

PRESSURE SCHUR COMPLEMENT PRECONDITIONERS FOR THE DISCRETE OSEEN PROBLEM *

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Abstract. We consider several preconditioners for the pressure Schur complement of the discrete steady Oseen problem. Two of the preconditioners are well known from the literature and the other is new. Supplemented with an appropriate approximate solve for an auxiliary velocity subproblem these approaches give rise to a family of the block preconditioners for the matrix of the discrete Oseen system. In the paper we critically review possible advantages and difficulties of using various Schur complement preconditioners. We recall existing eigenvalue bounds for preconditioned Schur complement and prove such with the newly proposed preconditioner. These bounds hold both for LBB stable and stabilized finite elements. Results of numerical experiments for several model 2D and 3D problems are presented. In experiments we use LBB stable finite element methods on uniform triangular and tetrahedral meshes. One particular conclusion is that in spite of essential improvement in comparison with “simple” scaled mass-matrix preconditioners in certain cases, none of the considered approaches provides satisfactory convergence rates in the case of small viscosity coefficients and sufficiently complex (e.g., circulating) advection vector field.

Key words. Oseen equations, saddle-point problems, finite elements, iterative methods, preconditioning, pressure Schur complement, Navier–Stokes equations

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1. Introduction. We consider the numerical solution of the steady Oseen problem: Given a divergence free advection velocity $\mathbf{w} : \Omega \rightarrow \mathbb{R}^d$, force field $\mathbf{f} : \Omega \rightarrow \mathbb{R}^d$ and boundary data $\mathbf{g} : \partial\Omega \rightarrow \mathbb{R}^d$, find a velocity $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ and a pressure $p : \Omega \rightarrow \mathbb{R}$ such that

$$-\nu\Delta\mathbf{u} + (\mathbf{w} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \quad (1.1)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \quad (1.2)$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \quad (1.3)$$

where $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is a bounded, connected domain with a piecewise smooth Lipschitz boundary $\partial\Omega$. The coefficient $\nu > 0$ is a given kinematic viscosity. For the sake of uniqueness of p one may impose some additional condition, such as $\int_{\Omega} p \, dx = 0$.

A necessity of solving Oseen equations numerically is commonly related to using Picard’s type iterative method to find a solution to steady Navier-Stokes problem. In this case \mathbf{w} is an approximation of velocity from previous iterative steps, so it is updated on every nonlinear iteration. Among another applications we mention Uzawa type algorithms for augmented variational inequality approach to the modeling of Bingham fluids, e.g. [22]. Again one may need to solve discrete Oseen system many times with different \mathbf{w} and \mathbf{f} . Thus there is a demand for efficient iterative solvers for the discrete Oseen problem. We note that besides (1.3) another boundary conditions may be imposed in various models. Further we will remark when using different boundary conditions requires a special attention.

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In this paper we consider a finite element method to discretize (1.1)–(1.3). However, the linear algebraic solvers discussed further can be applied in finite difference or finite volume context in the same manner. We assume that finite element velocity (non necessarily conforming) and pressure spaces \mathbb{V}_h and \mathbb{Q}_h approximate $\mathbf{H}_0^1(\Omega)$ and $L_2^0(\Omega) := \{q \in L_2(\Omega) : (q, 1) = 0\}$, respectively. Consider the following finite element problem: Find $\mathbf{u}_h \in \mathbb{V}_h$ and $p_h \in \mathbb{Q}_h$ satisfying

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) + (q_h, \operatorname{div} \mathbf{u}_h) + c_h(p_h, q_h) \\ = (\mathbf{f}_h, \mathbf{v}_h) + (g_h, q_h) \quad \forall \mathbf{v}_h \in \mathbb{V}_h, q_h \in \mathbb{Q}_h. \end{aligned} \quad (1.4)$$

The bilinear form $a_h(\cdot, \cdot)$ may include some stabilizing terms for the advection dominated case. The non-negative bilinear form $c_h(\cdot, \cdot)$ may be included in the finite element formulation if \mathbb{V}_h and \mathbb{Q}_h form an LBB unstable pair [14], otherwise $c_h(\cdot, \cdot) \equiv 0$. Denote by $(\cdot, \cdot)_{\mathbb{V}}$ the energy scalar product on \mathbb{V}_h satisfying $(\psi, \phi)_{\mathbb{V}} = (\nabla \psi, \nabla \phi)$ for $\psi, \phi \in \mathbb{V}_h \cap \mathbf{H}_0^1$. For the bilinear forms a_h and c_h we assume ellipticity, continuity and stability conditions:

$$\alpha_1 \|\mathbf{v}_h\|_{\mathbb{V}}^2 \leq a_h(\mathbf{v}_h, \mathbf{v}_h), \quad a_h(\mathbf{v}_h, \mathbf{u}_h) \leq \alpha_2 \|\mathbf{v}_h\|_{\mathbb{V}} \|\mathbf{u}_h\|_{\mathbb{V}} \quad \forall \mathbf{v}_h, \mathbf{u}_h \in \mathbb{V}_h \quad (1.5)$$

$$\gamma_1^2 \|q_h\|^2 \leq \sup_{\mathbf{v}_h \in \mathbb{V}_h} \frac{(q_h, \operatorname{div} \mathbf{v}_h)^2}{\|\mathbf{v}_h\|_{\mathbb{V}}^2} + c_h(q_h, q_h) \quad \forall q_h \in \mathbb{Q}_h, \quad (1.6)$$

$$c_h(q_h, p_h) \leq \gamma_2 \|q_h\| \|p_h\|, \quad (q_h, \operatorname{div} \mathbf{v}_h) \leq \gamma_3 \|q_h\| \|\mathbf{v}_h\|_{\mathbb{V}} \quad \forall q_h, p_h \in \mathbb{Q}_h, \mathbf{v}_h \in \mathbb{V}_h \quad (1.7)$$

with positive mesh independent constants α_1 , α_2 , γ_1 , γ_2 , and γ_3 . We note that conditions (1.6) and (1.7) are common for the pressure stabilized finite element methods, see e.g. the recent studies in [7]; for the LBB stable pairs (1.6) and (1.7) trivially hold.

Let $\{\phi_i\}_{1 \leq i \leq n}$ and $\{\psi_j\}_{1 \leq j \leq m}$ be bases of \mathbb{V}_h and \mathbb{Q}_h , respectively. Define the following matrices:

$$A_{i,j} = a_h(\phi_j, \phi_i), \quad B_{i,j} = -(\operatorname{div} \phi_j, \psi_i), \quad C_{i,j} = c_h(\psi_j, \psi_i)$$

The linear algebraic system corresponding to (1.4) (*the discrete Oseen system*) takes the form:

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \quad (1.8)$$

We are interested in solving (1.8) by a preconditioned iterative method. Following the conventional approach [14], [26], [10], we consider the block triangular preconditioner for the system (1.8):

$$\mathcal{P} = \begin{pmatrix} \hat{A} & B^T \\ O & -\hat{S} \end{pmatrix}, \quad (1.9)$$

The matrix \hat{A} is a preconditioner for the matrix A , such that \hat{A}^{-1} may be considered as an inexact solver for linear systems involving A . The matrix \hat{S} is a preconditioner for the pressure Schur complement of (1.8) $S = BA^{-1}B^T + C$. In an iterative algorithm one needs the actions of \hat{A}^{-1} and \hat{S}^{-1} on subvectors, rather than the matrices \hat{A} , \hat{S} explicitly. Once good preconditioners for A and S are given, an appropriate Krylov

subspace iterative method for (1.8) with the block preconditioner (1.9) is an efficient solver. In a literature one can find geometric or algebraic multigrid (see, e.g. [14] and references therein) or domain decomposition [21, 30] iterative algorithms which provide effective preconditioners \hat{A} for a wide range of ν and various meshes. However, despite a considerable recent effort and progress, building a preconditioner for S , which is robust for a wide range of parameters (especially viscosity), discretizations, and meshes, is still a challenge.

