Mesh and model adaptivity for frictional contact problems

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Abstract

The article focuses on adaptive finite element methods for frictional contact problems. The approach is based on a reformulation of the mixed form of the underlying Signorini problem with friction as a nonlinear variational equation using nonlinear complimentarity (NCP) functions. The usual dual weighted residual (DWR) framework for a posteriori error estimation is applied. However, we have to take into account the nonsmoothness of the problem formulation. Error identities for measuring the discretization as well as the model error with respect to a model hierarchy of friction laws are derived and a method for the numerical evaluation of them is proposed. The estimates are utilized in an adaptive framework, which balances the discretization and the model error. Several numerical examples substantiate the accuracy of the proposed estimates and the efficiency of the adaptive method.

Keywords: Signorini's problem with friction, mixed finite element method, goal-oriented a posteriori error estimation, mesh and model adaptivity

1 Introduction

In the modelling of many physical or engineering processes, contact problems with friction frequently occur, see, for instance, [20, 32]. Hence, the development of efficient and accurate numerical solution techniques for frictional contact problems has been of special interest in the last decades. One main ingredient is given by efficient solution algorithms for the arising discrete problems. Furthermore, adaptive algorithms lead to an optimal convergence behavior of the discretizations, which cannot be achieved by uniform methods due to the missing regularity of frictional contact problems. They are based on accurate a posteriori error estimators, which should control the error in user-defined quantities of interest involving in our case the contact and frictional forces.

In literature, the obstacle problem, as model contact problem, is frequently studied. A posteriori error estimates in the energy norm are derived, for instance, in [2, 4, 15, 18, 28, 31, 37, 54] using different techniques. Even the convergence of adaptive algorithms in the context of obstacle problems is proven in [17, 16, 50]. Signorini's problem is studied, e.g., in [19, 26, 35, 47, 55], where a posteriori error estimates in the energy norm are discussed. Moreover, multibody contact problems are in the focus of [33, 57]. The dual weighted residual (DWR) method, see, e.g., [3, 5] is a popular approach to derive a posteriori error estimates, which control the error in user-defined quantities of interest. The approach is based on the representation of the quantity of interest by the solution of a so-called dual problem. Similar arguments are used in [40, 41]. The DWR framework was applied to contact problems in [10, 11] for the first time. The results are summarized in [53]. Here, a dual variational inequality is used to represent linear quantities of interest in the displacement. In [49], an alternative procedure is used, which is based on a linear dual problem representing also nonlinear quantities of interest in the displacement. It is extended to frictional contact problems and quantities of interest also in the Lagrange multiplier in [43]. There, a linear mixed dual problem, which does not depend on the primal problem, is used to represent the quantities of interest in the displacement as well as the Lagrange multiplier, which coincides with the contact forces. This approach also leads to an improved localization of the error estimate. In both approaches, the (frictional) contact conditions lead to extra additive terms in the estimates, which is some product of the dual solution and the error of the primal solution. However, the estimates do not directly measure the error in the (frictional) contact conditions and the accurate numerical approximation of the extra terms include several difficulties, which lead to involved and numerically costly algorithms. In [42], an approach is presented, which overcomes these drawbacks for Signorini's problem. Its starting point is a reformulation of Signorini's problem in mixed formulation as a nonlinear and nonsmooth variational equality based on a nonlinear complementarity (NCP) function, see, for instance, [27]. Here, the dual problem is also a linear mixed problem. However, it is determined by the active and inactive set of the primal problem. The usual error identites can be derived based on the DWR framework. However, the nonsmoothness of the underlying problem leads to remainder terms, which are of first order in the error of the discrete active set. Here, we extend this approach to frictional contact problems. Since the NCP function for friction includes combinations of nonlinear functions in contrast to the one for contact, the derivation is more complex and leads to several remainder terms. The basic idea is to seperate the smooth and nonsmooth parts using fixed active sets. The results presented in this article can be applied on a wide range of discretization schemes. For mixed discretization schemes like [24, 30], the application of the developed framework is straight forward. If semi-smooth Newton methods are used for solving the discrete contact problem, the dual problem coincide with the transposed system of the last Newton step. In displacement based discretization schemes like [8, 34, 58], an approximation to the Lagrange multiplier has to be calculated in a post processing step, cf. e.g. [15]. The derived error identities cannot be evaluated numerically. Thus, a numerical approximation scheme depending on the different discretization approaches has to be realized. We exemplify such a strategy for the mixed discretization introduced in [24].

