Edge-oriented stabilisation for nonconforming finite element methods

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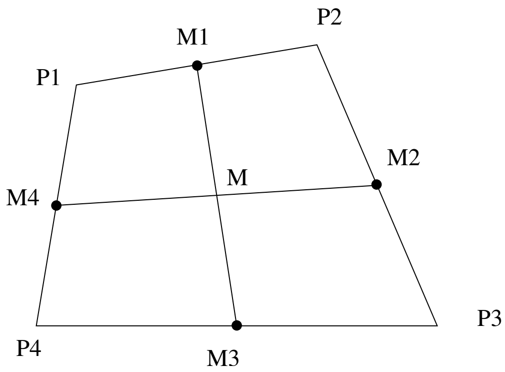
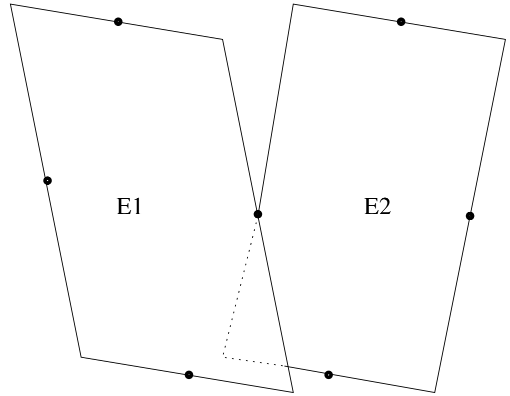
Spatial discretization

- Quadrilateral Rannacher-Turek Stokes Element





- Advantage
  - Stable and efficient for incompressible flow.
  - Compact data structures.

- Disadvantage
  - Not satisfying **classical** discrete Korn's inequality

$$\sum_{\tau \in \mathcal{T}_h} \|v\|_{H^1(\tau)} \leq c(\|v\|_{L^2}^2 + \|D(v)\|_{L^2}^2)^{\frac{1}{2}}$$

  - Numerical oscillations for convection dominated problem

Aspects of this contribution

- **Aspect of necessity:** When does NCFEM fail !?
  - Lack of coercivity for nonconforming low order approximations fo symmetric deformation formulation (small Re number)
  - Whenever convective operators are dominant (medium and high Re number)

- **Aspect of robustness:** The ability of edge stabilisation to overcome this wide range of Re number !?

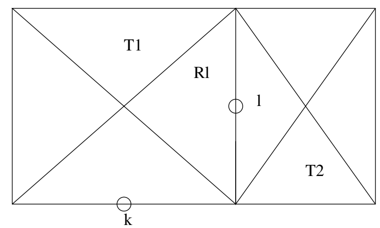
- **Aspect of efficiency:**
  - New sparsity of the matrix and its contradiction to the FE philosophy
  - Multigrid for the edge stabilisation for NCFEM

Classical Stabilisation methods

- Streamline diffusion

$$S = \sum_{\tau \in \tau_h} \delta_\tau \int_\tau (\boldsymbol{u}_h \cdot \nabla \boldsymbol{v}_h)(\boldsymbol{u}_h \cdot \nabla \boldsymbol{w}_h) dx$$

- Samarski's upwind



$$S = \sum_l \sum_{k \in \Lambda_l} \oint_{\Gamma_{lk}} \boldsymbol{u}_h \cdot \boldsymbol{n}_{lk} d\gamma [1 - \lambda_{lk}(\boldsymbol{u}_h)(\boldsymbol{v}_h(m_k) - \boldsymbol{v}_h(m_l))] w_h(m_l).$$

Based on the local Reynold number $Re_\tau = \frac{\|\boldsymbol{u}\|_\tau \cdot h_h}{\nu}$, we can either define

$$\delta_\tau = \delta^* \cdot \frac{h_\tau}{\|\boldsymbol{u}\|_\Omega} \cdot \frac{2Re_\tau}{1 + Re_\tau}, \quad \lambda_{lk}(u_h) = \begin{cases} \dfrac{\frac{1}{2} + \delta^* Re_\tau}{1 + \delta^* Re_\tau} & \textbf{\textit{if }} Re_\tau \geq 0 \\ \dfrac{1}{2(1 - \delta^* Re_\tau)} & \textbf{\textit{otherwise}} \end{cases}$$

