Numerical Methods and Simulation Techniques for flow with Shear and Pressure dependent Viscosity

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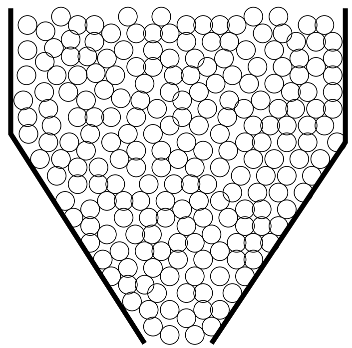
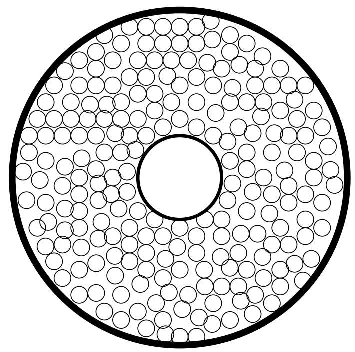
Motivation of this work

- **The flow of granular materials**
  - Example of application
    Pharmaceutical Industry, Food Processing, Soil Mechanics ...



Granular material storage



Couette flow

- **What about the viscosity !!?**
  - From engineering point of view this material does not have any viscosity!!
  - From mathematical and numerical point of view we are able to set this type of fluid in the same range of flow with generalized viscosity, since it exibits the same difficulties !?

Equations of motion

The general equation of motion for incompressible powders

- **Conservation of mass**

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\boldsymbol{u}) = 0$$

$\frac{D*}{Dt}$ is the material derivative and $\boldsymbol{u}$ is the velocity vector

- **For an incompressible material**
  the bulk density, ρ, is a constant thus

$$\nabla \cdot \boldsymbol{u} = 0$$

- **The equation of motion**

$$\rho\frac{Du}{Dt} = -\nabla \cdot \mathbf{T} + \rho g$$

with, $\mathbf{T} = \mathbf{S} + p\mathbf{I}$, the deviatoric stress.

Constitutive equation

The constitutive equation is devoted to correlate between the deviatoric tensor, $\mathbf{S}$, and the velocity, throught the rate of deformation $\mathbf{D} = -\frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$, and assure the closure of equations.

- **Newtonian law**

$$\mathbf{S} = 2\nu_{\mathsf{o}}\mathbf{D}$$

- **Power law**

$$\mathbf{S} = 2\nu(D_{\|})\mathbf{D}, \quad \nu(z) = z^{\frac{r}{2}-1}, \quad r > 1$$

- **Schaeffer's law (1987):** For a powder a constitutive equation first introduced by Schaeffer (1997), which has to obey a
  - yield condition; $\|\mathbf{S}\| = \sqrt{2}p\sin\phi$, and
  - flow rule; $\mathbf{S} = \lambda\mathbf{D}$

  we use this correlation to obtain the constitutive equation

$$\mathbf{S} = \sqrt{2}p\sin\phi\frac{\mathbf{D}}{\|\mathbf{D}\|}$$

Generalized Navier-Stokes Equations

- The generalized incompressible Navier-Stokes problem

$$\rho \frac{Du}{Dt} = -\nabla p + \nabla \cdot (\nu(p, D_{\mathrm{II}})\mathbf{D}) + \rho g, \quad \nabla \cdot \boldsymbol{u} = 0$$

If we define the nonlinear ''pseudo viscosity'' $\nu(\cdot, \cdot)$ as a function of $D_{\mathrm{II}}(u) = \frac{1}{2}\mathbf{D} : \mathbf{D}$ and p, then we can show that different materials could be ranged with different viscosity law including powder;

  - Power law defined for
$$\nu(z, p) = \nu_0 z^{\frac{r}{2}-1}$$
  - Bingham law defined for
$$\nu(z, p) = \nu_0 z^{-\frac{1}{2}}$$
  - Schaeffer's law (including the pressure) defined for
$$\nu(z, p) = \sqrt{2} \sin\phi \, p z^{-\frac{1}{2}}$$

New problems

In what follows we will show how to deal with the following problems

- **Discretization method:**
  How to use nonconforming finite element methods for problems involving rate of deformation tensor rather than the gradient !?