In this paper we recall two recent approaches to construct a preconditioner for S . One is due to Kay, Loghin and Wathen [25], another is from Elman and coauthors [16, 18]. Further, in attempt to overcome some difficulties associated with these approaches we consider a new preconditioner for S . We give motivations for different choices of \hat{S}^{-1} , prove eigenvalue bounds and present results of several numerical experiments.

As already mentioned, a good preconditioner \hat{S} is necessary for building the block triangular preconditioner (1.9). Furthermore, there exist other numerical methods for incompressible Navier-Stokes equations, where finding a proper approximation to S is vital. These methods are based on Uzawa method and its variants [20, 32, 9], Arrow-Hurwicz [20, 1], SOR [12], and special factorizations [6, 4] for the linearized problems. Moreover, the performance of the widely used splitting algorithms like SIMPLE, projection or pressure correction methods for the time integration of unsteady problem is also closely related with issue of the Schur complement preconditioning [35, 38, 37].

There are also iterative methods for solving (1.8) which do not require the consideration of the Schur complement S or its preconditioner, at least explicitly. Among them are coupled multigrid methods of Vanka type [24, 38], methods based on Hermitian splitting [2], augmented lagrangian based preconditioning [5], implicit - factorization preconditioning [13, 11], see also the review article [4]. It is not the intension of this paper to discuss or compare these methods. We only note that implementing some of them in a purely algebraic manner may experience serious difficulties [33]. Thus the pressure Schur complement based block solvers remain attractive for treating ‘real-life’ engineering problems and have a potential to develop in black-box algorithms.

The remainder of the paper is organized as follows. In section 2 we consider two well-known preconditions for S and present a new approach. In section 3 we prove eigenvalue bounds. For the new preconditioner the h -independent bounds both for the LBB stable and the pressure stabilized discretizations are shown. In section 4 results of numerical experiments with different preconditioners are given for 2D and 3D problems discretized on simplicial meshes.

2. Schur complement preconditioners. In this paper all variants of preconditioner \hat{S} are defined through their inverses. Before proceeding to the preconditioners we define the pressure mass matrix, the velocity mass, and laplacian matrices:

$$(M_p)_{i,j} = (\psi_j, \psi_i), \quad (M_u)_{i,j} = (\phi_j, \phi_i), \quad L_{i,j} = (\phi_j, \phi_i)_{\mathbb{V}}.$$

2.1. PCD. We first consider the *pressure convection diffusion* (PCD) preconditioner, proposed by Kay, Loghin, Wathen [25] and studied by these and other authors (see [14]):

$$\hat{S}^{-1} := \hat{M}_p^{-1} A_p L_p^{-1}. \quad (2.1)$$

Here \hat{M}_p^{-1} denotes an approximate solve with the pressure mass matrix. Matrices A_p and L_p are approximations to convection-diffusion and laplacian operators in \mathbb{Q}_h ,

respectively. Note that both A_p and L_p need some boundary conditions to be prescribed.

In discretizations of (1.1)–(1.3) with continuous pressure approximations one can use the conforming discretization of pressure Poisson problem with Neumann’s boundary conditions. The corresponding finite element formulation for the case of the Neumann conditions is standard:

$$(\nabla p_h, \nabla q_h) = (f, q_h), \quad \forall q_h \in \mathbb{Q}_h \subset H^1(\Omega).$$

Neumann boundary conditions are conventionally set for the convection-diffusion problem on \mathbb{Q}_h .

An alternative way to define L_p is to set $L_p = (B\hat{M}_u^{-1}B^T)$, where \hat{M}_u is a diagonal approximation to the velocity mass matrix. Although a diagonal matrix might be a poor approximation to M_u in the case of anisotropic grids, this operator can be seen as a mixed discretization of the pressure Poisson problem with Neumann’s boundary conditions. Using $(B\hat{M}_u^{-1}B^T)$ is convenient in the case of discontinuous pressures. It is not straightforward to define A_p for the case of discontinuous pressures, see section 4 for a definition of A_p in the case of iso P_2 - P_0 elements.

Analysis of the PCD preconditioner and numerical results found in a literature (see also section 4) recover several specific features of the method. In particular the advantages of using the preconditioner are the following:

1. The PCD preconditioner provides mesh-independent convergence rates for moderate values of ν ;
2. Dependence of the convergence rates on ν^{-1} is significantly improved in comparison to a scaled pressure mass matrix;
3. Numerical results [34, 27] suggest that the preconditioner is not very sensitive to grid anisotropy, at least for some discretizations;
4. Some theoretical analysis of the preconditioner is available [15];
5. The preconditioner can be used both for LBB stable and pressure stabilized discretizations.

There are also some open questions associated with using this method:

1. A degradation of the convergence rates as $\nu \rightarrow 0$ is well seen even for the simplest constant flow: $\mathbf{w} = (1, 0)$, or $\mathbf{w} = (1, 0, 0)$.
2. The issue of a proper boundary conditions in L_p and A_p is not very well understood. We address this question below in more details;
3. A_p matrix does not naturally arise in the original problem. Some effort may be needed to build it, especially for discontinuous pressure approximations;
4. The preconditioner is specifically oriented to the Oseen problem. Its extension to similar problems, like Newton linearization, quasi-Newtonian fluids, the Navier-Stokes system coupled with other equations, or general problems having the same 2×2 structure as (1.8), is not immediately clear.

The choice of boundary conditions for the definition of A_p and L_p depends on boundary conditions in the Oseen problem. In [14] it is recommended that Neumann’s boundary conditions should be prescribed on those parts of $\partial\Omega$ where in the original formulation of the Oseen problem one has Dirichlet’s boundary conditions for \mathbf{u} , and Dirichlet’s boundary conditions in A_p and L_p should be used on those parts of $\partial\Omega$ where in the original formulation one has outflow boundary conditions for the stress tensor. A motivation for these recommendations comes from the symmetric case of the unsteady Stokes problem, where this choice is proved to work well for the Cahouet-Chabard preconditioner. Our experiments and analysis suggest that for the case of

dominated convection ($\nu \rightarrow 0$) this choice is not optimal. Below we give arguments that on the inflow boundary for small ν one may prefer using Dirichlet's homogeneous fictitious boundary for constructing A_p . Experimentally we observed that the change of boundary conditions on the outflow part of the boundary is not crucial. However, on the characteristic parts of $\partial\Omega$ and outflow parts with conditions on stress tensor (so called "do-nothing" b.c.) the Neumann's boundary conditions in A_p and L_p are more appropriate.