Edge Stabilisation methods

The following jump terms were introduced

- to achieve the same accuracy for SDNCFEM as SDCFEM by John et al.(1997)
- to guarantee discrete Korn's inequality by Hansbo and Larson (2002) and by Brenner (2004)

$$j_1(\boldsymbol{u},\boldsymbol{v}) = \sum_{\text{edge E}} \gamma\nu \frac{1}{|E|}\int_E [\boldsymbol{u}][\boldsymbol{v}]d\sigma$$

- to stabilize convection dominated problem by Burman and Hansbo (2003)

$$j_{2,\alpha}(\boldsymbol{u},\boldsymbol{v}) = \sum_{\text{edge E}} \gamma|E|^\alpha \int_E [\nabla\boldsymbol{u}][\nabla\boldsymbol{v}]d\sigma$$

$$j_{3,\alpha}(\boldsymbol{u},\boldsymbol{v}) = \sum_{\text{edge E}} \gamma|E|^\alpha \int_E [\boldsymbol{n}\cdot\nabla\boldsymbol{u}][\boldsymbol{n}\cdot\nabla\boldsymbol{v}]d\sigma$$

Edge Stabilisation methods

The following jump terms were introduced

- to control the nonconformity arising from the pressure term in Darcy's law by Burman and Hansbo (2003)

$$j(\boldsymbol{u},\boldsymbol{v}) = \sum_{\text{edge E}} \gamma\nu \frac{1}{|E|} \int_E [\boldsymbol{n}\cdot\boldsymbol{u}][\boldsymbol{n}\cdot\boldsymbol{v}]d\sigma$$

- by the midpoint continuity of the bilinear rotated element and the use Taylor expansion

$$j_{4,\alpha}(\boldsymbol{u},\boldsymbol{v}) = \sum_{\text{edge E}} \gamma|E|^{\alpha} \int_E [\boldsymbol{t}\cdot\nabla\boldsymbol{u}][\boldsymbol{t}\cdot\nabla\boldsymbol{v}]d\sigma$$

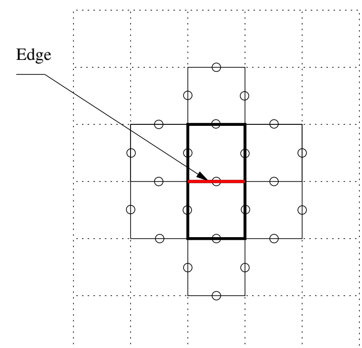
$$j_{5,\alpha}(\boldsymbol{u},\boldsymbol{v}) = \sum_{\text{edge E}} \gamma|E|^{\alpha} \int_E [(\boldsymbol{t}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n}][(\boldsymbol{t}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{n}]d\sigma$$

$$j_{6,\alpha}(\boldsymbol{u},\boldsymbol{v}) = \sum_{\text{edge E}} \gamma|E|^{\alpha} \int_E [(\boldsymbol{n}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{t}][(\boldsymbol{n}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{t}]d\sigma$$

Linear solver

- Multigrid for velocity and pressure simultanously:
  - Vanca-like block Gauss-Seidel scheme as smoother and solver
  - adaptive step length control for correction step (with F-cycle)
  - macro-elementwise interpolation for grid-transfer

- "Reduced" sparsity of the matrix S



- Defect correction method:

$$\begin{bmatrix} \boldsymbol{u}^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{l} \\ p^{l} \end{bmatrix} + \omega^l \sum_i \begin{pmatrix} S^*_{|\Omega_i} & B_{|\Omega_i} \\ B^T_{|\Omega_i} & 0 \end{pmatrix}^{-1} \begin{bmatrix} \tilde{R}_u(\boldsymbol{u}^l, p^l) \\ \tilde{R}_p(\boldsymbol{u}^l, p^l) \end{bmatrix}$$