The performance of the solution algorithm of frictional contact problems depends on the chosen friction model. One can save a large amount of computation time and gain a more stable algorithm by choosing a different model. The idea is now to select the model out of a predefined model hierarchy based on an a posteriori error estimate corresponding to the desired accuracy. In literature, one finds only few contributions to model adaptive algorithms. Dimension adaptivity is considered in [1, 7, 12, 51, 52]. In these papers, volume elements are combined with shells or plates. The automatic selection of the local model is one subject in [38, 39], where heterogeneous linear elastic models and their homogenization are included in the model hierarchy. The underlying a posteriori error estimates include the error in the energy norm as well as in linear quantities of interest. Models for different physical processes are adaptively coupled in [36] by means of problems from electrocardialogy. The basic DWR idea is extended to control modelling errors in [13]. Here, the model error is basically given by entering the solution to the coarse model into the fine one weighted by the dual solution. In [13], diffusion-reaction-equations with highly oscillating coeffients are considered. Further applications are given by time dependent problems in [14] as well as by problems from elasticity in [22]. In this article, we use the ideas from [13] to derive a posteriori estimates of the model error with respect to different friction laws, where the nonsmoothness of the underlying problems complicates the derivation. In the model adaptive algorithm, we globally balance the model and the discretization error, which cannot be done locally due to the structure of the problem.

The article is structured as follows: In Section 2, we introduce the strong and the mixed formulation of Signorini's problem with friction as well as the reformulation as a nonlinear variational equation. Furthermore, the assumptions on the model adaptive discretization are formulated. Section 3 focuses on the derivation of the error identities involving the model as well as the discretization error. At first, we consider the model and the discretization error, seperately. Afterwards, an identity for both is derived. In Section 4, we outline the ideas for the numerical approximation of the error identites and exemplify them for a concrete mixed discretization. Section 5 is devoted to numerical results, which substantiate the accuracy of the presented error estimates and the efficiency of the adaptive schemes. We conclude the article with a discussion of the results and an outlook on further tasks.

2 Problem formulation

In this section, we introduce the continuous problem formulation and a general model adaptive discretization.

2.1 Continuous problem formulation

We consider Signorini's problem with nonlinear friction laws on domains $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with sufficiently smooth boundary $\Gamma := \partial\Omega$. Homogeneous Dirichlet boundary conditions are assumend on $\Gamma_D \subset \Gamma$, where Γ_D is closed with positive measure. The possible contact boundary is given by $\Gamma_C \subset \Gamma\backslash\Gamma_D$ with $\overline{\Gamma}_C \subsetneq \Gamma\backslash\Gamma_D$. Furthermore, we have the part $\Gamma_N = \Gamma\backslash\left(\Gamma_D \cup \Gamma_C\right)$, where Neumann boundary conditions are prescribed. The usual Sobolev spaces are denoted by $L^2(\Omega)$, $H^l(\Omega)$ with $l \geq 1$, and $H^{1/2}(\Gamma_C)$. We set $H^1_D(\Omega) := \left\{v \in H^1(\Omega) \,|\, \gamma(v) = 0 \text{ on } \Gamma_D\right\}$ and $V := \left(H^1_D(\Omega)\right)^d$ with the trace operator γ. The topological dual space of $H^{1/2}(\Gamma_C)$ is given by $\tilde{H}^{-1/2}(\Gamma_C)$ with the norms $\|\cdot\|_{-1/2,\Gamma_C}$ and $\|\cdot\|_{1/2,\Gamma_C}$, respectively. The L^2-scalar products on $\omega \subset \Omega$ and $\Gamma' \subset \Gamma$ are denoted by $(\cdot,\cdot)_{0,\omega}$ and $(\cdot,\cdot)_{0,\Gamma'}$. The linear and bounded mapping $\gamma_C := \gamma_{|\Gamma_C} : H^1_D(\Omega) \to H^{1/2}(\Gamma_C)$ is surjective due to the assumptions on Γ_C, see [32, page 88]. We define $v_n := \gamma_C(v)n$ and $v_{t,j} := \gamma_C(v)t_j$, where n denotes the vector-valued function describing the outer unit normal vector with respect to Γ and t the $k \times (k-1)$-matrix-valued function containing the tangential vectors. In the following, we use the inequality symbols $\geq$ and $\leq$ for functions in $L^2(\Gamma_C)$, where the symbols are defined as "almost everywhere". We set $H^{1/2}_+(\Gamma_C) := \left\{v \in H^{1/2}(\Gamma_C) \middle| v \geq 0\right\}$. The dual cone of $H^{1/2}_+(\Gamma_C)$ is $\Lambda_n := \left\{\mu \in \tilde{H}^{-1/2}(\Gamma_C) \middle| \forall v \in H^{1/2}_+(\Gamma_C) : \langle \mu, v\rangle \geq 0\right\}$. Furthermore, we set

