Efficient Numerical Methods and Simulation Techniques for Granular flow

Abderrahim Ouazzi, Stefan Turek

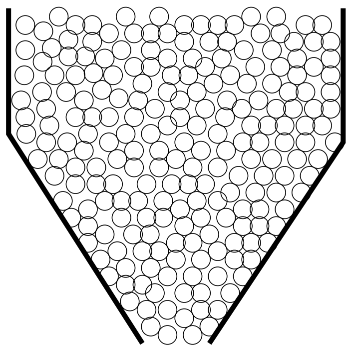
Institute for Applied Mathematics, LS III,

Dortmund University,

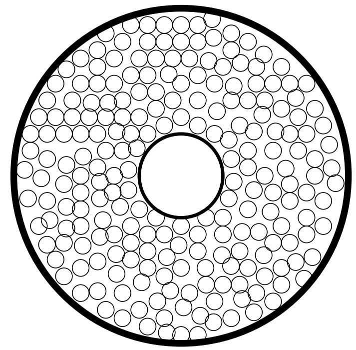
Germany

Motivation of this work

- **The flow of granular materials**
  - Example of application
    Pharmaceutical Industry, Food Processing, Soil Mechanics ...



Granular material storage

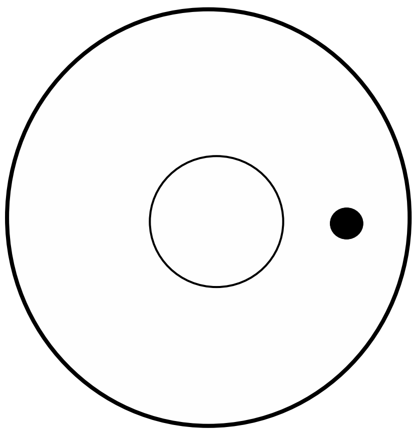


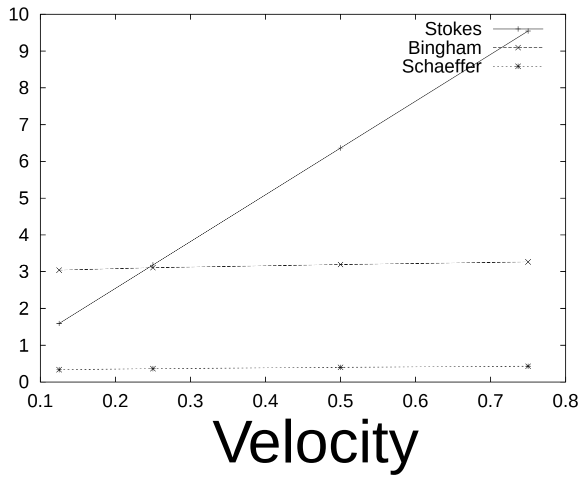
Couette flow

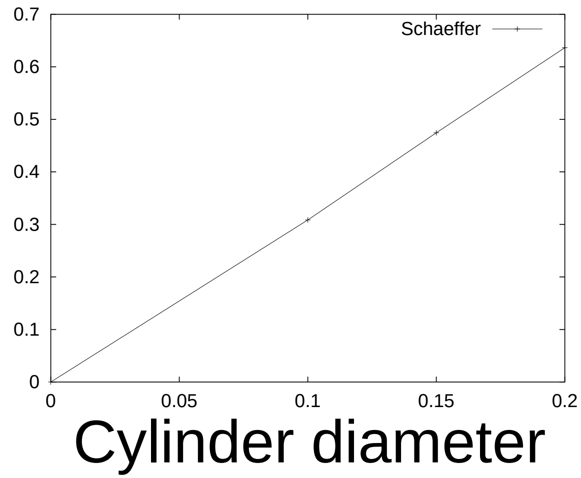
- **What about the viscosity !!?**
  - From engineering point of view this material do not exibit viscosity!!
  - From mathematical and numerical point of view we are able to set this type of problem in the same range of flow with generalized viscosity, since it exibits the same difficulties !?

Flow around an immersed cylinder

- The dependence of **drag force** with grain **velocity** in a couette flow around a cylinder for different material and for different cylinder diameter







- The **drag force** for schaeffer and Bingham flow acting on cylinder is **independent** of the grain **velocity**, contrary to the stokes flow

This goes ahead with experiments and gives more advantageous position to our handling of granular flow in the framework of fluid mechanics !

Equations of motion

The general equation of motion for incompressible powders

- **Conservation of mass**

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

  $\frac{D*}{Dt}$ is the material derivative and $\boldsymbol{u}$ is the velocity vector

- **For an incompressible material**
  the bulk density, ρ, is a constant thus

$$\nabla \cdot \boldsymbol{u} = 0$$

- **The equation of motion**

$$\rho \frac{Du}{Dt} = -\nabla \cdot \mathbf{T} + \rho g$$

  with, $\mathbf{T} = \mathbf{S} + p\mathbf{I}$.

Constitutive equation

The constitutive equation is devoted to correlate between the deviatoric tensor, **S**, and the velocity, throught the rate of deformation $\mathbf{D} = -\frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$, and assure the closure of equations.