- **Nonlinear solver:**
  How to apply Newton linearization technique for this highly nonlinear and irregular problem !?

- **Linear multigrid solver:**
  In connection with the first two problems, how to keep the efficiency of the linear multigrid solver!?

Nonlinear Solver: Newton iteration

Let $\boldsymbol{u}^l$ being the initial state, the (continuous) Newton method consists of finding $\boldsymbol{u}$ such that

$$\int_\Omega 2\nu(D_{\mathbf{II}}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})dx$$

$$+ \int_\Omega 2\partial_1\nu(D_{\mathbf{I}}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{u})][\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})]dx$$

$$+ \boxed{\int_\Omega 2\partial_2\nu(D_{\mathbf{II}}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})]pdx}$$

$$= \int_\Omega \boldsymbol{f}\boldsymbol{v} - \int_\Omega 2\nu(D_{\mathbf{II}}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})dx, \quad \forall \boldsymbol{v}, \quad (1)$$

where $\partial_i\nu(\cdot,\cdot); i = 1, 2$ is the partial derivative of ν related to the first and second variable, respectively.

New Linear Algebraic Problem

The algorithm consists of finding $(\boldsymbol{u}, p)$ as solution of the linear system

$$\begin{cases} A(\boldsymbol{u}^l, p^l)\boldsymbol{u} + \delta_d A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u} + Bp + \delta_p B^*(\boldsymbol{u}^l, p^l)p & = R_u(\boldsymbol{u}^l, p^l), \\ B^T\boldsymbol{u} & = R_p(\boldsymbol{u}^l, p^l), \end{cases} \tag{2}$$

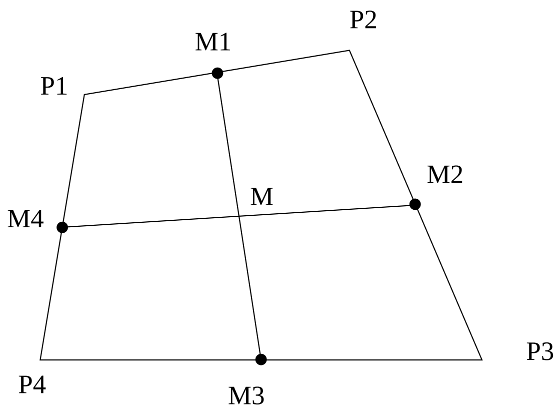
where $R_u(\cdot,\cdot)$ and $R_p(\cdot,\cdot)$ denote the corresponding nonlinear residual terms for the momentum and continuity equations, and the matrix $A^*(\boldsymbol{u}^l, p^l)$ and $B^*(\boldsymbol{u}^l, p^l)$ are defined as follows, respectively

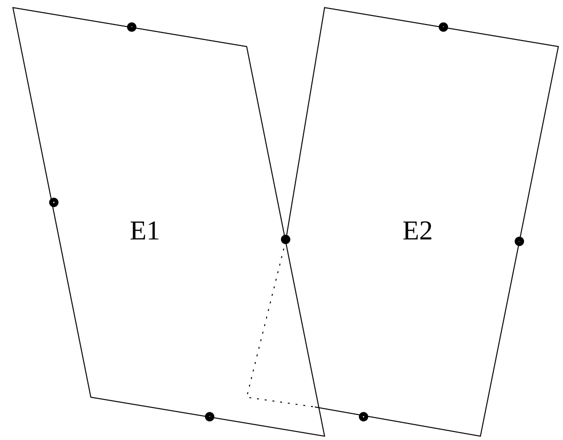
$$\langle A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega 2\partial_1\nu(D_{\rm II}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{u})][D(\boldsymbol{u}^l) : D(\boldsymbol{v})]dx. \tag{3}$$

$$\boxed{\langle B^*(\boldsymbol{u}^l, p^l)p, \boldsymbol{v}\rangle = \int_\Omega 2\partial_2\nu(D_{\rm II}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{v})]pdx.} \tag{4}$$

