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In this note, we review some new approaches to enforcing discrete maximum principles in continuous high-order finite element discretizations of hyperbolic conservation laws. As a model problem, we consider the linear advection equation

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) = 0 \qquad \text{in } \Omega, \tag{1}$$

where $\mathbf{v}$ is a given velocity field and Ω is the domain of interest. A homogeneous flux boundary condition is imposed weakly on the inflow boundary of Ω. The standard Galerkin discretization leads to the semi-discrete system [5, 6]

$$M_C \frac{\mathrm{d}u}{\mathrm{d}t} + Au = 0, \tag{2}$$

where $M_C = \{m_{ij}\}$ is the consistent mass matrix, $A = \{a_{ij}\}$ is the discrete transport operator, and $u = \{u_i\}$ is the vector of time-dependent nodal values associated with globally continuous Lagrange or Bernstein basis functions $\varphi_1, \ldots, \varphi_{N_{\mathrm{dof}}}$.

Following the derivation of *algebraic flux correction* (AFC) schemes [5] for $\mathbb{P}_1$ and $\mathbb{Q}_1$ finite element discretizations, we write system (2) in the form

$$M_L \frac{\mathrm{d}u}{\mathrm{d}t} + (A - D)u = f\left(u, \frac{\mathrm{d}u}{\mathrm{d}t}\right), \tag{3}$$

where $M_L = \{\delta_{ij} \sum_j m_{ij}\}$ is the lumped mass matrix and $D = \{d_{ij}\}$ is a symmetric perturbation matrix defined in terms of the artificial diffusion coefficients

$$d_{ij} = \begin{cases} \max\{a_{ij}, 0, a_{ji}\} & \text{if } j \neq i, \\ -\sum_{k \neq i} d_{ik} & \text{if } j = i. \end{cases} \tag{4}$$

By definition of M_L and D, the low-order *discrete upwind* approximation

$$M_L \frac{\mathrm{d}u}{\mathrm{d}t} + (A - D)u = 0 \tag{5}$$

is bound-preserving [5, 8]. Hence, any violations of discrete maximum principles are caused by the antidiffusive term $f(u, \dot{u}) = (M_L - M_C)\dot{u} - Du$. To suppress undershoots and overshoots, we decompose $f_i = \sum_e f_i^e$ into edge or element contributions f_i^e and apply adaptively chosen correction factors $\alpha^e \in [0, 1]$.

The simplest algebraic limiting techniques enforce local maximum principles via postprocessing based on the following predictor-corrector strategy [4, 5, 8]:

1. Calculate a bound-preserving low-order approximation $\bar{u}^{n+1}$ using (5).
2. Add a sum of limited antidiffusive edge/element contributions to obtain

$$u_i^{n+1} = \bar{u}_i^{n+1} + \frac{\Delta t}{m_i} \sum_e \alpha^e f_i^e, \tag{6}$$

where Δt is the time step and m_i is the i-th diagonal entry of M_L. Adopting the design philosophy of *flux-corrected transport* (FCT) algorithms, the correction factors α^e are chosen so as to guarantee that $\bar{u}_i^{\min} \leq u_i^{n+1} \leq \bar{u}_i^{\max}$, where the bounds $\bar{u}_i^{\max}$ and $\bar{u}_i^{\min}$ are defined as local maxima and minima of $\bar{u}^{n+1}$ [8]. Hence, the limited antidiffusive correction is *local extremum diminishing* (LED).

The predictor-corrector limiting strategy is ideally suited for numerical solution of evolutionary problems using small time steps. However, it is not to be recommended for steady-state computations since direct manipulation of the degrees of freedom at the antidiffusive correction stage prevents convergence to stationary solutions and the levels of numerical diffusion depend on the (pseudo-)time step. Instead of calculating and correcting a low-order predictor, a limited antidiffusive term $\bar{f}$ can be incorporated into the residual of the nonlinear system

$$\bar{M}\frac{\mathrm{d}u}{\mathrm{d}t} + \bar{A}u = \bar{f}\left(u, \frac{\mathrm{d}u}{\mathrm{d}t}\right). \tag{7}$$

The solution-dependent correction factors α^e for the edge/element contributions $f_i^e(u,\dot{u})$ to $\bar{f}_i = \sum_e \alpha^e f_i^e$ are defined so that $\bar{f}_i = 0$ whenever u_i is a local maximum or minimum. In contrast to predictor-corrector approaches, monolithic limiting techniques of this kind constrain the perturbation of the Galerkin system (2) in an iterative manner [2, 5]. The recently developed theory of algebraic flux correction schemes [3] provides a set of sufficient conditions for well-posedness of nonlinear discrete problems and convergence of iterative solvers in the steady state limit. Limiter functions that possess all desired theoretical properties in the context of $\mathbb{P}_1$ and $\mathbb{Q}_1$ finite element approximations can be found in [6].