For the preconditioning step only a part of the matrix, i.e. S^*, is taken

Aim

- **Looking at the literature**
  - different jump terms for different problems
  - different free constants γ and order $|E|^\alpha$
- **Aim:** one jump term and one parameter γ for different problems
  - discrete Korn's inequality
  - problem with medium and high Reynold number
  - problem with variable viscosity
- **Result:** work with the following formula

$$\max(\gamma\nu h, \gamma^* h^2) \sum_{\text{edge E}} \int_E [\nabla \boldsymbol{u}][\nabla \boldsymbol{v}] d\sigma$$

with $\gamma, \gamma^* \in [0.0001, 0.1]$

Stationary flow around cylinder: $Re = 20$

The drag and lift coefficients for gradient formulation with various stabilization techniques

gradient formulation								
Stab.	$j_{2,2}(\boldsymbol{u}, \boldsymbol{v})$			SD		UPW		Central
Level	γ			δ^*				
	0.0001	0.001	0.01	0.1	0.5	0.1	1.0	
Drag ($C_D = 5.5795$)								
4	5.5855	5.5864	5.5901	5.6417	5.7977	5.6005	5.7460	5.6040
5	5.5813	5.5815	5.5823	5.6020	5.6655	5.5841	5.6197	5.5862
6	5.5800	5.5800	5.5803	5.5868	5.6092	5.5806	5.5882	5.5812
Lift ($C_L = 0.01061$)								
4	0.009698	0.009806	0.010022	0.008633	0.007506	0.009697	0.007025	0.008604
5	0.010382	0.010398	0.010436	0.009914	0.009227	0.010483	0.010232	0.010043
6	0.010560	0.010562	0.010566	0.010394	0.010065	0.010598	0.010733	0.010471
NL/AVMG								
4	12/3	12/3	12/11	12/3	11/2	11/3	10/3	17/2
5	12/2	12/2	12/9	12/2	12/2	12/2	11/3	12/2
6	12/2	12/2	12/8	12/2	12/2	12/2	12/2	12/2

- Streamline and Upwind are more sensitive w.r.t. the free δ^*
- The accuracy for edge stabilisation remains independent of γ but the multigrid solver is slightly sensitive to over and under stabilisation

Stationary flow around cylinder: $Re = 20$

The drag and lift coefficients for deformation formulation with various stabilization techniques

deformation formulation								
Stab.	$\gamma j_{2,2}(\boldsymbol{u}, \boldsymbol{v})$			SD+$\gamma j_1(\boldsymbol{u}, \boldsymbol{v})$		UPW+$\gamma j_1(\boldsymbol{u}, \boldsymbol{v})$		central
Level	γ			$(\delta^* = 0.1)\gamma$				
	0.0001	0.001	0.01	0.0	0.1	0.0	0.1	
Drag ($C_D = 5.5795$)								
4	5.5846	5.5838	5.5811	5.6261	5.6264	5.5847	5.5850	5.5865
5	5.5810	5.5807	5.5790	5.5974	5.5975	5.5810	5.5811	5.5814
6	5.5799	5.5798	5.5793	5.5856	5.5856	5.5799	5.5799	5.5800
Lift ($C_L = 0.01061$)								
4	0.009893	0.009956	0.010120	0.009402	0.009412	0.009883	0.009894	0.009581
5	0.010432	0.010441	0.010464	0.010170	0.010173	0.010431	0.010434	0.010330
6	0.010572	0.010573	0.010574	0.010465	0.010465	0.010572	0.010573	0.010545
NL/AVMG								
4	12/3	12/2	12/12	12/2	12/2	12/5	12/2	19/2
5	12/3	12/2	12/8	12/5	12/2	12/11	12/2	21/2
6	12/4	12/2	12/8	12/9	12/2	12/12	12/2	26/4

- Streamline and Upwind requires additional stabilisation for deformatio formulation for the multigrid solver only
- For the edge stabilisation $j_{2,2}(\boldsymbol{u}, \boldsymbol{v})$, there is no need for any additiona stabilisation for deformation formulation

