Efficient FEM-multigrid solver for granular material

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1 Introduction
2 Mathematical formulation
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4 Conclusion and outlook
Generalized Navier-Stokes equation

Conservation of mass

\[ \nabla \cdot u = 0 \quad \text{in } \Omega \]

Conservation of momentum

\[ \rho \frac{Du}{Dt} = -\nabla p + \nabla \cdot \sigma + f \quad \text{in } \Omega \]

along with some initial and boundary conditions

Flow Rheology

\[ \sigma = 2\eta(\gamma^\parallel, p)\dot{\gamma}; \]

where \( \dot{\gamma} = \frac{1}{2}(\nabla u + \nabla u^T) \),

and \( 2\gamma^\parallel = \dot{\gamma} : \dot{\gamma} = |\dot{\gamma}|^2 = tr[(\dot{\gamma})^2] \)

Nonlinear and coupled equations
Fluid models

\[ \eta(\gamma, p) = \eta_0, \]
\[ = \eta_0(|\dot{\gamma}|^2 + \epsilon)^{\frac{m}{2} - 1}, \]

for Newtonian fluid

for Power law fluid

Granular materials

\[ \eta(\gamma, p) = \sqrt{2}p \sin \phi \frac{1}{|\dot{\gamma}|}, \]
\[ = \sqrt{2}p(\sin \phi + b \cos \phi |\dot{\gamma}|^n) \frac{1}{|\dot{\gamma}|}, \]
\[ = \frac{\sqrt{2}}{2} \left( \frac{\alpha p}{|\dot{\gamma}|} + \frac{\beta dp}{\delta \sqrt{\frac{p}{\rho} + |\dot{\gamma}|d}} \right), \]

Schaeffer model

Schaeffer-Tardos model

Poliquen model
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Bilinear Form

Find \((u, p) \in X \times M\) such that

\[
\int_{\Omega} 2\eta(\gamma II, p)D(u) : D(v)dx + \int_{\Omega} (u \nabla u)vdx + \int_{\Omega} p \text{div} \ vdx = \int_{\Omega} fvdx, \quad \forall v \in X;
\]

with \(\int_{\Omega} q \text{div} \ u dx = 0, \quad \forall q \in M\)

Compact form

Find \((u, p) \in X \times M\) such that

\[
\langle Lu + Nu, v \rangle + \langle Bp, v \rangle = \int_{\Omega} fvdx, \quad \forall v \in X;
\]

\[
\langle q, B^T u \rangle = 0, \quad \forall q \in M
\]
FEM discretization

We take $X_h \subset X$ and $M_h \subset M$

Find $(\tilde{u}, \tilde{p}) \in X_h \times M_h$ such that

$$
\begin{pmatrix}
A & B \\
B^T & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{u} \\
\tilde{p}
\end{pmatrix}
=
\begin{pmatrix}
g \\
0
\end{pmatrix}
$$

Nonlinear saddle point problem!
Newton solver

Motivation

- Strongly coupled problem
- Solution \( x^{n+1} = (\tilde{u}, p) \), Residual equation \( R(x^n) \)

\[
    x^{n+1} = x^n + w^n \left[ \frac{\partial R(x^n)}{\partial x} \right]^{-1} R(x^n)
\]

- Automatic damping control \( w^n \) for each nonlinear step
- Quadratic convergence when iterative solutions are close to the actual one
Jacobian with respect to the Diffusive term

\[
\int_\Omega 2\eta(\gamma_{II}(u^l), p^l) D(u) : D(v) \, dx \\
+ \int_\Omega 2\partial_1\eta(\gamma_{II}(u^l), p^l)[D(u^l) : D(u)][D(u^l) : D(v)] \, dx \\
+ \int_\Omega 2\partial_2\eta(\gamma_{II}(u^l), p^l)[D(u^l) : D(v)]p \, dx
\]

\[
= \int_\Omega fv \, dx - \int_\Omega 2\eta(\gamma_{II}(u^l), p^l) D(u^l) : D(v) \, dx, \forall v \in V
\]

Jacobian with respect to the Convective term

\[
\int_\Omega (u^l \cdot \nabla u) v \, dx + \int_\Omega (u \cdot \nabla u^l) v \, dx \\
\forall v \in X,
\]
Newton solver

