Numerical Methods and Simulation Techniques for Flow with Shear and Pressure dependent Viscosity

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Summary. In this note we present some of our recent results concerning flows with pressure and shear dependent viscosity. From the numerical point of view several problems arise, first from the difficulty of approximating incompressible velocity fields and, second, from poor conditioning and possible lack of differentiability of the involved nonlinear functions due to the material laws. The lack of differentiability can be treated by regularisation. Then, Newton-like methods as linearization technique can be applied; however the presence of the pressure in the viscosity function leads to an additional term introducing a new non-classical linear saddle point problem. The difficulty related to the approximation of incompressible velocity fields is treated by applying the nonconforming Rannacher-Turek Stokes element. However, then we are facing another problem related to the nonconforming approximation for problems involving the symmetric part of gradient: the classical discrete 'Korn's Inequality' is not satisfied. A new and more general approach which involves the jump across the inter-element boundaries should be used, which requires a small modification of the discrete bilinear form by adding an interface term, penalizing the jump of the velocity over edges. This is achieved via a modified procedure in the derivation of a Discontinuous Galerkin formulation. As a solver for the discrete nonlinear systems, a Newton variant is discussed while a 'Vanka-like' smoother as defect correction inside of a direct multigrid approach is presented. The results of some computational experiments for realistic flow configurations are provided, which contain a pressure dependent viscosity, too.

1 Introduction

The flowing of powders brings a new challenging and interesting problem to the CFD community: at very high concentrations and low rate-of-strain, grains are in permanent contact, rolling on each other. Therefore a frictional stress model must be taken into account. This can be done using plasticity and similar theories in which the material behavior is assumed to be independent of the velocity gradient or the rate-of-strain. This is in contrast to viscous Newtonian flow where stress specifically depends on a rate-of-strain. Furthermore, flowing powders do not exhibit viscosity and, again, this shows that a Newtonian rheology cannot describe granular flow accurately. It is assumed that the material is incompressible, dry, cohesionless, and perfectly rigid-plastic. Such properties are relevant for modelling the granular flows via special models for continuum mechanics, as for instance the Schaeffer model [9].

1.1 Equations of motion

The general equations of describing the motion of incompressible powders read:
Conservation of mass: $\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0$, $\frac{D*}{Dt}$ is the material derivative and $\boldsymbol{u}$ is the velocity vector.
Incompressible material: The bulk density, ρ, is a constant, so that $\nabla\cdot\boldsymbol{u} = 0$.
Equation of motion: $\rho\frac{Du}{Dt} = -\nabla\cdot\mathbf{T} + \rho g$ with $\mathbf{T} = \mathbf{S} + p\mathbf{I}$.

1.2 Constitutive equations

The constitutive equation is devoted to correlate between the deviatoric tensor, $\mathbf{S}$, and the velocity, through the second invariant of the rate deformation $D_{\mathrm{II}} = \frac{1}{2}\mathbf{D}:\mathbf{D}$, where the rate of deformation is given by $\mathbf{D} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u})$.
Newtonian law: $\mathbf{S} = 2\nu_{\mathrm{o}}\mathbf{D}$
Power law: $\mathbf{S} = 2\nu(D_{\mathrm{II}})\mathbf{D}, \quad \nu(z) = z^{\frac{r}{2}-1}, \quad r \geq 1$
Schaeffer's law: For a powder a constitutive equation which was first introduced by Schaeffer [9], has to obey a

- yield condition; $\|\mathbf{S}\| = \sqrt{2}p\sin\phi$,
- flow rule; $\mathbf{S} = \lambda\mathbf{D}$.

We use this correlation to obtain the constitutive equation $\mathbf{T} = \sqrt{2}p\sin\phi\frac{\mathbf{D}}{\|\mathbf{D}\|} + pI$.