Spatial discretization

- **Quadrilateral Rannacher-Turek Stokes Element**



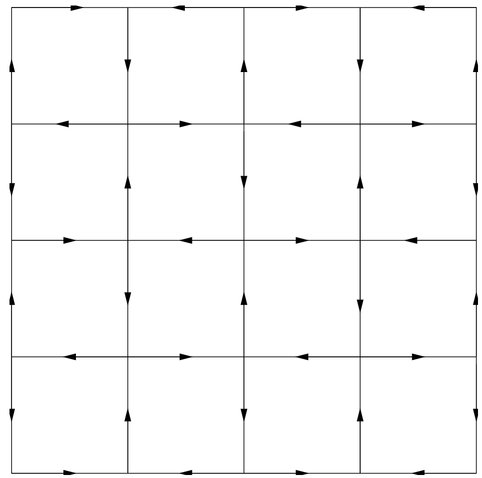


- **Advantage:**
  - Stable and efficient for incompressible flow.
  - Compact data structures.
- **Disadvantage:** Not satisfying discrete Korn's inequality

$$\sum_{\tau \in \tau_h} \|v\|_{H^1(\tau)} \leq c(\|v\|_{L^2}^2 + \|D(v)\|_{L^2}^2)^{\frac{1}{2}} \tag{5}$$

Kernel function

- The specific kernel function takes -1 or 1 in midpoints



- The constants in Korn's inequality for gradient, tensor and the stabilized tensor

NEL	$\|u_h\|_G$	$\|u_h\|_T$	$\|u_h\|_{ST}$
256	1.3	6.9×10^{-13}	0.85
1024	1.4	1.4×10^{-13}	0.90
4096	1.4	3.0×10^{-12}	0.92
16384	1.4	6.1×10^{-12}	0.93

Stabilized Rannacher-Turek Stokes Element

- **Remedy**: Stabilized R-T FEM (Hansbo et. al)

The stabilization consists of adding the following bilinear form

$$\sum_{E \in E_I \cup E_D} \frac{1}{|E|} \int_E [\phi_i][\phi_j] ds \tag{6}$$

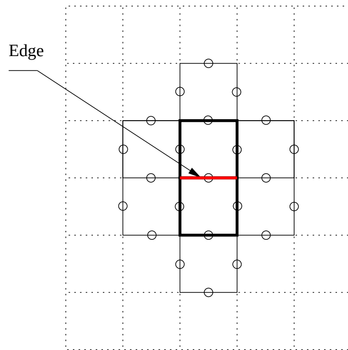
for all basis function ϕ_i and ϕ_j with a weighted parameter $s = s(\nu)$, then the corresponding matrix S is defined as:

$$< Su, v >= \sum_{E \in E_I \cup E_D} \frac{1}{|E|} \int_E [u][v] ds \tag{7}$$

Linear solver

- **Multigrid for velocity and pressure simultanously:**
  - Vanca-like block Gauss-Seidel scheme as smoother and solver
  - adaptive step length control for correction step (with F-cycle)
  - macro-elementwise interpolation for grid-transfer