Diffusive term

\[ \int_{\Omega} 2\eta(\gamma_{II}(u^l), p^l) D(u) : D(v) \, dx \quad L \]
\[ + \int_{\Omega} 2\partial_1 \eta(\gamma_{II}(u^l), p^l)[D(u^l) : D(u)][D(u^l) : D(v)] \, dx \quad L^* \]
\[ + \int_{\Omega} 2\partial_2 \eta(\gamma_{II}(u^l), p^l)[D(u^l) : D(v)]p \, dx \quad B^* \]
\[ = \int_{\Omega} fv \, dx - \int_{\Omega} 2\eta(\gamma_{II}(u^l), p^l)D(u^l) : D(v) \, dx, \forall v \in V \]

Convective term

\[ \int_{\Omega} (u^l \cdot \nabla u) v \, dx \quad N \quad \int_{\Omega} (u \cdot \nabla u^l) v \, dx \quad N^* \quad \forall v \in X, \]
Final discrete problem

Compute $\tilde{u}$ and $\tilde{p}$ by solving

\[
\begin{pmatrix}
A & \tilde{B} \\
B^T & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{u} \\
\tilde{p}
\end{pmatrix} =
\begin{pmatrix}
\text{Res}_{\tilde{u}} \\
\text{Res}_{\tilde{p}}
\end{pmatrix}
\]

where

\[
A\tilde{u} = [(L + \delta_d L^*)(\tilde{u}^l, \tilde{p}^l) + (N + \delta_c N^*)(\tilde{u}^l)]\tilde{u},
\]

\[
\tilde{B}\tilde{p} = [B + \delta_{\tilde{p}} B^*(\tilde{u}^l, \tilde{p}^l)]\tilde{p}
\]

Unusual saddle point problem!
Multigrid techniques

- Direct Gauss elimination as coarse-grid solver
- General VANKA smoother with block-diagonal preconditioner
- F-cycle multigrid
- Intergrid transfer and coarse grid correction based on the underlying mesh hierarchy and the finite elements
Benchmark problem

Figure: Geometry for the ‘flow around cylinder’ configuration

Figure: Computational mesh for the ‘flow around cylinder’ configuration
Why Featflow

- Basic flow solver for incompressible fluids
- Supports higher order (space/time) FEM
- Use of unstructured meshes
- Dynamic adaptive grid formulation
- FEM based tools

FEM characteristics

- Stable FE spaces for velocity/pressure and velocity/stress interpolation; e.g. $Q2/P1$
- Special treatments of the convective term - EO FEM, TVD/FCT
Inner solver

Solver

- Nonlinearity handled by Newton method
- Monolithic Multigrid techniques for the auxiliary linearized problem
- Vanishing shear rate taken care by the regularization parameter
- Appropriate module to ensure unique and positive pressure

Advantages

- Inf-sup stable (LBB condition) for velocity and pressure
- Higher order is good for accuracy
- Discontinuous pressure is good for the solver
Results

Shear-thinning fluid

\[ \eta(\gamma, \eta) = \eta_0 (|\dot{\gamma}|^2 + \epsilon)^{m/2 - 1}, \quad \eta_0 = 10^{-3}, \quad m = 1.5, \quad \epsilon = 0.1 \]

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Efficient nonlinear and linear solver

Shear-thickening fluid

\[ \eta(\gamma, p) = \eta_0 \left( |\gamma|^2 + \epsilon \right)^{\frac{m}{2} - 1}, \quad \eta_0 = 10^{-3}, \ m = 3.0, \ \epsilon = 0.1 \]

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Results

Test case

\[ \eta(\gamma_\parallel, p) = \eta_0 \exp(p), \quad \eta_0 = 10^{-3} \]

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Conclusion

Summary

- Newton solver for nonlinearity
- Multigrid techniques for the linearized problem
- Numerical examples
  - non-Newtonian fluids including shear thinning and shear thickening fluid
  - Test case with pressure dependant viscosity
Outlook

Current work

- Benchmarking of Poliquen model
- Test with split-bottom geometry
- To incorporate Tardos model

\[ \rho \frac{Du}{Dt} = -\nabla p + \nabla \cdot \left[ \frac{q(p, \rho)}{||D - \frac{1}{n} \nabla \cdot ul||} \left( D - \frac{1}{n} \nabla \cdot ul \right) \right] + \rho g; \quad n = 2, 3 \]

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \]

\[ \nabla \cdot u = \frac{\partial q(p, \rho)}{\partial p} \left( D - \frac{1}{n} \nabla \cdot ul \right) \]

Questions

- How to fix the pressure range?
- 2D version of split-bottom geometry?
THANK YOU!