We consider the continuous operator counterpart of the Schur complement and rewrite the equation $q = Sp$ in the form

$$-\nu\Delta\mathbf{v} + (\mathbf{w} \cdot \nabla)\mathbf{v} + \nabla p = \mathbf{0}, \quad -\operatorname{div}\mathbf{v} = q \quad \text{in } \Omega \quad (2.2)$$

$$\mathbf{v}|_{\partial\Omega} = \mathbf{0} \quad (2.3)$$

Let $\nu = 0$ and prescribe boundary condition (2.3) only on the inflow part Γ_{in} of $\partial\Omega$, i.e. $\Gamma_{in} = \{\mathbf{x} \in \partial\Omega : \mathbf{w}(\mathbf{x}) \cdot \mathbf{n} < 0\}$. For the sake of simplicity we assume the following: the plain patch Γ_{in} is orthogonal to x -axis, \mathbf{w} is sufficiently smooth, orthogonal to Γ_{in} at each point and stays parallel to x -axis in some neighborhood $\mathcal{O} \subset \Omega$ of Γ_{in} . For simplicity, we consider the 2D case (3D case is considered similarly). From the first equation in (2.2) and condition $\mathbf{v}|_{\Gamma_{in}} = \mathbf{0}$ one gets the equality $\mathbf{v}(x, y) = \int_0^x \nabla p(s, y) ds$ in \mathcal{O} . The second equation in (2.2) implies $q = -\partial p / \partial x - \int_0^x \partial^2 p / \partial y^2 ds$ in \mathcal{O} . Therefore, we get

$$q = -\frac{\partial p}{\partial x} \quad \text{on } \Gamma_{in}. \quad (2.4)$$

On the other hand, the PCD approach suggests to approximate $p = S^{-1}q$ by

$$p = -(\mathbf{w} \cdot \nabla)\Delta^{-1}q, \quad (2.5)$$

where Δ^{-1} is a solution operator to the Poisson problem with *some* boundary conditions. We want to define these conditions on Γ_{in} in a way consistent with (2.4). To this end we rewrite (2.5) in \mathcal{O} :

$$p = -\frac{\partial r}{\partial x}, \quad \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = q. \quad (2.6)$$

Relations (2.4) and (2.6) lead to $\partial^2 r / \partial y^2 = 0$ on Γ_{in} . The corresponding homogeneous boundary condition in the definition of Δ^{-1} is $r = 0$ on Γ_{in} . Therefore, for the practically important case when the flow \mathbf{w} is orthogonal to Γ_{in} , the reasonable boundary condition for the PCD preconditioner on the *inflow* is *homogeneous Dirichlet's boundary conditions*, at least for the case of small ν . From implementation standpoint, Dirichlet's boundary conditions may not be imposed for the nodes at Γ_{in} since these nodes have to contribute to the set of pressure degrees of freedom. For this reason one introduces outside Ω a fictitious one-cell layer attached to Γ_{in} . Dirichlet's boundary conditions are assigned at layer nodes not belonging to Γ_{in} .

Results of numerical experiments suggest that the choice of boundary condition on $\Gamma_{out} = \{\mathbf{x} \in \partial\Omega : \mathbf{w}(\mathbf{x}) \cdot \mathbf{n} > 0\}$ in the preconditioner does not effect its performance in any substantial way. If in the Oseen problem (1.1)–(1.3) instead of Dirichlet's conditions one sets normal component of the stress tensor equals zero on the outflow:

$$-\nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) \cdot \mathbf{n} + 2p\mathbf{n} = 0 \quad \text{on } \Gamma_{out}, \quad (2.7)$$

then for $\nu \rightarrow 0$ one gets $p = 0$ on Γ_{out} . Now relation (2.5) immediately gives $(\mathbf{w} \cdot \nabla)r = 0$ on Γ_{out} . For the case when \mathbf{w} is orthogonal to Γ_{out} this results in the *homogeneous Neumann's boundary conditions* for the matrices in the PCD preconditioner.

Using similar arguments one can show that Neumann's boundary conditions "inside" \hat{S} are appropriate for the characteristic part of $\partial\Omega$.

2.2. BFBt. Next we consider the BFBt preconditioner, proposed by Elman [16] and further developed by Elman and coauthors in [18]. The best available modification from [18] is

$$\hat{S}^{-1} := (B\hat{M}_u^{-1}B^T)^{-1}B\hat{M}_u^{-1}A\hat{M}_u^{-1}B^T(B\hat{M}_u^{-1}B^T)^{-1} \quad (2.8)$$

where \hat{M}_u is a diagonal approximation of M_u . In the case of continuous pressure elements BFBt preconditioner may be reformulated as

$$\hat{S}^{-1} := L_p^{-1}B\hat{M}_u^{-1}A\hat{M}_u^{-1}B^TL_p^{-1}.$$

Below some observations on BFBt preconditioner are listed. On the positive side one has:

1. In contrast to the PCD method, the preconditioner (2.8) is build from the matrices readily available. Indeed, matrices A and B are already in the system (1.8), and \hat{M}_u is fairly easy to construct from M_u by the lumping procedure;
2. The preconditioner can be defined for general problems like (1.8), although in general case one may need to find some other scaling matrices instead of velocity mass matrix;
3. We found BFBt preconditioner to be robust with respect to ν for simplest parallel constant wind, $\mathbf{w} = (1, 0)$ and continuous pressure elements;
4. The issue of proper pressure boundary conditions does not arise explicitly.

On the other side:

1. The dependence on ν^{-1} is still observed for more complicated flows, like circulating flows;
2. h - independence of the convergence rate is observed for special case of small ν , parallel constant wind, iso P_2 - P_1 elements. Otherwise BFBt method shows some h -dependence;
3. Two pressure Poisson problems should be solved instead of one, as for the PCD preconditioner.

We have no clear explanation why for certain discretizations the mesh dependent convergence rates occur for the BFBt preconditioner (see some arguments in § 2.3). In [16] and [17] Elman observed some h -dependence in convergence rate of the GEMRES method using BFBt preconditioner for the finite difference (MAC) and finite element Q_2 - Q_1 discretizations. Some h -dependence was also observed by Vainikko and Graham in [31] with Q_2 - P_{-1} elements and by Hemmingsson and Wathen in [23] with a finite difference method. In all these papers the variant of the preconditioner with identity matrices instead of \hat{M}_u in (2.8) was used. In the recent paper [18] it was noted that introducing \hat{M}_u in (2.8) improves the situation significantly in the case of Q_2 - Q_1 finite element discretizations, leading to virtually no h -dependence. The explanation of this phenomena was partially heuristic. Furthermore, if one considers a uniform grid and apply finite differences or finite elements with piecewise linear velocity, then \hat{M}_u is a scaled identity matrix (at least for the case of Dirichlet's boundary conditions in (1.3)). Thus in this case the matrix \hat{M}_u in (2.8) has no effect on the preconditioner. We note that a possible deterioration of convergence rates is consistent with

available eigenvalue estimates for this preconditioner, see (3.2), which also show the h -dependence.

An attempt to extend the preconditioner for the pressure-stabilized elements was recently done by Elman and coauthors [19]. It leads to more complicate definition of \hat{S}^{-1} .