1.3 Generalized Navier-Stokes equations

The problem can be stated in the framework of the generalized incompressible Navier-Stokes equations:

$$\rho\frac{Du}{Dt} = -\nabla p + \nabla\cdot(\nu(p, D_{\mathrm{II}})\mathbf{D}) + \rho g, \quad \nabla\cdot\,\boldsymbol{u} = 0$$

If we define the nonlinear pseudo viscosity $\nu(\cdot,\cdot)$ as a function of the second invariant of the rate deformation D_{II} and the 'pressure' p, we can show that different materials can be ranged within different viscosity laws including powder;

- Power law defined for $\nu(z,p) = \nu_{\mathrm{o}}z^{\frac{r}{2}-1}$
- Bingham law defined for $\nu(z,p) = \nu_{\mathrm{o}}z^{-\frac{1}{2}}$
- Schaeffer's law (including the 'pressure') defined for $\nu(z,p) = \sqrt{2}\sin\phi\; pz^{-\frac{1}{2}}$

2 Problem formulation

Let us consider the flow of the stationary (!) generalized Navier-Stokes problem in (1.3) in a bounded domain $\Omega\subset\mathbb{R}^2$. If we restrict the set V of test functions to be divergence-free and if we take the constitutive laws into account, the above equations from (1.3) lead to:

$$\int_\Omega 2\nu(D_{\mathrm{II}}(\boldsymbol{u}),p)\mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{v})\,dx + \int_\Omega(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\boldsymbol{v}\,dx = \int_\Omega \boldsymbol{f}\boldsymbol{v}\,dx, \qquad \forall\boldsymbol{v}\in V \qquad (1)$$

It is straightforward to penalize the constraint $\operatorname{div}\boldsymbol{v} = 0$ to derive the equivalent mixed formulations of (1):

Find $(\boldsymbol{u}, p) \in X \times M$ (with the spaces $X = H_0^1(\Omega)$ and $M = L^2(\Omega)$) such that:

$$\begin{aligned}
\int_\Omega 2\nu(D_{\mathrm{II}}(\boldsymbol{u}), p)\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})\, dx + \int_\Omega (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) v\, dx \quad &+ \int_\Omega p \operatorname{div} \boldsymbol{v}\, dx \\
= \int_\Omega \boldsymbol{f} \boldsymbol{v}\, dx, \qquad & \forall \boldsymbol{v} \in X, \\
\int_\Omega q \operatorname{div} \boldsymbol{u}\, dx = 0, \qquad & \forall q \in M,
\end{aligned} \tag{2}$$

2.1 Nonlinear solver: Newton iteration

In this approach, the nonlinearity is first handled on the continuous level. Let $\boldsymbol{u}^l$ being the initial state, the (continuous) Newton method consists of finding $\boldsymbol{u} \in V$ such that

$$\begin{aligned}
&\int_\Omega 2\nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v}) dx \\
&\qquad + \int_\Omega 2\partial_1 \nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{u})][\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})] dx \\
&\qquad\qquad + \boxed{\int_\Omega 2\partial_2 \nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})] p dx} \\
&\qquad\qquad = \int_\Omega \boldsymbol{f}\boldsymbol{v} - \int_\Omega 2\nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v}) dx, \quad \forall \boldsymbol{v} \in V\,,
\end{aligned} \tag{3}$$

where $\partial_i \nu(\cdot,\cdot); i = 1,2$ is the partial derivative of ν related to the first and second variables, respectively. To see this, set $\boldsymbol{X} = \mathbf{D}(\boldsymbol{u}^l), \boldsymbol{x} = \mathbf{D}(\boldsymbol{u}), Y = p^l, y = p, F(\boldsymbol{x}, y) = \nu(\frac{1}{2}|\boldsymbol{x}|^2, y)\boldsymbol{x}$ and $f(t) = F(\boldsymbol{X} + t\boldsymbol{x}, Y + ty)$, so that