- **Newtonian law**

$$\mathbf{S} = 2\nu_0 \mathbf{D}$$

- **Power law**

$$\mathbf{S} = 2\nu(D_{\|})\mathbf{D}, \quad \nu(z) = z^{\frac{r}{2}-1}, \quad r > 1$$

- **Schaeffer's law (1997):** For a powder a constitutive equation first introduced by Schaeffer (1997), which has to obey a
  - yield condition; $\|\mathbf{S}\| = \sqrt{2}p\sin\phi$, and
  - flow rule; $\mathbf{S} = \lambda\mathbf{D}$

  we use this correlation to obtain the constitutive equation

$$\mathbf{S} = \sqrt{2}p\sin\phi\frac{\mathbf{D}}{\|\mathbf{D}\|}$$

Generalized Navier-Stokes Equations

- The generalized incompressible Navier-Stokes problem

$$\rho \frac{Du}{Dt} = -\nabla p + \nabla \cdot (\nu(p, D_{\mathrm{I}})\mathbf{D}) + \rho g, \quad \nabla \cdot \boldsymbol{u} = 0$$

If we define the nonlinear ''pseudo viscosity'' $\nu(\cdot, \cdot)$ as a function of $D_{\mathrm{I}}(u) = \frac{1}{2}\mathbf{D} : \mathbf{D}$ and p, then we can show that different materials could be ranged with different viscosity law including powder;

  - Power law defined for

$$\nu(z, p) = \nu_0 z^{\frac{r}{2}-1}$$

  - Bingham law defined for

$$\nu(z, p) = \nu_0 z^{-\frac{1}{2}}$$

  - Schaeffer's law (including the pressure) defined for

$$\nu(z, p) = \sqrt{2} \sin\phi \, p z^{-\frac{1}{2}}$$

New mathematical problems

In what follows we will show how to deal with the following problems

- **Discretization method:**
  How to use nonconforming finite element methods for problems involving rate of deformation tensor rather than the gradient !?

- **Singular viscosity:**
  Stabilization of singular phenomena due to nonlinear viscosity !?

- **Nonlinear solver:**
  How to apply Newton linearization technique for this highly nonlinear and irregular problem !?

- **Linear multigrid solver:**
  In connection with the first two problems, how to keep the efficiency of the linear multigrid solver!?

Nonlinear Solver: Newton iteration

Let $\boldsymbol{u}^l$ being the initial state, the (continuous) Newton method consists of finding $\boldsymbol{u}$ such that

$$
\begin{aligned}
&\int_\Omega 2\nu(D_{\rm II}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})dx \\
&\qquad + \int_\Omega 2\partial_1\nu(D_{\rm II}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{u})][\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})]dx \\
&\qquad\qquad + \boxed{\int_\Omega 2\partial_2\nu(D_{\rm II}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})]p dx} \\
&\qquad\qquad\quad = \int_\Omega \boldsymbol{f}\boldsymbol{v} - \int_\Omega 2\nu(D_{\rm II}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})dx, \quad \forall \boldsymbol{v}, \qquad (1)
\end{aligned}
$$

where $\partial_i\nu(\cdot,\cdot); i = 1,2$ is the partial derivative of ν related to the first and second variable, respectively.

New Linear Algebraic Problem

The algorithm consists of finding $(\boldsymbol{u}, p)$ as solution of the linear system

$$\left\{ \begin{array}{ll} A(\boldsymbol{u}^l, p^l)\boldsymbol{u} + \delta_d A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u} + Bp + \delta_p B^*(\boldsymbol{u}^l, p^l)p & = R_u(\boldsymbol{u}^l, p^l), \\ B^T \boldsymbol{u} & = R_p(\boldsymbol{u}^l, p^l), \end{array} \right. \tag{2}$$

where $R_u(\cdot, \cdot)$ and $R_p(\cdot, \cdot)$ denote the corresponding nonlinear residual terms for the momentum and continuity equations, and the matrix $A^*(\boldsymbol{u}^l, p^l)$ and $B^*(\boldsymbol{u}^l, p^l)$ are defined as follows, respectively

$$\langle A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega 2\partial_1 \nu(D_{II}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{u})][D(\boldsymbol{u}^l) : D(\boldsymbol{v})]dx. \tag{3}$$

$$\boxed{\langle B^*(\boldsymbol{u}^l, p^l)p, \boldsymbol{v}\rangle = \int_\Omega 2\partial_2 \nu(D_{II}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{v})]p dx.} \tag{4}$$

Power law case

In this case the nonlinear viscosity has the form $\nu(z) = \nu_0 z^{\frac{r}{2}-1}$, $z = D_\parallel$, the gradient and tensor formulation are not equivalent any more. The quality of the solution is checked by the comparison with the stable conforming Q_2/P_1 approximation,

Level	Elements	Drag	Lift	$\triangle p$	Drag	Lift	$\triangle p$
Power		$r = 1.5$			$r = 1.1$		
4	$\tilde{Q}_1/Q_0$	1594.20	14.25	24.56	916.02	3.7381	15.74
	Q_2/P_1	1635.80	14.39	25.09	953.94	3.9217	15.82
5	$\tilde{Q}_1/Q_0$	1615.60	14.43	24.81	935.13	3.9954	15.82
	Q_2/P_1	1637.60	14.44	25.07	957.64	4.0587	15.87
6	$\tilde{Q}_1/Q_0$	1626.20	14.46	24.94	946.22	4.0592	15.85