We have not found in a literature results about performance of the BFBt preconditioner in the case of stretched grids.

2.3. More preconditioners. Our motivation is to build a preconditioner based on available matrices (blocks) in the spirit of BFBt with similar “robustness” properties with respect to ν , but without possible convergence failures for small h . To this end, let us consider the preconditioner (2.8) once again. If one ignores the velocity stabilization terms in A , the continuous counterpart of the BFBt preconditioner can be written as

$$\Delta_N^{-1} \operatorname{div} (\nu \Delta + \mathbf{w} \cdot \nabla) \nabla \Delta_N^{-1}, \quad (2.9)$$

where Δ_N^{-1} is the solution operator to the Poisson problem with Neumann’s boundary conditions. Consider the operator (2.9) acting on some $p \in L_2(\Omega)$. A trouble might be caused by the lack of a proper tangential boundary conditions for $\mathbf{v} = \nabla \Delta_N^{-1} p$. Indeed, for the normal component of \mathbf{v} we have $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$, but the tangential component does not necessarily vanish on $\partial\Omega$. At the same time the computing of $(\nu \Delta + \mathbf{w} \cdot \nabla) \mathbf{v}$ requires $\mathbf{v} = 0$ on $\partial\Omega$, since the discrete counterpart of $(\nu \Delta + \mathbf{w} \cdot \nabla)$ – matrix A – was built assuming Dirichlet’s homogeneous boundary conditions for velocity. Similarly, for the vector function $\mathbf{u} = (\nu \Delta + \mathbf{w} \cdot \nabla) \mathbf{v}$ the normal condition $\mathbf{u} \cdot \mathbf{n} = 0$ is not ensured, although this condition was used to define the matrix B – a discrete counterpart of div . Another relevant question is whether the operator in (2.9) is well defined in a bounded domain as an operator from L_0^2 to L_0^2 . It can be shown that this requires the H^2 -regularity assumption for the Poisson problem and proper homogeneous boundary conditions for the intermediate function \mathbf{u} as discussed above.

It is necessary to point out that such regularity and boundary conditions issues formally do not arise on the discrete level. However, the failure of the discrete operator to approximate a well-posed continuous counterpart as $h \rightarrow 0$ may be a reason for the h -dependence of the BFBt preconditioner for some discretizations.

The suggested remedy is to commute ∇ and div with Δ_N^{-1} in (2.9). We obtain

$$\operatorname{div} \Delta_0^{-1} (\nu \Delta + \mathbf{w} \cdot \nabla) \Delta_0^{-1} \nabla, \quad (2.10)$$

where Δ_0^{-1} is a solution operator for the velocity vector Poisson problem with Dirichlet’s boundary conditions. Note that the commutation property which we used holds only with special boundary conditions, e.g. periodic; this property is even more questionable for discrete operators. However, variants of such commutation arguments are often used in a literature to deduce PCD and BFBt preconditioners. The discrete operator corresponding to (2.10) is

$$\hat{S}^{-1} := \hat{M}_p^{-1} B L^{-1} A L^{-1} B^T \hat{M}_p^{-1} \quad (2.11)$$

Here L^{-1} is an approximate solve for the discrete velocity vector Poisson problem.

Our observations about this preconditioner are the following. On the positive side one has:

1. The preconditioner (2.11) is build from the matrices already available: matrix L^{-1} is the diffusion part of matrix A ;
2. The action of L^{-1} may be performed using the same technology as that of \hat{A}^{-1} (MG, AMG, etc.);
3. Since the preconditioner does not use a discrete pressure Poisson solver, the issue of an appropriate boundary conditions does not arise;
4. Preconditioner from (2.11) can be easily extended for more general linearized Navier-Stokes type problems;
5. In our experiments preconditioner (2.11) shows h -independent convergence results in a wider set of cases: both FE choices, various convection fields, up to $\nu = 10^{-3}$. This is supported by the h -independent eigenvalue estimates, which we prove in the next section.

On the other side:

1. The dependence on ν^{-1} is observed for more complicated flows, like circulating flows;
2. For the diffusion dominated case the condition number of the preconditioned matrix $\hat{S}^{-1}S$ is squared comparing to the optimal mass matrix preconditioner. Indeed, if $\mathbf{w} = 0, \nu = 1$ and L^{-1} is exact we have $A = L$ and

$$\hat{S}^{-1}S = \hat{M}_p^{-1}BL^{-1}AL^{-1}B^T\hat{M}_p^{-1}S = (\hat{M}_p^{-1}S)^2;$$

This results in nearly doubling of iteration numbers for diffusion dominated case.

3. Compared to the PCD and BFBt preconditioner, the matrix L in (2.11) has larger dimension than L_p or $(B\hat{M}_u^{-1}B^T)$.

REMARK 2.1. Similar to BFBt the new preconditioner (2.11) can not be immediately used for the LBB unstable finite elements. However for this case it admits a simple modification:

$$S^{-1} := \hat{M}_p^{-1}(BL^{-1}AL^{-1}B^T + C)\hat{M}_p^{-1}.$$

It is easy to show that for the symmetric case this modification ensures h -independent convergence. For the non-symmetric case h -independent eigenvalue bounds will be proved in the next section.

Let us also mention a method from [8] for constructing a preconditioner \hat{S} . The approach is quite different from the techniques considered above. It is based on hierarchical matrix technique for building approximate inverses for matrices. However the numerical experiments in [8] show a large setup time in this approach, which makes it rather expensive in practice. Finally, we remark that effective pressure Schur complement preconditioners can be build for the linearized Navier-Stokes equations with non-linear terms written in the rotation form [28, 29].

3. Eigenvalues estimates. It is well known that characterizing the rate of convergence of nonsymmetric preconditioned iterations can be a difficult task. In particular, eigenvalue information alone may not be sufficient to give meaningful estimates of the convergence rate of a method like preconditioned GMRES. The situation is even more complicated for a method like BiCGStab, for which virtually no convergence theory exists. Nevertheless, experience shows that for many linear systems arising in practice, a well-clustered spectrum (away from zero) usually results in rapid convergence of the preconditioned iteration. Therefore, in this section we recall some

known estimates for the eigenvalues of the preconditioned Schur complement with PCD (2.1) and BFBt (2.8) preconditioners for LBB stable elements. Bounds for the PCD preconditioning will be extended to the pressure stabilized case. Also we prove analogous estimates for the new preconditioning (2.11).

Below we use the following notations. $\|\cdot\|$, $\langle \cdot, \cdot \rangle$ denotes the Euclidian norm and scalar product. We also define the norm $\|q\|_* := \langle M_p^{-1}q, q \rangle^{\frac{1}{2}}$. Note that for a matrix $D \in \mathbb{R}^{m \times m}$ and corresponding matrix norms it holds $\|D\|_* = \|M_p^{-\frac{1}{2}}DM_p^{\frac{1}{2}}\|$. Furthermore, the PCD preconditioner from (2.1) we denote by S_1 , the BFBt preconditioner from (2.8) we denote by S_2 , and S_3 will be the new preconditioner from (2.11).