$$\begin{aligned}
\partial_{x_j} F_i(\boldsymbol{x}, y) &= \partial_{x_j} \nu(\tfrac{1}{2}|\boldsymbol{x}|^2, y) x_j x_i + \nu(\tfrac{1}{2}|\boldsymbol{x}|^2, y)\delta_{ij} \\
\partial_y F_i(\boldsymbol{x}, y) &= \partial_y \nu(\tfrac{1}{2}|\boldsymbol{x}|^2, y) x_i
\end{aligned} \tag{4}$$

where δ_{ij} stands for the standard Kronecker symbol. Having

$$\begin{aligned}
f_i'(t) &= \textstyle\sum_j \partial_{x_j} F_i(\boldsymbol{X} + t\boldsymbol{x}, Y + ty) x_j + \partial_y F_i(\boldsymbol{X} + t\boldsymbol{x}, Y + ty) y \\
&= \nu(\tfrac{1}{2}|\boldsymbol{X} + t\boldsymbol{x}|^2, Y + ty)\boldsymbol{x}_i \\
&+ \partial_1 \nu(\tfrac{1}{2}|\boldsymbol{X} + t\boldsymbol{x}|^2, Y + ty)\langle \boldsymbol{X} + t\boldsymbol{x}, \boldsymbol{x}\rangle(\boldsymbol{X}_i + t\boldsymbol{x}_i) \\
&+ \partial_2 \nu(\tfrac{1}{2}|\boldsymbol{X} + t\boldsymbol{x}|^2, Y + ty) y (\boldsymbol{X}_i + t\boldsymbol{x}_i)
\end{aligned} \tag{5}$$

we decrease t towards zero, such that we obtain the Frechet derivative:

$$\begin{aligned}
\nabla \cdot [\ & 2\nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}) \\
&+ 2\partial_1 \nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)(\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{u}))\mathbf{D}(\boldsymbol{u}^l) \\
&+ 2\partial_2 \nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l) p \mathbf{D}(\boldsymbol{u}^l)]
\end{aligned} \tag{6}$$

2.2 New linear auxiliary problem

The resulting auxiliary subproblems in each Newton step consist of finding $(\boldsymbol{u}, p) \in X \times M$ as solutions of the linear (discretetized) systems

$$\begin{cases} A(\boldsymbol{u}^l, p^l)\boldsymbol{u} + \delta_d A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u} + Bp + \delta_p B^*(\boldsymbol{u}^l, p^l)p = R_u(\boldsymbol{u}^l, p^l), \\ B^T\boldsymbol{u} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = R_p(\boldsymbol{u}^l, p^l), \end{cases} \tag{7}$$

where $R_u(\cdot,\cdot)$ and $R_p(\cdot,\cdot)$ denote the corresponding nonlinear residual terms for the momentum and continuity equations, and the operators $A^(\boldsymbol{u}^l, p^l), B, A^*(\boldsymbol{u}^l, p^l)$ and $B^*(\boldsymbol{u}^l, p^l)$ are defined as follows:

$$\langle A(\boldsymbol{u}^l, p^l)\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega 2\nu(D_{\rm I\!I}(\boldsymbol{u}), p)\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})\, dx \tag{8}$$

$$\langle Bp, \boldsymbol{v}\rangle = \int_\Omega p\nabla \cdot \boldsymbol{v}\, dx \tag{9}$$

$$\langle A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega 2\partial_1\nu(D_{\rm I\!I}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{u})][D(\boldsymbol{u}^l) : D(\boldsymbol{v})]\, dx \tag{10}$$

$$\langle B^*(\boldsymbol{u}^l, p^l)\boldsymbol{v}, p\rangle = \int_\Omega 2\partial_2\nu(D_{\rm I\!I}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{v})]p\, dx \tag{11}$$