- **The accuracy of the nonconforming FEM is saved with stabilized tensor discretization!**
- **More efficient solver for nonconforming FEM availible !**

Pressure dependent viscosity

In this case the nonlinear pseudo viscosity has the form $\nu(p,z) = Q(p) z^{\frac{r}{2}-1}$

- The corresponding matrix of the linear problem
  - can no longer be fitted into classical saddle point problems;

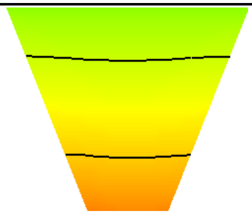
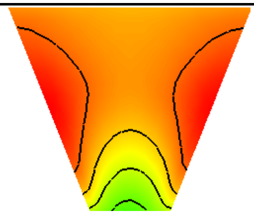
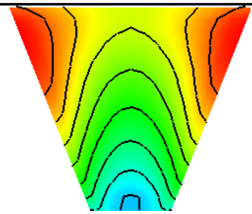
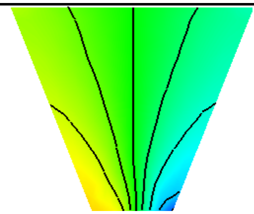
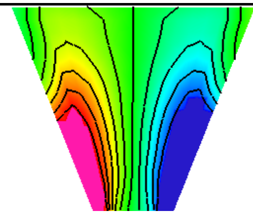
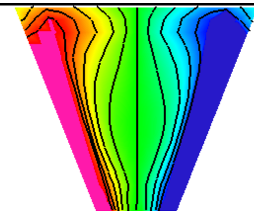
$$M_{\delta_p}(\tilde{\boldsymbol{u}}, \tilde{p}) = \begin{pmatrix} A & B + \delta_p B^* \\ B^T & 0 \end{pmatrix}$$

  - the solution is relative to the choice of imposing the uniqueness, since $\dim(\text{null}(M_{\delta_p})) = 1$
- **Efficiency of the solver:** we increase the nonlinearity and list the number of resulting nonlinear iterations and the averaged number of multigrid sweeps per nonlinear iteration for both Newton and fixpoint

$\nu(z,p) = \exp(\beta p)$		Fixpoint			Newton		
Level	β	0.1	0.3	0.5	0.1	0.3	0.5
5	stab. tensor	$6/2$	$12/2$	$33/2$	$3/3$	$4/2$	$4/3$
6	stab. tensor	$5/3$	$11/3$	$65/2$	$3/3$	$3/3$	$3/3$

Schaeffer's law

- **Schaeffer's law:** The time dependent equations are **linearly ill-posed** according to Schaeffer; in the Navier-Stokes equations, the pressure force associated to the constraint $\operatorname{div} v = 0$ can do no work. By contrast, the pressure force in equation of granular flow can do work, and for plane waves in certain directions, it does so.
- **Pseudo compressibility:** This scheme is an effort to regularize the instability in order to study the shear-band

average stress			shear stress		
					
$t1$	$t2$	$t3$	$t1$	$t2$	$t3$

The plot of the average stress and shear stress in a hopper shows the development of instability which leads to shear-banding.

Chronogram and curent work

- Since last spring we are working closely with Gabriel I. Tardos et. all (of University of New York) in order to:
  - develop theoretical models for powder with natural extension from fluid-mechanic field
  - use simple couette flow configuration as a benchmark and check theory with experimental and numerical results

- This with the main objective of coming up with Kolymbas model with our collaboration with the group G. Rombach and F. Neumann of TUHH.

Curent work

We have been able to develop the numerical methods and techniques to simulate a dense granular flow, in futur we want to cover a wide range of granular materials

- **General equation of motion for a powder**

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \nabla \cdot \left[\frac{q(p,\rho)}{\left\|\mathbf{D} - \frac{1}{n}\nabla\cdot\boldsymbol{u}I\right\|} \left(\mathbf{D} - \frac{1}{n}\nabla\cdot\boldsymbol{u}I\right)\right] + \rho g, \text{ with}$$

- **Continuity equation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \text{ and}$$

- **Normality condition**

$$\nabla \cdot \boldsymbol{u} = \frac{\partial q(p,\rho)}{\partial p} \left\|\mathbf{D} - \frac{1}{n}\nabla\cdot\boldsymbol{u}I\right\|$$

- **the yield condition $q(p, \rho)$ is given by:**

Powder properties	Non-cohesive	Cohesive
Incompressible	$p \sin\phi$	$p\sin\phi + c\cos\phi$
Compressible	$p\sin\phi\left[2 - \frac{p}{\rho^{\frac{1}{\beta}}}\right]$	$p\sin\phi\rho^{\frac{1}{\beta}} - C\frac{(p-\rho^{\frac{1}{\beta}})^2}{\rho^{\frac{1}{\beta}}}$