Assume a quasi-uniform discretization (partition into triangles or quads) of Ω . Let h denote a maximum diameter of elements. Assume that a finite elements spaces \mathbb{V}_h and \mathbb{Q}_h satisfy standard approximation properties and inverse inequalities. For the LBB stable case in [15] the following bounds for the eigenvalues of the preconditioned Schur complement were proved

$$c_1 \leq |\lambda(SS_1^{-1})| \leq C_1 \quad (3.1)$$

$$c_2 \leq |\lambda(SS_2^{-1})| \leq C_2 h^{-2} \quad (3.2)$$

with positive constants c_1, c_2, C_1, C_2 independent of the meshsize h . For the pressure stabilized case ($C \neq 0$) we do not find in the literature any eigenvalue estimates with the PCD preconditioner S_1 . Hence we give the proof of such bounds below in theorem 3.2. As we already mentioned, using the BFBt preconditioner S_2 in the pressure stabilized case is not straightforward and requires some modifications [19], this modified preconditioner is not considered in the paper. Also no eigenvalue estimates are known for the modified preconditioning.

REMARK 3.1. The constants in (3.1), (3.2) may depend on other parameters, in particular they depend on viscosity ν . It is hard to find this dependence in an optimal way. At the same time, one dimensional analysis from [16] suggests that the upper bound in (3.2) may be tight with respect to h at least for some discretizations.

The following theorem extends result in (3.1) for the LBB unstable case and provides h -independent bounds for preconditioner (2.11). For the sake of brevity we prove the theorem assuming the exact inverses M_p^{-1} and L^{-1} . More practical choice of using spectrally equivalent preconditioners does not change the result.

THEOREM 3.2. *With the above assumption on the mesh and finite element spaces \mathbb{V}_h and \mathbb{Q}_h , forming **not necessarily** LBB stable pair, the following estimates hold*

$$c_1 \leq |\lambda(SS_1^{-1})| \leq C_1 \quad (3.3)$$

$$c_3 \leq |\lambda(SS_3^{-1})| \leq C_3 \quad (3.4)$$

with positive constants c_1, C_1, c_3, C_3 independent of the meshsize h .

For proving the theorem we will need several auxiliary estimates, which we put together in the following lemma.

LEMMA 3.3. *The ellipticity, continuity, and stability assumptions (1.5) – (1.7)*

yield the following estimates involving matrices A, L, B, C and M_p :

$$\alpha_1 \alpha_2^{-2} \|z\|^2 \leq \langle A^{-1} L^{\frac{1}{2}} z, L^{\frac{1}{2}} z \rangle \quad \forall z \in \mathbb{R}^n \quad (3.5)$$

$$\|L^{\frac{1}{2}} A^{-1} L^{\frac{1}{2}}\| \leq \alpha_1^{-1} \quad (3.6)$$

$$\|L^{-\frac{1}{2}} A L^{-\frac{1}{2}}\| \leq \alpha_2 \quad (3.7)$$

$$\|M_p^{-\frac{1}{2}} B L^{-\frac{1}{2}}\| = \|L^{-\frac{1}{2}} B^T M_p^{-\frac{1}{2}}\| \leq \gamma_3 \quad (3.8)$$

$$\|M_p^{-\frac{1}{2}} C M_p^{-\frac{1}{2}}\| \leq \gamma_2 \quad (3.9)$$

$$\langle (B L^{-1} B^T + C) q, q \rangle \geq \gamma_1^2 \langle M_p q, q \rangle \quad \forall q \in \mathbb{R}^m \quad (3.10)$$

Proof. Note that the continuity and ellipticity estimate from (1.5) can be rewritten in the matrix-vector notations:

$$\langle Au, v \rangle \leq \alpha_2 \|L^{\frac{1}{2}} u\| \|L^{\frac{1}{2}} v\| \quad \forall u, v \in \mathbb{R}^n \quad (3.11)$$

$$\alpha_1 \|L^{\frac{1}{2}} u\|^2 \leq \langle Au, u \rangle \quad \forall u \in \mathbb{R}^n \quad (3.12)$$

For arbitrary $z \in \mathbb{R}^n$ consider $u = A^{-1}z$ and $v = L^{-1}z$. Due to (3.11) and (3.12) one gets

$$\begin{aligned} \langle L^{-1}z, z \rangle &= \langle v, z \rangle = \langle v, Au \rangle \leq \alpha_2 \|L^{\frac{1}{2}} u\| \|L^{\frac{1}{2}} v\| = \alpha_2 \|L^{\frac{1}{2}} u\| \langle L^{-1}z, z \rangle^{\frac{1}{2}} \\ &\leq \alpha_1^{-\frac{1}{2}} \alpha_2 \langle Au, u \rangle^{\frac{1}{2}} \langle L^{-1}z, z \rangle^{\frac{1}{2}} = \alpha_1^{-\frac{1}{2}} \alpha_2 \langle A^{-1}z, z \rangle^{\frac{1}{2}} \langle L^{-1}z, z \rangle^{\frac{1}{2}}. \end{aligned}$$

This yields $\langle L^{-1}z, z \rangle \leq \alpha_1^{-1} \alpha_2^2 \langle A^{-1}z, z \rangle$ for any $z \in \mathbb{R}^n$ which is equivalent to (3.5). Further consider the following relations:

$$\begin{aligned} \langle A^{-1}z, z \rangle &= \langle u, z \rangle = \langle u, Lv \rangle \leq \|L^{\frac{1}{2}} u\| \|L^{\frac{1}{2}} v\| \leq \alpha_1^{-\frac{1}{2}} \langle Au, u \rangle^{\frac{1}{2}} \|L^{\frac{1}{2}} v\| \\ &= \alpha_1^{-\frac{1}{2}} \langle A^{-1}z, z \rangle^{\frac{1}{2}} \langle L^{-1}z, z \rangle^{\frac{1}{2}}. \end{aligned}$$

Thus we obtain

$$\alpha_1 \langle A^{-1}z, z \rangle \leq \langle L^{-1}z, z \rangle \quad \forall z \in \mathbb{R}^n.$$

We use this inequality and (3.12) to check

$$\langle A^{-1}z, y \rangle \leq \alpha_1^{-1} \langle L^{-1}z, z \rangle^{\frac{1}{2}} \langle L^{-1}y, y \rangle^{\frac{1}{2}}, \quad \forall y, z \in \mathbb{R}^n. \quad (3.13)$$

Indeed, denoting $u = A^{-1}z$ and $v = L^{-1}y$ we have

$$\begin{aligned} \langle A^{-1}z, y \rangle &= \langle u, y \rangle = \langle u, Lv \rangle \leq \|L^{\frac{1}{2}} u\| \|L^{\frac{1}{2}} v\| \leq \alpha_1^{-\frac{1}{2}} \langle Au, u \rangle^{\frac{1}{2}} \langle L^{-1}y, y \rangle^{\frac{1}{2}} \\ &= \alpha_1^{-\frac{1}{2}} \langle A^{-1}z, z \rangle^{\frac{1}{2}} \langle L^{-1}y, y \rangle^{\frac{1}{2}} \leq \alpha_1^{-1} \langle L^{-1}z, z \rangle^{\frac{1}{2}} \langle L^{-1}y, y \rangle^{\frac{1}{2}}. \end{aligned}$$