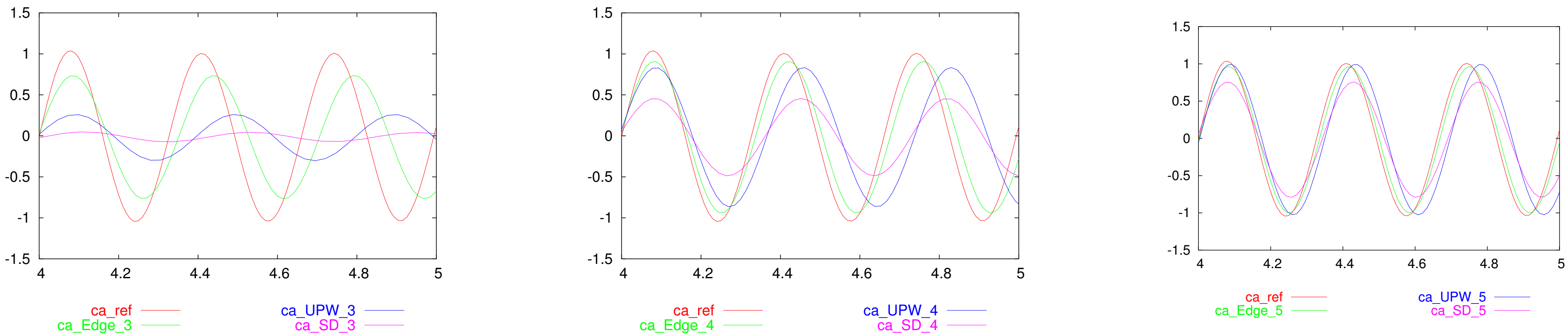
Stationary flow around cylinder: $Re = 20$

Remarks

- **Streamline and Samarski's upwind:**
  - The amount of artificial viscosity depends on the local Reynold number and on the value of the user-defined parameter δ^*
  - The multigrid behaves much better with the additional edge stabilisation $j_1(\boldsymbol{u}, \boldsymbol{v})$ for the deformation formulation
- **Edge Stabilisation $j_{2,2}(\boldsymbol{u}, \boldsymbol{v})$:**
  - The accuracy is less sensitive w.r.t the value of the user-defined parameter γ
  - For both gradient and deformation formulations the multigrid is or slightly sensitive to over and under stabilisation
  - No need for the additional edge stabilisation $j_1(\boldsymbol{u}, \boldsymbol{v})$ for the deformation formulation

Nonstationary flow around cylinder: $Re = 100$

- Lift coefficient for periodically oscillating flow



- maximun and minimum amplitude and the Strouhal of the lift for the periodically oscillating flow

Stab.	$j_{2,2}(\boldsymbol{u}, \boldsymbol{v})$		***SD*** ($\delta^* = 0.5$)		***UPW*** ($\delta^* = 0.1$)	
Level	***Max ampl***	***Strouhal***	***Max ampl***	***Strouhal***	***Max ampl***	***Strouhal***
3	0.7364	0.281	0.3219	0.257	0.2581	0.257
4	0.9075	0.296	0.4488	0.279	0.8295	0.261
5	0.9630	0.295	0.7549	0.280	0.9933	0.286
ref.	***0.99996***	***0.29976***	***0.99996***	***0.29976***	***0.99996***	***0.29976***

- **Samarski's upwind:** Good results for the amplitude (level 5)
- **Edge Stabilisation:** Good results for the amplitude and the frequency

Standing vortex $Re = \infty$

We consider the incompressible Navier-Stokes equations for inviscid flow ($Re = \infty$) in a unit square

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0, \quad \nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega = (0,1) \times (0,1). \qquad (1$$

