I}{\Delta t} \\
\hline
 & & & & & \dots & \dots
\end{array}\right)
$$

$\Rightarrow$ Block-Jacobi preconditioner:

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-I & \frac{I}{\Delta t}+N^* & & & & & \\
\hline
 & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & & & \\
 & & -I & \frac{I}{\Delta t}+N^* & & & \\
\hline
 & & & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & \\
 & & & & -I & \frac{I}{\Delta t}+N^* & \\
\hline
 & & & & & & \dots
\end{array}\right)
$$

$\tilde{A}$ in compressed form (omitting ∇ and $\nabla\cdot$ here):

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I+N & & & & & & \\
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\hline
-\frac{I}{\Delta t} & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & & & \\
 & & -I & \frac{I}{\Delta t}+N^* & & -\frac{I}{\Delta t} & \\
\hline
 & & -\frac{I}{\Delta t} & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & \\
 & & & & -I & \frac{I}{\Delta t}+N^* & -\frac{I}{\Delta t} \\
\hline
 & & & & & \cdots & \cdots
\end{array}\right)$$

$\Rightarrow$ Block-Gauß-Seidel preconditioner:

$$P := P_{GS} := \left(\begin{array}{cc|cc|cc|c}
I+N & & & & & & \\
-I & \frac{I}{\Delta t}+N^* & & & & & \\
\hline
-\frac{I}{\Delta t} & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & & & \\
 & & -I & \frac{I}{\Delta t}+N^* & & & \\
\hline
 & & -\frac{I}{\Delta t} & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & \\
 & & & & -I & \frac{I}{\Delta t}+N^* & \\
\hline
 & & & & & \cdots & \cdots
\end{array}\right)$$

Space–time discretisation and preconditioner

$\tilde{A}$ in compressed form (omitting ∇ and $\nabla\cdot$ here):

$$
\left(\begin{array}{cc|cc|cc|c}
I+N & & & & & & \\
-I & \frac{I}{\Delta t}+N^* & & -\frac{I}{\Delta t} & & & \\
\hline
-\frac{I}{\Delta t} & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & & & \\
 & & -I & \frac{I}{\Delta t}+N^* & & -\frac{I}{\Delta t} & \\
\hline
 & & -\frac{I}{\Delta t} & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & \\
 & & & & -I & \frac{I}{\Delta t}+N^* & -\frac{I}{\Delta t} \\
\hline
 & & & & & \dots & \dots
\end{array}\right)
$$

$\Rightarrow$ Reversed Block-Gauß-Seidel preconditioner:

$$
P := P_{GS}^r := \left(\begin{array}{cc|cc|cc|c}
I+N & & & & & & \\
-I & \frac{I}{\Delta t}+N^* & & -\frac{I}{\Delta t} & & & \\
\hline
 & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & & & \\
 & & -I & \frac{I}{\Delta t}+N^* & & -\frac{I}{\Delta t} & \\
\hline
 & & & & \frac{I}{\Delta t}+N & \frac{I}{\alpha} & \\
 & & & & -I & \frac{I}{\Delta t}+N^* & \\
\hline
 & & & & & \dots & \dots
\end{array}\right)
$$

In each timestep:

$$A_{ii}(x)x_i = \left(\begin{array}{cc|cc} \frac{I}{\Delta t} + N_i & \frac{I}{\alpha} & \nabla & \\ -I & \frac{I}{\Delta t} + N_i^* & & \nabla \\ \hline -\nabla\cdot & & 0 & \\ & -\nabla\cdot & & 0 \end{array}\right) \begin{pmatrix} y_i \\ \lambda_i \\ p_i \\ \xi_i \end{pmatrix} = \begin{pmatrix} \mathrm{rhs}_i^y \\ \mathrm{rhs}_i^\lambda \\ \mathrm{rhs}_i^p \\ \mathrm{rhs}_i^\xi \end{pmatrix}$$

- Saddle-point character
- Application of A_{ii}^{-1}:
  - monolithic Multigrid solver in space
  - ‘local Pressure-Schur complement’ techniques

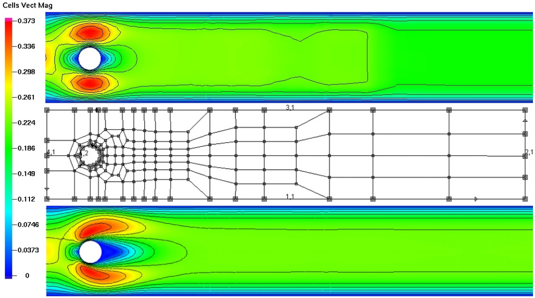
Discretisation:

- Finite Elements ($\rightarrow$ `FeatFlow`)

Numerical example

Flow-around-cylinder

- Target flow z: Stationary Stokes flow



Stokes, Target

Mesh

uncontrolled Nav.St.,
Initial condition

- Optimal control problem: Navier–Stokes, $Re = 20$, $t \in [0, 0.3]$.
- Coarse mesh: Standard DFG benchmark
  $\rightarrow$ 1404 DOF's in space, 16 timesteps, $\Delta t := 3/160$
  $\Rightarrow$ 23 868 DOF's, $\times 8$ per level

Convergence of the Newton solver

	Simulation			Optimisation			
lv.	#⊘NL	#⊘MG	T_{sim}	#NL	#MG	T_{opt}	$\frac{T_{opt}}{T_{sim}}$
2	3	11	2	5	16	112	56
3	3	11	15	5	27	1258	83
4	3	13	161	5	28	10226	63
5	3	15	1771	5	29	91529	52

- Nonlinear solver gained 8 digits
- Linear Solver: Space-Time-MG; gained 2 digits per step
- Space-preconditioner gained 2 digits per step
- $\alpha = 0.01$, $\gamma = 0.0$, FBSimSmoother

Pure single-grid solver

	Opt. with MG			Opt. without MG		
lv.	#NL	#MG	T_{opt}	#NL	#ite	T_{opt}
2	5	16	112	5	72	80
3	5	27	1258	5	160	1303
4	5	28	10226	5	360	23242

- Nonlinear solver gained 8 digits
- Space-time linear solver gained 2 digits per step
- Space-preconditioner gained 2 digits per step
- $\alpha = 0.01$, $\gamma = 0.0$, FBSimSmoother

$\Rightarrow$ Multigrid necessary

The role of the MG cycle

Choice of the MG cycle + hierarchy is crucial for the complexity!

Example (for simplicity): Heat equation

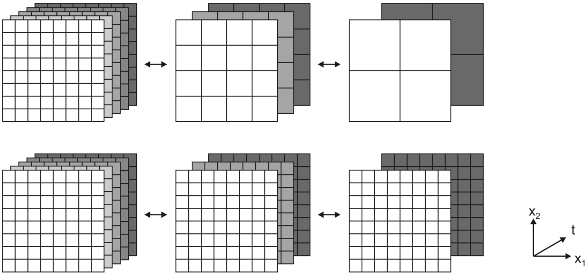
$$y_t - \Delta y = -\frac{1}{\alpha}\lambda \quad \text{in } Q$$

$$-\lambda_t - \Delta\lambda = (y - z) \quad \text{in } Q$$

Discr.: Space=Q_1, Time=IE or CN. Refinement/Coarsening:

1.) Space-time MG

2.) Pure time MG



The role of the MG cycle

- Time for NSM smoothing steps on level n: T_s^n.
- Total time for smoothing per MG step on level 2,...,n:

a) V-cycle: $T_s(2,\dots,n) = T_s^n + T_s^{n-1} + T_s^{n-2} + \dots + T_s^2$

b) W-cycle: $T_s(2,\dots,n) = T_s^n + 2T_s^{n-1} + 4T_s^{n-2} + \dots + 2^{n-2}T_s^2$

- Relationship of smoother time between levels (2D):

a) Full space-time multigrid: $T_s^{n-1} \approx 1/8\, T_s^n$

b) Time-multigrid: $T_s^{n-1} \approx 1/2\, T_s^n$

- Total numerical effort for smoothing $T_s(2,\dots,n)$:

	space-time MG	pure time-MG
V-cycle	$\leq 8/7\, T_s^n$	$\leq 2\, T_s^n$
W-cycle	$\leq 4/3\, T_s^n$	$\approx (n-1)\, T_s^n$

The role of the MG cycle

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	space-time MG	pure time-MG
V-cycle	$\leq\; 8/7\;T_s^n$	$\leq 2\;T_s^n$
W-cycle	$\leq\; 4/3\;T_s^n$	$\approx (n-1)\;T_s^n$

The role of the MG cycle

In practise:

- $\Omega = [0,1]^2$, $[0, T] = [0,1]$, $\alpha = 0.001$, $\gamma = 1000$.
- analyt. solution: $y = t^2(1-t)^2 x_1$. λ, z appropriately.
- space=Q_1, time=IE
- MG-solver with BiCGStab(Block-GS)-smoother.