Now (3.6) follows from (3.13) through

$$\|L^{\frac{1}{2}} A^{-1} L^{\frac{1}{2}}\| = \sup_{x \neq 0} \sup_{y \neq 0} \frac{\langle L^{\frac{1}{2}} A^{-1} L^{\frac{1}{2}} x, y \rangle}{\|x\| \|y\|} = \sup_{x \neq 0} \sup_{y \neq 0} \frac{\langle A^{-1}x, y \rangle}{\|L^{-\frac{1}{2}} x\| \|L^{-\frac{1}{2}} y\|} \leq \alpha_1^{-1}.$$

In the same way (3.7) follows from (3.11):

$$\|L^{-\frac{1}{2}} A L^{-\frac{1}{2}}\| = \sup_{u \neq 0} \sup_{v \neq 0} \frac{\langle L^{-\frac{1}{2}} A L^{-\frac{1}{2}} u, v \rangle}{\|u\| \|v\|} = \sup_{u \neq 0} \sup_{v \neq 0} \frac{\langle Au, v \rangle}{\|L^{\frac{1}{2}} u\| \|L^{\frac{1}{2}} v\|} \leq \alpha_2.$$

Finally, thanks to (1.7) we get

$$\begin{aligned} & \|L^{-\frac{1}{2}}B^T M_p^{-\frac{1}{2}}\| \\ &= \sup_{q \neq 0} \frac{\|L^{-\frac{1}{2}}B^T q\|}{\|M_p^{\frac{1}{2}}q\|} = \sup_{q \neq 0} \sup_{v \neq 0} \frac{\langle L^{-\frac{1}{2}}B^T q, v \rangle^{\frac{1}{2}}}{\|M_p^{\frac{1}{2}}q\| \|v\|} = \sup_{q \neq 0} \sup_{v \neq 0} \frac{\langle q, Bv \rangle^{\frac{1}{2}}}{\|M_p^{\frac{1}{2}}q\| \|L^{\frac{1}{2}}v\|} \leq \gamma_3 \end{aligned}$$

and $\|M_p^{-\frac{1}{2}}BL^{-\frac{1}{2}}\| = \|(L^{-\frac{1}{2}}B^T M_p^{-\frac{1}{2}})^T\| \leq \gamma_3$.

Inequalities (3.9) and (3.10) easily follow from the conditions (1.7) and (1.6), respectively.

□

Now we are in position to prove theorem 3.2.

Proof. The proof uses the technique of norm equivalence developed in [15]. In particular we will show that

$$c_5 \|M_p q\|_* \leq \|S q\|_* \leq C_5 \|M_p q\|_* \quad \forall q \in \mathbb{Q}_h \quad (3.14)$$

and prove the estimates for S_k :

$$c_4 \|M_p q\|_* \leq \|S_k q\|_* \leq C_4 \|M_p q\|_*, \quad k = 1, 3 \quad \forall q \in \mathbb{Q}_h \quad (3.15)$$

From (3.15) and (3.14) one obtains the norm equivalence

$$c \|S_k q\|_* \leq \|S q\|_* \leq C \|S_k q\|_*, \quad k = 1, 3 \quad \forall q \in \mathbb{Q}_h \quad (3.16)$$

with mesh independent positive constants c and C . To complete the proof one may consider the obvious inequalities

$$\|S_k S^{-1}\|_*^{-1} \leq |\lambda(SS_k^{-1})| \leq \|SS_k^{-1}\|_* \quad (3.17)$$

As a consequence of (3.16) and (3.17) the estimate (3.4) follows for the eigenvalues of $S_k^{-1}S$ with some constants c_k, C_k independent of the meshsize h .

Therefore we can focus on checking (3.14) and (3.15). First we prove (3.14). The upper bound in (3.14) follows from

$$\begin{aligned} \|SM_p^{-1}\|_* &= \|M_p^{-\frac{1}{2}}(BA^{-1}B^T + C)M_p^{-\frac{1}{2}}\| \\ &\leq \|M_p^{-\frac{1}{2}}BL^{-\frac{1}{2}}\| \|L^{\frac{1}{2}}A^{-1}L^{\frac{1}{2}}\| \|L^{-\frac{1}{2}}B^T M_p^{-\frac{1}{2}}\| + \|M_p^{-\frac{1}{2}}CM_p^{-\frac{1}{2}}\| \leq \gamma_3 \alpha_1^{-1} + \gamma_2, \end{aligned}$$

here we used estimates (3.6), (3.8), (3.9). Next we find a lower bound for $\frac{\|SM_p^{-1}q\|_*}{\|q\|_*}$:

$$\begin{aligned} \inf_{q \neq 0} \frac{\|SM_p^{-1}q\|_*}{\|q\|_*} &= \inf_{q \neq 0} \frac{\|M_p^{-\frac{1}{2}}SM_p^{-\frac{1}{2}}q\|}{\|q\|} \geq \inf_{q \neq 0} \frac{\langle M_p^{-\frac{1}{2}}SM_p^{-\frac{1}{2}}q, q \rangle}{\|q\|^2} \\ &= \inf_{q \neq 0} \frac{\langle A^{-1}B^T M_p^{-\frac{1}{2}}q, B^T M_p^{-\frac{1}{2}}q \rangle + \langle CM_p^{-\frac{1}{2}}q, M_p^{-\frac{1}{2}}q \rangle}{\|q\|^2}. \quad (3.18) \end{aligned}$$

Denoting $u = B^T M_p^{-\frac{1}{2}}q$ and applying (3.5) and (3.10) we continue with (3.18):

$$\begin{aligned} \inf_{q \neq 0} \frac{\|SM_p^{-1}q\|_*}{\|q\|_*} &\geq \inf_{q \neq 0} \frac{\alpha_1 \alpha_2^{-2} \langle L^{-1}B^T M_p^{-\frac{1}{2}}q, B^T M_p^{-\frac{1}{2}}q \rangle + \langle CM_p^{-\frac{1}{2}}q, M_p^{-\frac{1}{2}}q \rangle}{\|q\|^2} \\ &= \inf_{q \neq 0} \frac{\langle (\alpha_1 \alpha_2^{-2} BL^{-1}B^T + C)q, q \rangle}{\|M_p^{\frac{1}{2}}q\|^2} \geq \min\{\alpha_1 \alpha_2^{-2}, 1\} \gamma_1^2. \end{aligned}$$

Thus the lower bound in (3.14) is proved with the h -independent constant $c_5 = \min\{\alpha_1\alpha_2^{-2}, 1\}\gamma_1^2$.

For $k = 1$ inequalities (3.15) were proved in [15]. The proof in [15] does not use the LBB stability assumption. Thus we can use this result and conclude that eigenvalue bounds (3.3) hold with h -independent constants c_1, C_1 .

To show (3.15) for $k = 3$ we first prove an upper bound on $\|M_p S_3^{-1}\|_*$. Thanks to (3.7), (3.8) and (3.9) one gets

$$\begin{aligned} \|M_p S_3^{-1}\|_* &= \|M_p^{-\frac{1}{2}}(BL^{-1}AL^{-1}B^T + C)M_p^{-\frac{1}{2}}\| \\ &\leq \|M_p^{-\frac{1}{2}}BL^{-\frac{1}{2}}\| \|L^{-\frac{1}{2}}AL^{-\frac{1}{2}}\| \|L^{-\frac{1}{2}}B^T M_p^{-\frac{1}{2}}\| + \|M_p^{-\frac{1}{2}}CM_p^{-\frac{1}{2}}\| \leq \gamma_3\alpha_2 + \gamma_2. \end{aligned}$$

Hence the low bound in (3.15) holds with the h -independent constant $c_4 = \frac{1}{(\gamma_3\alpha_2 + \gamma_2)}$.