$$\Lambda_t(\lambda^r_n) := \left\{\mu \in \left(\tilde{H}^{-1/2}(\Gamma_C)\right)^{d-1} \middle| \langle \mu, v_t\rangle \leq \langle s^r(\lambda^r_n), |v_t|\rangle,\, v_t \in \left(H^{1/2}(\Gamma_C)\right)^{d-1}\right\}$$

with the euclidian norm $|\cdot|$. Here, $s^r : \Lambda_n \to \Lambda_n$ denotes the possible nonlinear friction law. The index r stands for reference friction law, its meaning will be clearified in the discussion of the modell adaptive approach. For a given displacement field $v \in V$, the linearized strain tensor is defined as $\varepsilon(v) := \frac{1}{2}\left(\nabla v + (\nabla v)^\top\right)$ and the stress tensor as $\sigma(v)_{ij} := \mathcal{C}_{ijkl}\varepsilon(v)_{kl}$ describing a linear-elastic material law with $\mathcal{C}_{ijkl} \in L^\infty(\Omega)$, $\mathcal{C}_{ijkl} = \mathcal{C}_{jilk} = \mathcal{C}_{klij}$ and $\mathcal{C}_{ijkl}\tau_{ij}\tau_{kl} \geq \kappa\tau^2_{ij}$ for $\tau \in L^2(\Omega)^{k\times k}_{\mathrm{sym}}$ and a $\kappa > 0$. We define $\sigma_n := \sigma n$, $\sigma_{nn} := n^\top\sigma n$, $\sigma_{nt,l} := t^\top_l\sigma n$.

The strong formulation of Signorini's problem with friction is to find a displacement field $u^r \in V \cap H^2(\Omega)$ such that

$$-\operatorname{div}\left(\sigma\left(u^r\right)\right) = f \text{ in } \Omega, \quad \sigma_n\left(u^r\right) = b \text{ on } \Gamma_N, \tag{1}$$

$$u^r_n - g \leq 0,\ \sigma^r_{nn} \leq 0,\ \sigma^r_{nn}\left(u^r_n - g\right) = 0 \text{ on } \Gamma_C, \tag{2}$$

$$|\sigma^r_{nt}| \leq s^r\left(\sigma^r_{nn}\right) \text{ with } \left\{\begin{array}{ll} |\sigma^r_{nt}| < s^r\left(\sigma^r_{nn}\right) & \Rightarrow u^r_t = 0 \\ |\sigma^r_{nt}| = s^r\left(\sigma^r_{nn}\right) & \Rightarrow \exists\zeta \in \mathbb{R}_{\geq 0} : u^r_t = -\zeta\sigma^r_{nt} \end{array}\right\} \text{ on } \Gamma_C. \tag{3}$$

Formula	Name
$s_0(\sigma_{nn}) = 0$	Friction less contact
$s_1(\sigma_{nn}) = C_T$	Tresca friction with a constant $C_T > 0$
$s_2(\sigma_{nn}) = -\mathcal{F}\sigma_{nn}$	Coulomb friction with a constant $\mathcal{F} > 0$
$s_3(\sigma_{nn}) = C_T \sqrt[n]{\tanh\left(\left(\frac{-\mathcal{F}\sigma_{nn}}{C_T}\right)^n\right)}$	Friction law of Betten with $n \in \mathbb{N}$