The extension of algebraic limiting to high-order piecewise-polynomial approximations calls for the use of finite element basis functions φ_i which guarantee that the numerical solution $u_h = \sum_i u_i \varphi_i$ is bounded by the maxima and minima of its coefficients u_i not only at the nodal points but also in-between. In contrast to high-order Lagrange elements, the Bernstein basis representation of u_h does provide this property. As shown in [1, 8], FCT-like limiting techniques are applicable to arbitrary-order Bernstein-Bèzier elements but require careful localization to preserve the high-order accuracy of the target scheme for smooth data.

Another promising approach to constraining high-order continuous Galerkin discretizations is the use of limiters in the basis functions of *partitioned* finite element spaces, as proposed in [7]. Let $V_{ph,p} = \mathrm{span}\{\varphi_1^H, \ldots, \varphi_{N_{\mathrm{dof}}}^H\}$ denote the space of continuous piecewise-polynomial functions such that $u_h|_K \in \mathbb{P}_p(K)$ or $u_h|_K \in \mathbb{Q}_p(K)$ for each $u_h \in V_{ph,p},\ p \in \mathbb{N}$ and each element $K \in \mathcal{T}_{ph}$ of a conforming finite element mesh $\mathcal{T}_{ph}$. Using a $\mathbb{P}_1/\mathbb{Q}_1$ approximation on elements of the embedded *submesh* $\mathcal{T}_h$, we define the space $V_{h,1} = \mathrm{span}\{\varphi_1^L, \ldots, \varphi_{N_{\mathrm{dof}}}^L\}$. To design an adaptive finite element scheme which employs the high-order approximation u_h^H in 'smooth' cells and the low-order approximation u_h^L in 'troubled' cells, we define the partitioned space $V_h(\alpha_h) := \mathrm{span}\{\varphi_1, \ldots, \varphi_{N_{\mathrm{dof}}}\} \subset V_{h,p+1}$ in terms of

$$\varphi_i(\mathbf{x}) = \alpha_h(\mathbf{x})\varphi_i^H(\mathbf{x}) + (1-\alpha_h(\mathbf{x}))\varphi_i^L(\mathbf{x}), \qquad \mathbf{x} \in \bar{\Omega}, \quad i = 1, \ldots, N_{\mathrm{dof}}, \tag{8}$$

where α_h is a blending function which yields a convex average of φ_i^H and φ_i^L. The traditional approach to hp-adaptivity in finite element methods is to use piecewise-constant basis selectors ($\alpha_h|_K \equiv 1$ or $\alpha_h|_K \equiv 0$ for $K \in \mathcal{T}_{ph}$). To ensure continuity of traces at common boundaries of adjacent mesh cells, this adaptation strategy requires special treatment of 'hanging' nodes and is difficult to implement. Moreover, the outcome depends on many binary decisions and small changes of the data may produce entirely different finite element spaces. To avoid the theoretical and practical difficulties associated with this methodology, the use of continuous blending functions $\alpha_h = \sum_i \alpha_i \varphi_i^L$ was proposed in [7]. The partition of unity (PU) parameters $\alpha_i \in [0, 1]$ may be defined using error indicators, smoothness criteria, and/or a priori information about location of internal/boundary layers. Moreover, the use of artificial diffusion operators and limiters can be restricted to troubled cells where the blending function α_h is set to zero by a smoothness indicator. Numerical examples illustrating the potential of using continuous blending functions in partitioned space and time discretizations can be found in [7].

In summary, a finite element approximation can be constrained to satisfy discrete maximum principles by using limiters to combine the degrees of freedom, edge/element contributions, and/or basis functions corresponding to a high-order target scheme and its bound-preserving low-order counterpart. The combination of limiting techniques with hp-adaptivity based on the PU approach provides a very general framework for robust, accurate, and physics-compatible discretization of conservation laws on general meshes. Further efforts are currently required to extend the theoretical foundations of algebraic limiting [3] to high-order / partitioned FEM and develop more efficient iterative solvers for nonlinear systems.

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