Finally we find an upper bound for $\|S_3 M_p^{-1}\|_*$:

$$\begin{aligned} \|S_3 M_p^{-1}\|_*^{-1} &= \|M_p^{-\frac{1}{2}} S_3 M_p^{-\frac{1}{2}}\|^{-1} \\ &= \inf_{q \neq 0} \sup_{p \neq 0} \frac{\langle M_p^{\frac{1}{2}} S_3^{-1} M_p^{\frac{1}{2}} q, p \rangle}{\|q\| \|p\|} \geq \inf_{q \neq 0} \frac{\langle M_p^{\frac{1}{2}} S_3^{-1} M_p^{\frac{1}{2}} q, q \rangle}{\|q\|^2} \\ &= \inf_{q \neq 0} \frac{\langle AL^{-1}B^T M_p^{-\frac{1}{2}} q, L^{-1}B^T M_p^{-\frac{1}{2}} q \rangle + \langle C M_p^{-\frac{1}{2}} q, M_p^{-\frac{1}{2}} q \rangle}{\|q\|^2}. \end{aligned} \quad (3.19)$$

Denote $u = L^{-\frac{1}{2}}B^T M_p^{-\frac{1}{2}} q$. The condition (3.12) yields $\langle AL^{-\frac{1}{2}}u, L^{-\frac{1}{2}}u \rangle \geq \alpha_1 \|u\|^2$. Therefore we continue with (3.19):

$$\begin{aligned} \|S_3 M_p^{-1}\|_*^{-1} &\geq \inf_{q \neq 0} \frac{\alpha_1 \langle L^{-\frac{1}{2}}B^T M_p^{-\frac{1}{2}} q, L^{-\frac{1}{2}}B^T M_p^{-\frac{1}{2}} q \rangle + \langle C M_p^{-\frac{1}{2}} q, M_p^{-\frac{1}{2}} q \rangle}{\|q\|^2} \\ &= \inf_{q \neq 0} \frac{\langle (\alpha_1 BL^{-1}B^T + C)q, q \rangle}{\|M_p^{\frac{1}{2}} q\|^2} \geq \min\{\alpha_1, 1\} \gamma_1^2. \end{aligned}$$

Thus the upper bound in (3.15) is proved with the h -independent constant $C_4 = (1 + \alpha_1^{-1})\gamma_1^{-2}$. \square

4. Numerical experiments. In this section we present numerical results for two model problems in $[0, 1]^d$ for $d = 2, 3$. In the first problem the wind is parallel x axis and constant:

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (4.1)$$

The second problem is the linearized ‘‘driven cavity’’ problem. In the 2D case the velocity function is suggested in [3]:

$$\mathbf{w} = \begin{pmatrix} \frac{r_2}{2\pi} \frac{e^{r_2 y}}{(e^{r_2} - 1)} \sin\left(\frac{2\pi(e^{r_2 y} - 1)}{e^{r_2} - 1}\right) \left(1 - \cos\left(\frac{2\pi(e^{r_1 x} - 1)}{e^{r_1} - 1}\right)\right) \\ -\frac{r_1}{2\pi} \frac{e^{r_1 x}}{(e^{r_1} - 1)} \sin\left(\frac{2\pi(e^{r_1 x} - 1)}{e^{r_1} - 1}\right) \left(1 - \cos\left(\frac{2\pi(e^{r_2 y} - 1)}{e^{r_2} - 1}\right)\right) \end{pmatrix} \quad (4.2)$$

where $r_1 = 4, r_2 = 0.1$. This type of convection simulates a rotating vortex, which center has coordinates (x_0, y_0) , $x_0 \cong 0.831, y_0 \cong 0.512$ and $\max_{\Omega} |\mathbf{w}| \cong 1$ (Fig.4.1,left). In the 3D case the convection velocity field \mathbf{w} is the solution of the the ‘‘driven cavity’’ Stokes problem (Fig.4.1,right).

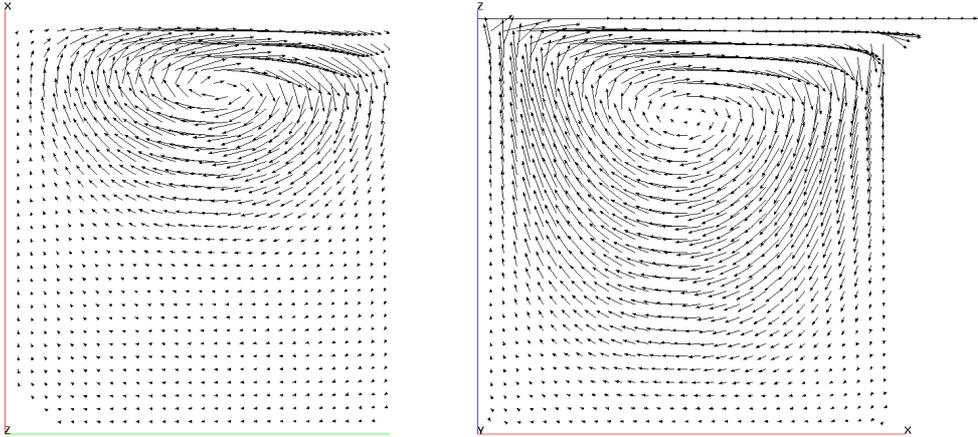


FIG. 4.1. *Left picture: the 2D velocity field (4.2), right picture: the central cross-section b Oxz -plane of the 3D velocity field.*

For the discretization method we use $isoP_2-P_0$ and $isoP_2-P_1$ finite elements defined on uniform triangulation (tetrahedrization) of a square (cubic) mesh in $[0, 1]^d$. The velocity triangulation is build by connecting the midpoints on the edges of triangles or tetrahedra. In all the cases the convection term is stabilized by SUPG [36]. In tables below h denotes the size for the pressure triangulations.

We use the block triangular matrix (1.9) as a right preconditioner in the Krylov subspace method for solving system (1.8). Some details of experiments may differ in the 2D and 3D case (cf. below), since we used two different FE software packages to treat two- and three-dimension problems, respectively. In the 2D experiments we use the BiCGStab and for 3D ones the full GMRES method is applied as an outer iterative solver. Note that the expense of one BiCGStab iteration approaches the cost of two GMRES iterations. The stopping criterion is the 10^{-6} decrease of the euclidian norm of the residual. The approximate inverses involved in application of the preconditioner (1.9) were computed as follows. The application of \hat{A}^{-1} to a vector is achieved via 3 multigrid (1,1)-cycles with alternating Gauss-Seidel method. In the 2D case W-cycle of geometric multigrid was used, whereas in the 3D case V-cycle of algebraic multigrid was adopted. Both choices give fairly good approximation to A^{-1} for all values of h and ν under consideration. Application of $(B\hat{M}_u^{-1}B^T)^{-1}$ and L_p^{-1} was evaluated using 10 V(4,4)-cycles in the 2D case and exact sparse factorization in the 3D. L^{-1} was evaluated using interior iterations to provide a very good approximation of the inversion. Thus all the inverses involved in all Schur complement preconditioners were evaluated with pretty high accuracy.