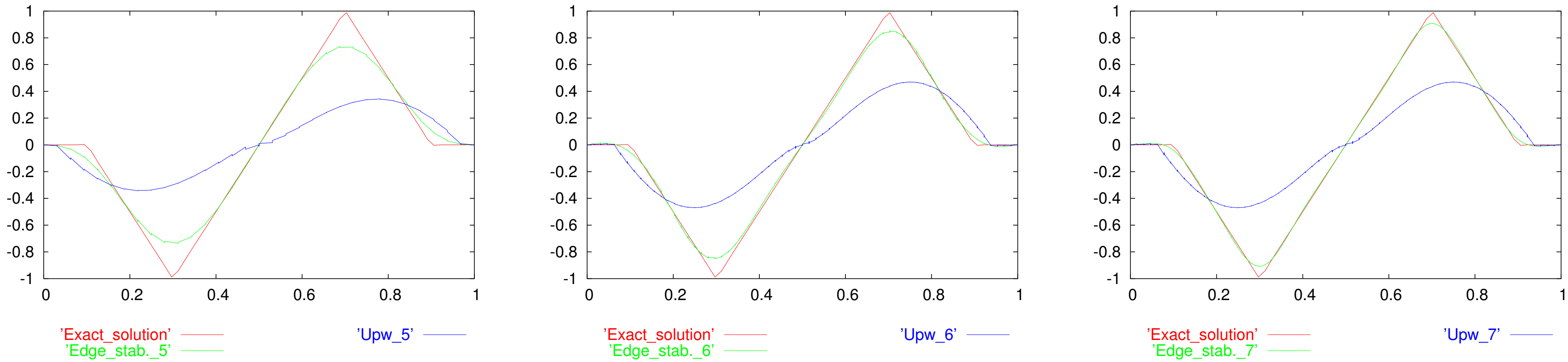
$$\boldsymbol{u}_r = 0, \quad \boldsymbol{u}_\theta = \begin{cases} 5r, & r < 0.2, \\ 2 - 5r, & 0.2 \le r \le 0.4, \\ 0, & r > 0.4, \end{cases} \qquad (1$$

where $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$ denotes the distance from the center

Which disrcetization schemes preserve the original vortex !?

Standing vortex $Re = \infty$

The cutline [(0.0, 0.5);(1.0,0.5)] of the solution at $t = 3s$



Remarks

- **Samarski's upwind:**
  - a significant smearing effect
  - strong dependency on the Re number; only first order upwind
- **Edge Stabilisation** $j_{2,2}(\boldsymbol{u}, \boldsymbol{v})$**:**
  - preserves "perfectly" the solution
  - no dependency on the Re number

Stationary flow around cylinder: Stokes

The drag and lift coefficients for gradient formulation with various stabilization techniques

gradient formulation							
Stab.	$j_{2,2}(\boldsymbol{u},\boldsymbol{v})$		**SD**		**UPW**		**Central**
Level	γ		δ^*				
	0.0001	**0.001**	**0.1**	**0.5**	**0.1**	**1.0**	
Drag ($C_D = 3142.4$)							
4	3127.4	3127.4	3127.4	3127.4	3127.4	3127.4	3127.4
5	3138.6	3138.6	3138.6	3138.6	3138.6	3138.6	3138.6
6	3141.5	3141.5	3141.5	3141.5	3141.5	3141.5	3141.5
Lift ($C_L = 30.865$)							
4	30.525	30.525	30.526	30.526	30.525	30.525	30.526
5	30.780	30.780	30.780	30.780	30.780	30.780	30.780
6	30.844	30.844	30.844	30.844	30.844	30.844	30.844
NL/AVMG							
4	3/2	3/2	3/2	3/2	3/2	3/2	3/2
5	4/2	4/2	4/2	4/2	4/2	4/2	4/2
6	4/2	4/2	4/2	4/2	4/2	4/2	4/2

- No need for any edge stabilisation, but no negative side effect

Stationary flow around cylinder: Stokes

The drag and lift coefficients for deformation formulation with various stabilization techniques

deformation formulation							
Stab.	$\gamma j_{2,1}(\boldsymbol{u},\boldsymbol{v})$		**SD**$+\gamma j_1(\boldsymbol{u},\boldsymbol{v})$		**UPW**$+\gamma j_1(\boldsymbol{u},\boldsymbol{v})$		**central**
Level	γ		$(\delta^* = 0.1)\gamma$				
	0.001	**0.01**	**0.0**	**0.1**	**0.0**	**0.1**	
Drag ($C_D = 3142.4$)							
4	3132.5	3133.6	3132.4	3133.1	3132.4	3133.1	3132.4
5	3139.9	3139.9	3139.9	3140.1	3139.9	3140.1	3139.9
6	3141.8	3141.6	3141.8	3141.8	3141.8	3141.8	3141.8
Lift ($C_L = 30.8657$)							
4	30.658	30.668	30.658	30.667	30.657	30.666	30.658
5	30.813	30.810	30.813	30.816	30.813	30.816	30.813
6	30.852	30.849	30.853	30.853	30.853	30.853	30.853
NL/AVMG							
4	4/3	4/2	4/29	4/2	4/29	4/2	4/29
5	4/3	4/2	5/98	4/2	4/99	4/2	5/98
6	4/3	4/2	5/154	4/2	5/154	4/2	5/154