- **Reduced sparsity of the matrix S**



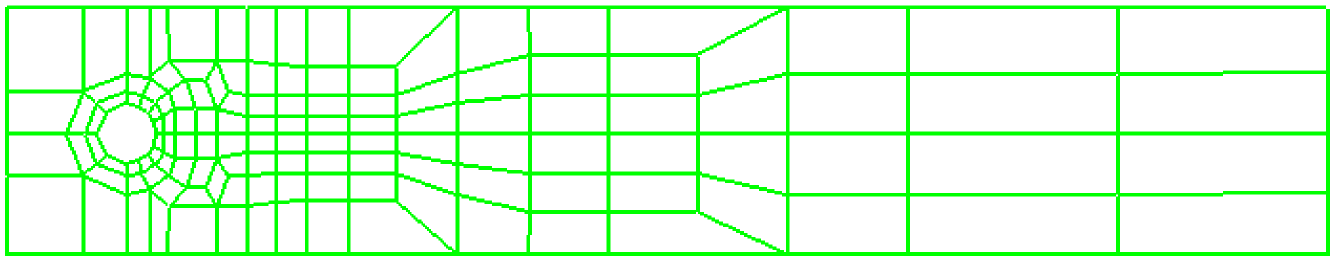
- **Defect correction method:**

$$\begin{bmatrix} \boldsymbol{u}^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{l} \\ p^{l} \end{bmatrix} + \omega^l \sum_i \begin{pmatrix} F + S^*_{|\Omega_i} & \tilde{B} + \delta_p \tilde{B}^*_{|\Omega_i} \\ \tilde{B}^T_{|\Omega_i} & 0 \end{pmatrix}^{-1} \begin{bmatrix} \tilde{R}_u(\boldsymbol{u}^l, p^l) \\ \tilde{R}_p(\boldsymbol{u}^l, p^l) \end{bmatrix}$$

For the preconditioning step only a part of the matrix, i.e. $F + S^*$, is taken

Newtonian case

In this case the gradient and tensor formulation are equivalent; the efficiency of the stabilized tensor discretization is checked by comparison with the gradient one on the **flow around cylinder benchmark**



Level 5				
$1/\nu$		grad	tensor	stab. tensor
1	Drag	31252×10^{-1}	31221×10^{-1}	31231×10^{-1}
	Lift	30898×10^{-3}	30924×10^{-3}	30936×10^{-3}
1000	Drag	55657×10^{-4}	55531×10^{-4}	55535×10^{-4}
$(Re = 20)$	Lift	10180×10^{-6}	10259×10^{-6}	10277×10^{-6}

- For all three formulations the **lift** and **drag** forces are similar

Power law case

In this case the nonlinear viscosity has the form $\nu(z) = \nu_0 z^{\frac{r}{2}-1}, z = D_{\rm II}$, the gradient and tensor formulation are not equivalent any more. The quality of the solution is checked by the comparison with the stable conforming Q_2/P_1 approximation,

Level	Elements	Drag	Lift	$\triangle\ p$	Drag	Lift	$\triangle\ p$
Power		$r = 1.5$			$r = 1.1$		
4	$\tilde{Q}_1/Q_0$	1594.20	14.25	24.56	916.02	3.7381	15.74
	Q_2/P_1	1635.80	14.39	25.09	953.94	3.9217	15.82
5	$\tilde{Q}_1/Q_0$	1615.60	14.43	24.81	935.13	3.9954	15.82
	Q_2/P_1	1637.60	14.44	25.07	957.64	4.0587	15.87
6	$\tilde{Q}_1/Q_0$	1626.20	14.46	24.94	946.22	4.0592	15.85

The accuracy of the FEM is saved with stabilized tensor discretization!

Effect of convection

	$1/\nu$	1	10	1000
Level	Formulation	NL/MG	NL/MG	NL/MG
4	grad tensor stab. tensor	3/3 3/**15** 3/3	4/3 5/**17** 5/3	11/4 11/**4** 11/4
5	grad tensor stab. tensor	3/3 4/**140** 4/3	4/3 5/**35** 5/3	11/3 11/**10** 11/3
6	grad tensor stab. tensor	3/3 7/**200** 3/3	4/3 4/**161** 4/3	11/3 11/**12** 11/3

- **gradient and stabilized tensor:** similar behaviour
- **tensor :** the number of linear multigrid sweeps
  - increases with refinement; **kernel funcion dominates**
  - decreases with the increases of the Reynold number; **convection dominates**

This may explain why people from the CFD community did not pay attention to this problem much more before!

Pressure dependent viscosity

In this case the nonlinear pseudo viscosity has the form $\nu(p,z) = Q(p) z^{\frac{r}{2}-1}$

- The corresponding matrix of the linear problem
  - can no longer be fitted into classical saddle point problems;

$$M_{\delta_p}(\tilde{\boldsymbol{u}}, \tilde{p}) = \begin{pmatrix} A & B + \delta_p B^* \\ B^T & 0 \end{pmatrix}$$