In the first experiment we illustrate the effect of different boundary conditions in A_p on the performance of the PCD preconditioner. Recall that our analysis in section 2.1 suggests that for the problem with \mathbf{w} from (4.1) one should implement Dirichlet's conditions at the inflow boundaries, while for the problem with \mathbf{w} from (4.2) one should use Neumann's conditions on the entire boundary. In table 4.1 we compare iteration counts for the PCD preconditioner with different boundary conditions in A_p in 2D and 3D and for two types of convection flow \mathbf{w} . For the case of \mathbf{w} from (4.1) we test two variants of boundary conditions in A_p : One consists in setting Neumann's

TABLE 4.1

PCD preconditioner; isoP₂-P₁ FE; Neumann's/Dirichlet's boundary conditions in A_p (for the constant wind boundary conditions are changed only at the inlet).

mesh size h	viscosity ν			
	0.1	0.01	10^{-3}	10^{-4}
w is 2D cavity vortex, #BiCGStab				
1/32	12 / 12	24 / 24	57 / 67	248 / 769
1/64	12 / 12	22 / 22	64 / 81	619 / 1267
w is constant 2D wind, #BiCGStab				
1/32	12 / 13	27 / 21	124 / 28	1027 / 48
1/64	11 / 13	23 / 24	116 / 33	1425 / 51
w is constant 3D wind, #GMRES				
1/8	28 / 36	69 / 58	368 / 201	548 / 285
1/16	28 / 42	45 / 39	232 / 162	1108 / 430

Number of the preconditioned iterations.

TABLE 4.2

Results for isoP₂-P₁ FE. PCD / BFBt / preconditioner (2.11).

mesh size h	viscosity ν			
	0.1	0.01	10^{-3}	10^{-4}
w is 2D constant wind, #BiCGStab				
1/32	13 / 8 / 22	21 / 6 / 35	28 / 8 / 39	48 / 10 / 46
1/64	13 / 12 / 27	24 / 8 / 34	33 / 7 / 34	51 / 12 / 40
1/128	13 / 17 / 23	24 / 16 / 30	41 / 6 / 37	62 / 9 / 37
w is 2D cavity vortex, #BiCGStab				
1/32	12 / 9 / 27	24 / 14 / 29	56 / 30 / 70	248 / 120 / 242
1/64	12 / 13 / 27	22 / 20 / 31	64 / 45 / 76	619 / 211 / 314
1/128	13 / 17 / 29	25 / 26 / 33	77 / 61 / 69	1053 / 349 / 446
w is 3D cavity vortex, #GMRES				
1/8	35 / 71 / 129	46 / 104 / 178	112 / 233 / 512	243 / 437 / 787
1/16	38 / 82 / 143	50 / 102 / 167	114 / 415 / 781	462 / 1402 / >2000

Number of preconditioned iterations.

boundary conditions on the whole boundary, alternatively we set Dirichlet's boundary conditions on the inflow and Neumann's conditions on the rest of the boundary. For the case of \mathbf{w} from (4.2) the problem has only characteristic boundaries, thus we test either setting Dirichlet's or Neumann's conditions in A_p on the whole boundary.

In L_p we use Neumann's conditions only. We found that changing boundary conditions in L_p does not improve convergence rates. Results of the experiments in table 4.1 are consistent with the analysis in § 2.1. The same phenomena was observed for discretizations with isoP₂-P₀ finite elements. Thus, further in the experiments we will always define A_p with Neumann's conditions for rotating \mathbf{w} and Dirichlet's conditions at the 'inflow' for \mathbf{w} from (4.1).

TABLE 4.3
Results for the *BFBt* preconditioner with *isoP₂-P₀* FE.

mesh size h	viscosity ν			
	0.1	0.01	10^{-3}	10^{-4}
w is 2D constant wind				
1/32	39	35	52	71
1/64	80	65	93	147
1/128	174	141	154	219

Number of preconditioned BiCGStab iterations.

TABLE 4.4
Results for *PCD* / (2.11) preconditioner with *isoP₂-P₀* FE .

mesh size h	viscosity ν			
	0.1	0.01	10^{-3}	10^{-4}
w is 2D constant wind				
1/32	9 / 17	11 / 24	21 / 36	68 / 58
1/64	9 / 13	12 / 23	21 / 32	57 / 66
1/128	9 / 11	14 / 23	25 / 31	50 / 59
w is 2D cavity vortex				
1/32	10 / 18	18 / 22	66 / 60	>2000 / 191
1/64	9 / 21	19 / 23	74 / 63	>2000 / 306
1/128	9 / 21	19 / 23	78 / 58	>2000 / 453

Number of preconditioned BiCGStab iterations.

In table 4.2 we compare convergence results for all three preconditioners tested for the *isoP₂-P₁* discretization of the Oseen problem with **w** from (4.1), (4.2), the latter case being examined both in 2D and 3D. All preconditioners demonstrate almost h -independent results except the case of small ν . In general, rotating flow with small viscosity turns out to be a hard problem for all preconditioners.

Next we proceed to approximations with *isoP₂-P₀* finite element method. For this discretization with discontinuous pressure approximation the *BFBt* preconditioner shows a strong h -dependence results even for the simplest constant parallel flow. This is illustrated in table 4.3.

In table 4.4 we examine *PCD* and (2.11) preconditioners which demonstrate h -independent convergence rate at least for the simplest case (4.1). We note that in the *PCD* preconditioner we now use mixed approximation for the pressure Poisson problem: $L_p = (B\hat{M}_u^{-1}B^T)$. To define A_p , we set $F_p := r_{u \rightarrow p} A_x r_{p \rightarrow u}$, where A_x is the x -subblock of A (may be with different boundary conditions), $r_{u \rightarrow p}$ and $r_{p \rightarrow u}$ are suitable mappings from \mathbb{Q}_h to \mathbb{V}_h and vice versa. The type of boundary conditions in A_p was taken the same as for *isoP₂-P₁* discretizations. We observe that up to $\nu = 10^{-3}$ both methods exhibits feasible convergence rate, and for $\nu = 10^{-4}$ the *PCD* method fails to converge.

5. Conclusions. The paper studies a preconditioning technique for finite element discretizations of the Oseen problem arising from Picard linearizations of the

steady Navier–Stokes equations. The preconditioner is block triangular and requires an approximation to the inverse of the pressure Schur complement matrix. We focus on several approaches for building the pressure Schur complement preconditioner. Two of them are well known from a literature and one is new. The preconditioners differ in implementation and performance for various discretizations and flow patterns. The paper accounts their properties and available theoretical results. We prove missing eigenvalue estimates and discuss some open implementation problems, such as the choice of an appropriate pressure boundary conditions in the method of Kay, Loghin, and Wathen. Numerical experiments show that all the methods work satisfactory (with mild dependence on ν) in the range of small and modest Reynolds numbers, however they may experience serious loss of efficiency when the Reynolds number is larger.

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