- Edge stabilisation is a must for deformation formulation

Stationary flow around cylinder: Stokes

The drag and lift coefficients for deformation formulation with various stabilization techniques

Deformation						
Stab.	$\gamma j_{2,2}(\boldsymbol{u},\boldsymbol{v})$			$\gamma j_{2,1}(\boldsymbol{u},\boldsymbol{v})$		
Level	γ					
	0.0001	**0.1**	**1.0**	**0.001**	**0.01**	**0.1**
Drag ($C_D = 3142.4$)						
4	3132.4	3132.7	3134.9	3132.5	3133.6	3147.7
5	3139.9	3139.9	3140.1	3139.9	3139.9	3142.2
6	3141.8	3141.8	3141.8	3141.8	3141.6	3141.5
Lift ($C_L = 30.8657$)						
4	30.657	30.660	30.687	30.658	30.668	30.908
5	30.813	30.814	30.816	30.813	30.810	30.863
6	30.853	30.853	30.853	30.852	30.849	30.857
NL/AVMG						
4	5/18	4/4	4/3	4/3	4/2	3/6
5	9/20	4/6	4/2	4/3	4/2	3/6
6	11/22	4/10	4/3	4/3	4/2	4/6

- The accuracy is independent of the relaxation parameter γ
- The multigrid solver needs more analysis

Stationary flow around cylinder: Stokes

Nonlinear iterations (NL)/Average multigrid sweeps (AVMG) per nonlinear iteration for linear Stokes problem ($Re = 1$) with deformation formulation ar various edge stabilizations, several levels of refinement (Level)

$\nu = 1$								
γ	***0.01***	***0.05***	***0.1***	***0.5***	***0.01***	***0.05***	***0.1***	***0.5***
Level	$j_{2,1}(\boldsymbol{u}, \boldsymbol{v})$				$j_{2,2}(\boldsymbol{u}, \boldsymbol{v})$			
3	***4/2***	***4/2***	***4/4***	***–***	***4/7***	***4/3***	***4/2***	***4/2***
4	***4/2***	***4/2***	***4/5***	***–***	***4/14***	***4/6***	***4/4***	***4/2***
5	***4/2***	***4/2***	***4/5***	***–***	***4/19***	***4/10***	***4/7***	***4/3***
6	***4/2***	***4/2***	***4/6***	***–***	***4/24***	***4/16***	***4/11***	***4/6***

- The multigrid solver for edge stabilisation $j_{2,1}(\boldsymbol{u}, \boldsymbol{v})$ (with the order γh is independent of the refinement
- The multigrid solver for edge stabilisation $j_{2,2}(\boldsymbol{u}, \boldsymbol{v})$ (with the order γh significantly depends on the refinement

Unified Stabilisation for all Re number

Finally, we test our proposed unified stabilisation term with a wide range of constants γ for the benchmark of flow around cylinder