3 Discretization

We consider a subdivision $T \in \mathcal{T}_h$ consisting of quadrilaterals in the domain $\Omega_h \in \mathbb{R}^2$, and we employ the rotated bilinear *Rannacher-Turek* element [5]. For any quadrilateral T, let (ξ, η) denote a local coordinate system obtained by joining the midpoints of the opposing faces of T. Then, in the *nonparametric* case, we set on each element T

$$\tilde{Q}_1(T) := span\left\{1, \xi, \eta, \xi^2 - \eta^2\right\}. \tag{12}$$

The degrees of freedom are determined by the nodal functionals $\{F_\Gamma^{(a,b)}(\cdot), \Gamma \subset \partial\mathcal{T}_h\}$,

$$F_\Gamma^a := |\Gamma|^{-1}\int_\Gamma v d\gamma \qquad \text{or} \qquad F_\Gamma^b := v(m_\Gamma) \quad (m_\Gamma \text{ midpoint of edge } \Gamma) \tag{13}$$

such that the finite element space can be written as

$$\begin{aligned} W_h^{a,b} := \{v \in L_2(\Omega_h), v \in \tilde{Q}_1(T), \forall T \in \mathcal{T}_h, v \text{ continuous w.r.t. all} \\ \text{nodal functionals } F_{\Gamma_{i,j}}^{a,b}(\cdot), \text{ and } F_{\Gamma_{i0}}^{a,b}(v) = 0, \forall \Gamma_{i0}\}. \end{aligned} \tag{14}$$

Here, $\Gamma_{i,j}$ denote all inner edges sharing the two elements i and j, while Γ_{i0} denote the boundary edges of $\partial\Omega_h$. In this paper, we always employ version 'a' with the integral mean values as degrees of freedom. Then, the corresponding discrete functions will be approximated in the spaces

$$V_h := W_h^{a,b} \times W_h^{a,b}, \; L_h := \left\{q_h \in L^2(\Omega), q_{h|T} = \text{const. }, \forall \mathcal{T} \in \mathcal{T}_h\right\}. \tag{15}$$

Due to the nonconformity of the discrete velocities, the classical discrete 'Korn's Inequality' is not satisfied which is important for problems involving the symmetric part of the gradient [4]. Therefore, appropriate edge-oriented stabilization techniques (see [1, 2, 8], have to be included which directly treat the jump across the inter-elementary boundaries via adding the following bilinear form

$$\sum_{edges E} \frac{1}{|E|} \int_E [\phi_i][\phi_j] d\sigma \tag{16}$$

for all basis functions ϕ_i and ϕ_j of $W_h^{a,b}$. Taking into account an additional relaxation parameter $s = s(\nu)$, the corresponding stiffness matrices are defined via:

$$\langle \mathbf{S}\boldsymbol{u}, \boldsymbol{v} \rangle = s \sum_{E \in E_I \cup E_D} \frac{1}{|E|} \int_E [\boldsymbol{u}][\boldsymbol{v}] d\sigma \tag{17}$$

Here, the jump of a function $\boldsymbol{u}$ on an edge E is given by

$$[\boldsymbol{u}] = \begin{cases} \boldsymbol{u}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{u}^- \cdot \boldsymbol{n}^- & \text{on internal edges } E_I \;, \\ \boldsymbol{u} \cdot \boldsymbol{n} & \text{on Dirichlet boundary edges } E_D, \\ 0 & \text{on Neumann boundary edges } E_N, \end{cases} \tag{18}$$

where $\boldsymbol{n}$ is the outward normal to the edge and $(\cdot)^+$ and $(\cdot)^-$ indicate the value of the generic quantity $(\cdot)$ on the two elements sharing the same edge.

4 Linear solver

This section is devoted to give a brief description of the involved solution techniques for the resulting linear systems. For the nonconforming Stokes element $\tilde{Q}_1/Q_0$, a 'local pressure Schur complement' preconditioner (see [7]) as generalization of so-called 'Vanka smoothers' is constructed on patches Ω_i which are ensembles of one single or several mesh cells, and this local preconditioner is embedded as global smoother into an outer block Jacobi/Gauss-Seidel iteration which acts directly on the coupled systems of generalized Stokes, resp., Oseen type as described in [8]. If we denote by $\tilde{R}_u$ and $\tilde{R}_p$ the discrete residuals for the momentum and continuity equation which include the complete stabilisation term due to the modified bilinear form $\mathbf{S}$ as described in (17), one smoothing step in defect-correction notation can be described as