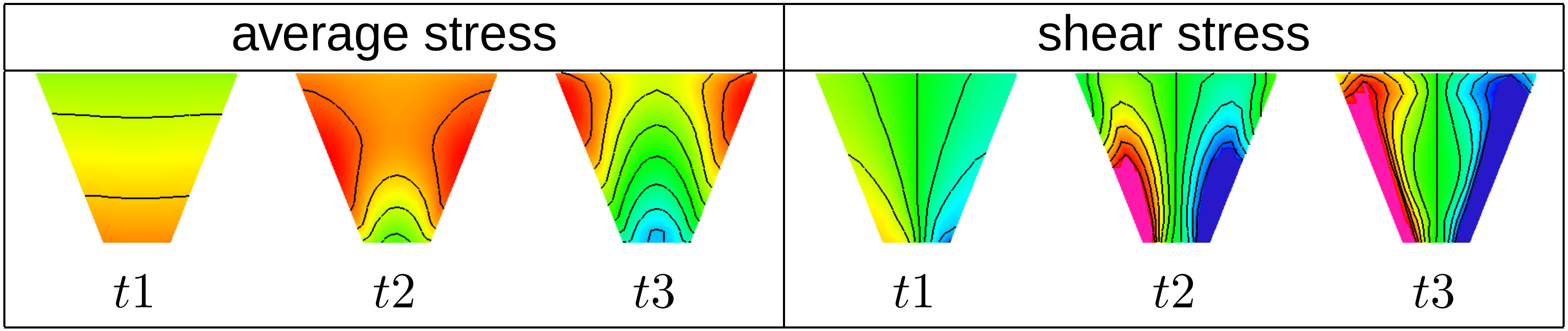
  - the solution is relative to the choice of imposing the uniqueness, since $\dim(\text{null}(M_{\delta_p})) = 1$

- **Efficiency of the solver:** we increase the nonlinearity and list the number of resulting nonlinear iterations and the averaged number of multigrid sweeps per nonlinear iteration for both Newton and fixpoint

$\nu(z,p) = \exp(\beta p)$		Fixpoint			Newton		
Level	β	0.1	0.3	0.5	0.1	0.3	0.5
5	stab. tensor	$6/2$	$12/2$	$33/2$	$3/3$	$4/2$	$4/3$
	gradient	$6/2$	$11/2$	$34/2$	$3/3$	$4/2$	$4/3$
6	stab. tensor	$5/3$	$11/3$	$65/2$	$3/3$	$3/3$	$3/3$
	gradient	$5/3$	$9/3$	$76/2$	$3/3$	$3/3$	$5/3$

Schaeffer's law

- **Schaeffer's law:** The time dependent equations are **linearly ill-posed** according to Schaeffer; in the Navier-Stokes equations, the pressure force associated to the constraint $\operatorname{div} v = 0$ can do no work. By contrast, the pressure force in equation of granular flow can do work, and for plane waves in certain directions, it does so.
- **Pseudo compressibility:** This scheme is an effort to regularize the instability in order to study the shear-band



The plot of the average stress and shear stress in a hopper shows the development of instability which leads to shear-banding.

Outlook

We have been able to develop the numerical methods and techniques to simulate a dense granular flow, in futur we want to cover a wide range of granular materials

- **General equation of motion for a powder**

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \nabla \cdot \left[\frac{q(p,\rho)}{\left\| \mathbf{D} - \frac{1}{n}\nabla \cdot \boldsymbol{u} I \right\|} \left(\mathbf{D} - \frac{1}{n}\nabla \cdot \boldsymbol{u} I \right) \right] + \rho g, \text{ with}$$

- **Continuity equation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \text{ and}$$

- **Normality condition**

$$\nabla \cdot \boldsymbol{u} = \frac{\partial q(p,\rho)}{\partial p} \left\| \mathbf{D} - \frac{1}{n}\nabla \cdot \boldsymbol{u} I \right\|$$

- **the yield condition $q(p, \rho)$ is given by:**

Powder properties	Non-cohesive	Cohesive
Incompressible	$p \sin \phi$	$p \sin \phi + c \cos \phi$
Compressible	$p \sin \phi \left[2 - \frac{p}{\rho^{\frac{1}{\beta}}} \right]$	$p \sin \phi \rho^{\frac{1}{\beta}} - C \frac{(p - \rho^{\frac{1}{\beta}})^2}{\rho^{\frac{1}{\beta}}}$