$\max(10\nu h, h^2)\sum_{\text{edge } E}\gamma\int_E[\nabla \boldsymbol{u}][\nabla \boldsymbol{v}]d\sigma$									
γ	0.0001	0.001	0.01	0.0001	0.001	0.01	0.0001	0.001	0.01
Level	drag			lift			solver		
Deformation, Stokes									
4	3132.5	3133.6	3147.7	30.658	30.668	30.908	4/3	4/2	3/6
5	3139.9	3139.9	3142.2	30.813	30.810	30.863	4/3	4/2	3/6
6	3141.8	3141.6	3141.5	30.852	30.849	30.857	4/4	4/2	4/12
gradient, Stokes									
4	3127.5	3128.2	3140.7	30.535	30.613	31.066	4/2	4/2	3/9
5	3138.6	3138.2	3139.2	30.783	30.813	30.966	4/2	4/2	3/9
6	3141.4	3141.0	3140.2	30.846	30.858	30.914	4/2	4/2	3/7
Deformation $Re = 20$									
4	5.5844	5.5815	5.5688	0.009890	0.009934	0.010087	12/2	12/2	12/11
5	5.5809	5.5795	5.5725	0.010430	0.010417	0.010375	12/2	12/2	12/8
6	5.5799	5.5793	5.5759	0.010571	0.010559	0.010501	12/2	12/2	12/8
gradient $Re = 20$									
4	5.5850	5.5818	5.5651	0.009698	0.009802	0.010075	12/3	12/3	12/11
5	5.5811	5.5797	5.5709	0.010379	0.010376	0.010376	12/2	12/2	12/9
6	5.5799	5.5794	5.5753	0.010558	0.010548	0.010503	12/2	12/2	12/8

- The accuracy is independent of the relaxation parameter γ
- The multigrid solver is also perfectly working up to a "small" sensitivit w.r.t over and under stabilisation

Generalized Navier-Stokes Equations

- The generalized incompressible Navier-Stokes problem reads

$$\rho \frac{Du}{Dt} = -\nabla p + \nabla \cdot (\nu(p, D_{\mathbf{I}})\mathbf{D}) + \rho g, \quad \nabla \cdot \boldsymbol{u} = 0$$

If we define the nonlinear ''pseudo viscosity'' $\nu(\cdot,\cdot)$ as a function of $D_{\mathbf{I}}(u) = \frac{1}{2}\mathbf{D} : \mathbf{D}$ and p, then we can show that different materials coul be ranged with different viscosity laws including ganular material;

  - Power law defined for

$$\nu(z,p) = \nu_0 z^{\frac{r}{2}-1}$$

  - Bingham law defined for

$$\nu(z,p) = \nu_0 z^{-\frac{1}{2}}$$

  - Schaeffer's law (including the pressure) defined for

$$\nu(z,p) = \sqrt{2}\sin\phi \, p z^{-\frac{1}{2}}$$

Nonlinear Solver: Newton iteration

Let $\boldsymbol{u}^l$ being the initial state, the (continuous) Newton method consists of finding $\boldsymbol{u}$ such that

$$\begin{aligned}
&\int_\Omega 2\nu(D_{\mathrm{I}}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})dx \\
&\qquad + \int_\Omega 2\partial_1\nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{u})][\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})]dx \\
&\qquad\qquad + \boxed{\int_\Omega 2\partial_2\nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})]p dx} \\
&\qquad\qquad\quad = \int_\Omega \boldsymbol{f}\boldsymbol{v} - \int_\Omega 2\nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})dx, \quad \forall \boldsymbol{v}, \quad (1
\end{aligned}$$

where $\partial_i\nu(\cdot,\cdot); i = 1, 2$ is the partial derivative of ν related to the first and second variable, respectively.

New Linear Algebraic Problem

The algorithm consists of finding $(\boldsymbol{u}, p)$ as solution of the linear system

$$\begin{cases} A(\boldsymbol{u}^l, p^l)\boldsymbol{u} + \delta_d A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u} + Bp + \delta_p B^*(\boldsymbol{u}^l, p^l)p & = R_u(\boldsymbol{u}^l, p^l), \\ B^T \boldsymbol{u} & = R_p(\boldsymbol{u}^l, p^l), \end{cases} \qquad (1$$

where $R_u(\cdot,\cdot)$ and $R_p(\cdot,\cdot)$ denote the corresponding nonlinear residual terr for the momentum and continuity equations, and the matrix $A^*(\boldsymbol{u}^l, p^l)$ and $B^*(\boldsymbol{u}^l, p^l)$ are defined as follows, respectively

$$\langle A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega 2\partial_1 \nu(D_{\|}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{u})][D(\boldsymbol{u}^l) : D(\boldsymbol{v})]dx. \qquad (1$$

$$\boxed{\langle B^*(\boldsymbol{u}^l, p^l)p, \boldsymbol{v}\rangle = \int_\Omega 2\partial_2 \nu(D_{\|}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{v})]p\,dx.} \qquad (1$$