$$\begin{bmatrix} \boldsymbol{u}^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{l} \\ p^{l} \end{bmatrix} + \omega^l \textstyle\sum_i \begin{pmatrix} F + S^*_{|\Omega_i} & \tilde{B} + \delta_p \tilde{B}^*_{|\Omega_i} \\ \tilde{B}^T_{|\Omega_i} & 0 \end{pmatrix}^{-1} \begin{bmatrix} \tilde{R}_u(\boldsymbol{u}^l, p^l) \\ \tilde{R}_p(\boldsymbol{u}^l, p^l) \end{bmatrix} \tag{19}$$

with matrix $F = \tilde{A} + \delta_d \tilde{A}^*$ and $\tilde{A}, \tilde{B}, \tilde{A}^*$ and $\tilde{B}^*$ are the discrete matrices corresponding to the operators in (8), (9), (10) and (11). For the preconditioning step only a part of the matrix, i.e. $F + S^*$, is involved. All other components in the multigrid approach, that means intergrid transfer, coarse grid correction and coarse grid solver, are the standard ones and are based on the underlying hierarchical mesh hierarchy and the properties of the nonconforming finite elements (see [7] and [8] for the details).

5 Numerical tests

5.1 Newtonian case

In this case, the gradient and tensor formulations are equivalent; the accuracy and efficiency of the stabilized tensor discretization is checked by comparisons with the gradient formulation (see Table 1); the tests have been performed for the 'flow around cylinder' benchmark configuration [10]. For all three formulations the lift and drag forces are very similar.

Table 1. Efficiency of the stabilized nonconforming FEM: Lift and Drag forces

Level 5				
$1/\nu$		grad	tensor	stab. tensor
1	Drag	31252×10^{-1}	31221×10^{-1}	31231×10^{-1}
	Lift	30898×10^{-3}	30924×10^{-3}	30936×10^{-3}
	NL/MG	3/3	7/200	3/3
1000	Drag	55657×10^{-4}	55531×10^{-4}	55535×10^{-4}
($Re = 20$)	Lift	10180×10^{-6}	10259×10^{-6}	10277×10^{-6}
	NL/MG	11/4	11/12	11/3

5.2 Effect of convection

The average number of inner multigrid sweeps (MG) per outer nonlinear sweep (NL) increases with mesh refinement (see Table 2), due to the more dominant influence of the kernel function in the second order differential operator. Since, in contrast, the convection

Table 2. Nonlinear iteration (NL)/Averaged multigrid sweeps (MG) per nonlinear iteration for different viscosity parameter (*Re* numbers) and various formulations (gradient, tensor and stabilized tensor) and for different mesh levels

	$1/\nu$	1	10	1000
Level	Formulation	NL/MG	NL/MG	NL/MG
4	grad	3/3	4/3	11/4
	tensor	3/**15**	5/**17**	11/**4**
	stab. tensor	3/3	5/3	11/4
5	grad	3/3	4/3	11/3
	tensor	4/**140**	5/**35**	11/**10**
	stab. tensor	4/3	5/3	11/3
6	grad	3/3	4/3	11/3
	tensor	7/**200**	4/**161**	11/**12**
	stab. tensor	3/3	4/3	11/3

dominates with the increase of the Reynolds number, the average number of multigrid sweeps per nonlinear sweep decreases, as the influence of the kernel function is getting irrelevant. This may explain why many people from the CFD community did not pay much attention to this problem before.

5.3 Power law case

In this case the nonlinear viscosity has the form $\nu(z) = \nu_o z^{\frac{r}{2}-1}$, $z = D_{\mathrm{II}}$, and the gradient and tensor formulation are not equivalent any more. The quality of the solution is checked by comparisons with the well-known and stable conforming Q_2/P_1 approximation; the extended description can be seen in [3]. The accuracy of the nonconforming FEM is saved with the stabilized tensor discretization, see Table 3.