	Full space-time MG				Time-MG			
	V-cycle		W-cycle		V-cycle		W-cycle	
	#ite	time	#ite	time	#ite	time	#ite	time
$T_s(5,...,5)$	4	245	4	245	3	181	3	182
$T_s(4,...,5)$	4	268	4	291	3	272	3	366
$T_s(3,...,5)$	4	269	4	298	3	320	3	531
$T_s(2,...,5)$	4	270	4	300	3	345	3	663
	$8/7 \approx \frac{270}{245}$		$4/3 \approx \frac{300}{245}$		$2 \approx \frac{345}{181}$		$n-1 \approx \frac{663}{182}$	

Summary and Outlook

Shown:

- Concept and realisation of an optimal control flow solver with optimal complexity

Key points:

- linear complexity (due to MG techniques)
- flexible (due to FEM approach, higher order + 3D possible)
- robust (FEM-stab. possible, e.g. EO-FEM/interior penalty)
- generalisation to more complex problems possible

  (Non-Newtonian + Non-isothermal flow,
  boundary control, constraint control)

Choice of the MG cycle + hierarchy is crucial for the complexity!

Example (for simplicity): Heat equation

$$
\begin{aligned}
y_t - \Delta y &= -\frac{1}{\alpha}\lambda && \text{in } Q \\
-\lambda_t - \Delta \lambda &= (y - z) && \text{in } Q
\end{aligned}
$$

- Discr.: Space=Q_1, Time=IE or CN. Refinement/Coarsening:

Variant 1: Full space-time MG

lv.	#steps	#DOF
1	5	50
2	10	162
3	20	578
4	40	2178
5	80	8450

Variant 2: Pure time MG

lv.	#steps	#DOF
1	5	8450
2	10	8450
3	20	8450
4	40	8450
5	80	8450

Convergence of the Newton solver

	Simulation			Optimisation			
lv.	#⊘NL	#⊘MG	T_{sim}	#NL	#MG	T_{opt}	$\frac{T_{opt}}{T_{sim}}$
2	3	11	4	4	8	308	73
3	3	11	24	4	8	1727	72
4	3	13	208	5	11	15355	74

- Nonlinear solver gained 8 digits
- Linear Solver: Space-Time-MG; gained 2 digits per step
- Space-preconditioner gained 2 digits per step
- $\alpha = 0.01$, $\gamma = 0.0$, BiCGStab(FBGSSmoother)

Convergence of the Newton solver

	Simulation			Optimisation			
lv.	#⊘NL	#⊘MG	T_{sim}	#NL	#MG	T_{opt}	$\frac{T_{opt}}{T_{sim}}$
2	3	11	2	4	5	51	25
3	3	11	15	4	6	455	30
4	3	13	161	4	7	4483	28
5	3	15	1771	4	7	39925	23

- Nonlinear solver gained 8 digits
- Linear Solver: Time-MG; gained 2 digits per step
- Space-preconditioner gained 2 digits per step
- $\alpha = 0.01$, $\gamma = 0.0$, FBSimSmoother

Convergence for different α/γ

α	1.0			0.1			0.01		
γ	#NL	#MG	Time	#NL	#MG	Time	#NL	#MG	Time
0.0	4	11	516	4	15	705	5	27	1258
0.1	4	14	653	4	17	800	5	36	1675
0.5	4	15	701	4	23	1082	div	div	div
1.0	4	17	795	4	24	1123	div	div	div

- Nonlinear solver gained 8 digits
- Linear Solver: Space-Time-MG; gained 2 digits per step
- Space-preconditioner gained 2 digits per step
- Space/time level 3, FBSimSmoother

Convergence for different α/γ

α	1.0			0.1			0.01		
γ	#NL	#MG	Time	#NL	#MG	Time	#NL	#MG	Time
0.0	3	4	303	3	4	309	4	6	454
0.1	4	6	437	4	6	455	4	5	382
0.5	4	6	440	4	6	465	div	div	div
1.0	4	6	440	4	6	463	div	div	div

- Nonlinear solver gained 8 digits
- Linear Solver: Time-MG; gained 2 digits per step
- Space-preconditioner gained 2 digits per step
- Space/time level 3, FBSimSmoother

Convergence for different α/γ

α	1.0			0.1			0.01		
γ	#NL	#MG	Time	#NL	#MG	Time	#NL	#MG	Time
0.0	4	5	1260	4	7	1757	5	10	2519
0.1	4	6	1490	4	9	2215	4	9	2247
0.5	3	5	1295	4	10	2445	4	11	2711
1.0	3	6	1499	4	11	2719	4	11	2743

- Nonlinear solver gained 8 digits
- Linear Solver: Space-Time-MG; gained 2 digits per step
- Space-preconditioner gained 2 digits per step
- Space/time level 3, BiCGStab(FBGSSmoother)

$\Rightarrow$ Much more stable, but much slower!