Power law case

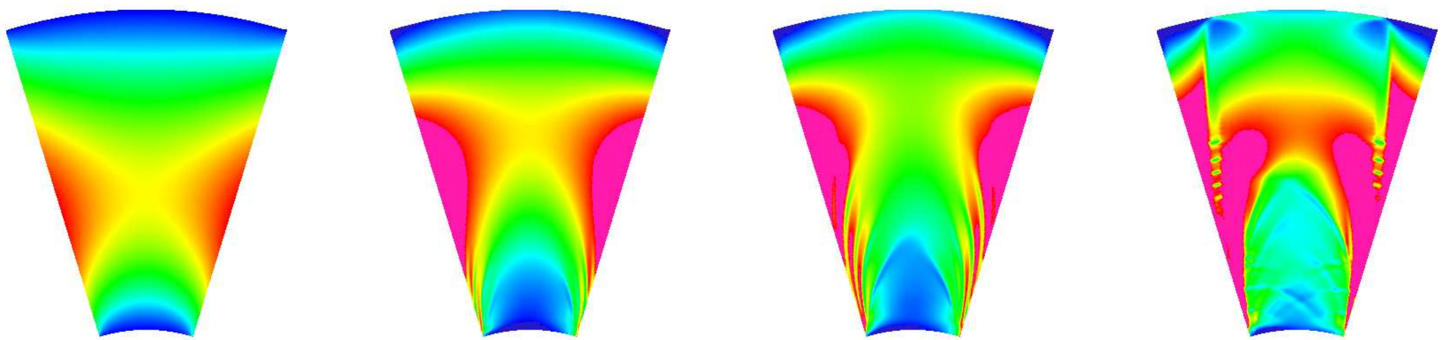
In this case the nonlinear viscosity has the form $\nu(z) = \nu_0 z^{\frac{r}{2}-1}, z = D_{II}$, the gradient and tensor formulation are not equivalent anymore. The quality of the solution is checked by reference values computed by the stable conforming Q_2/P_1 approximation, we use only quasi-Newton linearisation and give the Nonlinear iterations (NL)/Average multigrid sweeps (AVMG)

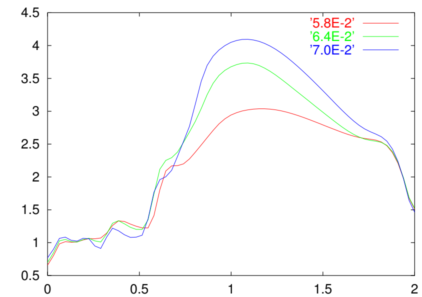
Level	Drag	Lift	NN/NL	Drag	Lift	NL/AVMG
	$r = 1.5$			$r = 1.1$		
3	1617.50	13.76	12/2	959.06	3.7057	38/1
4	1628.90	14.26	10/2	955.81	4.0701	41/2
5	1633.60	14.39	10/2	952.93	4.1257	41/2
6	1635.90	14.44	10/2	952.27	4.1009	39/2
7	1637.00	14.45	10/2	953.05	4.0832	40/3
ref	**1637.60**	**14.44**		**957.64**	**4.0587**	

- **The accuracy of the nonconforming FEM is saved with stabilized tensor discretization!**
- **Efficient solvers for nonconforming FEM available !**

Incompressible Powder: Schaeffer model

- **Hopper configuration**
  - Development of the pressure





  - Appearence of well-known shear bands

Summary

We conclude for edge stabilisation methods based on interior penalty and low order nonconforming FEM:

- we can handle problems in the limit of inviscid flow
- we stabilise the lack of coercivity for problems formulated in terms of symmetric part of the velocity gradient
- we unified stabilisation for all relevant Re numbers
- we have succesfully applied this stabilisation techniques in conjunctio with multigrid solver in standard FEM codes saving the same data structure
- we have succesfully derived useful tools to simulate nonnewtonian flo of power law, Bingham and Schaeffer types

The proposed stabilisation is a candidate for a black box tool and can be extended to variable viscosity (granular flow, turbulence)!