Table 3. Comparison of the aproximation results for lift, drag and pressure difference for two FEM approaches, the stabilized nonconforming $\tilde{Q}_1/Q_0$ and the classical conforming Q_2/P_1 (see [3]).

Level	Elements	Drag	Lift	$\triangle\ p$	NN/NL	Drag	Lift	$\triangle\ p$	NN/NL
Power		$r = 1.5$				$r = 1.1$			
4	$\tilde{Q}_1/Q_0$	1594.20	14.25	24.56	9/2	916.02	3.7381	15.74	12/2
	Q_2/P_1	1635.80	14.39	25.09	8/140	953.94	3.9217	15.82	19/294
5	$\tilde{Q}_1/Q_0$	1615.60	14.43	24.81	8/2	935.13	3.9954	15.82	15/3
	Q_2/P_1	1637.60	14.44	25.07	9/723	957.64	4.0587	15.87	18/1162
6	$\tilde{Q}_1/Q_0$	1626.20	14.46	24.94	8/2	946.22	4.0592	15.85	13/5

5.4 Pressure dependent viscosity

Finally, the nonlinear (pseudo) viscosity has the form $\nu(p, z) = \exp(\beta p)$, and we list the number of resulting nonlinear iterations and the averaged number of multigrid sweeps per nonlinear iteration for both Newton and Fixpoint methods as outer nonlinear solver. Table 4 shows that the presence of the new linear operator B^* cannot be ignored; otherwise, we destroy the efficiency of the Newton method which is necessary for the robust treatment of the significant nonlinearity.

Table 4. Corresponding results for the number of nonlinear iterations and the averaged number of linear sweeps per nonlinear cycle

$\nu(z,p) = \exp(\beta p)$		Fixpoint			Newton		
Level	β	0.1	0.3	0.5	0.1	0.3	0.5
5	stab. tensor	6/2	12/2	33/2	3/3	4/2	4/3
	gradient	6/2	11/2	34/2	3/3	4/2	4/3
6	stab. tensor	5/3	11/3	65/2	3/3	3/3	3/3
	gradient	5/3	9/3	76/2	3/3	3/3	5/3

6 Conclusion and outlook

We can conclude our present numerical analysis as follows:

- The proposed stabilization technique is stable and accurate for the used FEM spaces.
- The full (!) Newton method seems to be necessary for this type of nonlinear problem.
- The multigrid convergence behaviour for this new class of auxiliary linear subproblems is
  - (almost) identical for both gradient and deformation tensor formulations: ***The stabilization for nonconforming FEM works fine!***
  - depending on the involved pressure terms for both fixed point and Newton methods: ***More investigation should focus on the linear algebraic problem, beside the nonlinear solution procedure!***

In future, we will cover a wider range of granular materials (see [6] for a discussion):

- **General equation of motion for a powder**

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \nabla \cdot \left[\frac{q(p,\rho)}{\left\| \mathbf{D} - \frac{1}{n}\nabla \cdot \boldsymbol{u} I \right\|} \left(\mathbf{D} - \frac{1}{n}\nabla \cdot \boldsymbol{u} I \right) \right] + \rho g, \text{ with}$$

- **Continuity equation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \text{ and}$$

- **Normality condition**

$$\nabla \cdot \boldsymbol{u} = \frac{\partial q(p,\rho)}{\partial p} \left\| \mathbf{D} - \frac{1}{n}\nabla \cdot \boldsymbol{u} I \right\|$$

- **The yield condition** $q(p,\rho)$ **is given by:**

Powder properties	Non-cohesive	Cohesive
Incompressible	$p \sin\phi$	$p\sin\phi + c\cos\phi$
Compressible	$p\sin\phi \left[2 - \frac{p}{\rho^{\frac{1}{\beta}}} \right]$	$p\sin\phi \rho^{\frac{1}{\beta}} - C \frac{(p-\rho^{\frac{1}{\beta}})^2}{\rho^{\frac{1}{\beta}}}$

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