Tab. 1: Some examples of friction laws

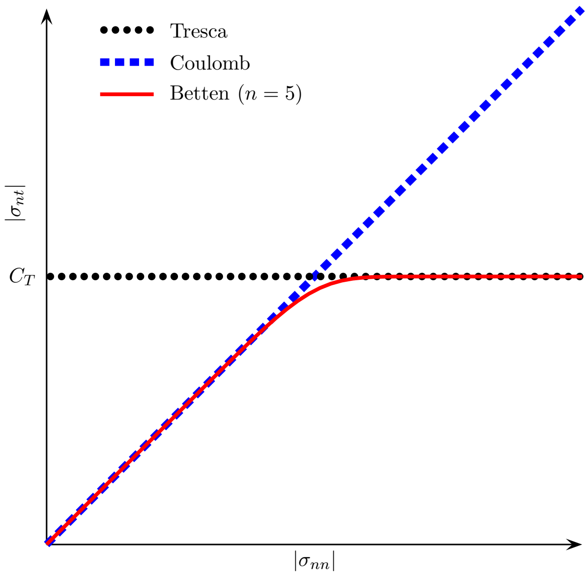


Fig. 1: Illustration of different friction laws

The equilibrium equation of linear elasticity is noted in (1), where the volume and surface loads are specified by f and b. In the following weak formulation, we assume $f \in \left(L^2(\Omega)\right)^d$, $b \in \left(L^2(\Gamma_N)\right)^d$. The geometrical contact conditions are described in (2). The possible contact boundary Γ_C is parametrized by a sufficiently smooth function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that, without loss of generality, the geometrical contact condition for a displacement field v in the d-th component is given by $\varphi(x) + v_d(x, \varphi(x)) \leq \psi(x_1 + v_1(x, \varphi(x)), \ldots, x_{d-1} + v_{d-1}(x, \varphi(x)))$ with $x := (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$. The sufficiently smooth function ψ describes the surface of the rigid obstacle. The linearization, presented for instance in [32, Chapter 2], of this condition results in $v_n \leq g$ in (2) with $g(x) := (\psi(x) - \varphi(x))(1 + (\nabla\varphi(x))^\top \nabla\varphi(x))^{-1/2}$. In the weak formulation, we assume $g \in H^{1/2}(\Gamma_C)$. The second condition, a sign condition on the outer normal stress, ensures that only pressure occurs. By the complementary condition in (2), we have either pressure or no contact. The frictional conditions are denoted in (3). They indicate that sticking occurs if the magnitude of the tangential forces is below a critical value given by the friction law s^r. If this critical value is reached, we obtain sliding, where the sliding direction corresponds to the negative direction of the tangential forces. Some examples of friction laws are given in Table 1, see for instance [59]. They are illustrated in Figure 1. From a physical point of view, Tresca's friction law fits well for high contact pressure, while Coulomb's law is accurate for low contact pressure. If low as well as high contact pressures occur, we need a nonlinear model handling both regions well. An example for such a type of friction law is given by Betten's law. For our analysis, we assume that the considered friction laws are two times continuously differentiable w.r.t. λ_n^r.

For the a posteriori error analysis, we use a mixed formulation of (1-3). The bilinear form of linear elasticity is given by $a(w,v) := (\sigma(w), \varepsilon(v))_0$ on $V \times V$, which is symmetric, continuous and due to Korn's inequality V-elliptic. The volume and surface loads are represented by $\langle \ell, v\rangle := (f,v)_0 + (b, v_N)_{0,\Gamma_N}$. A function $w^r := (u^r, \lambda_n^r, \lambda_t^r) \in V \times \Lambda_n \times \Lambda_t(\lambda_n^r)$ is a saddle point of the frictional contact problem (1-3) if and only if,

$$a(u^r, v) + \langle \lambda_n^r, v_n\rangle + \langle \lambda_t^r, v_t\rangle = \langle \ell, v\rangle, \tag{4}$$

$$\langle \mu_n - \lambda_n^r, u_n^r - g\rangle + \langle \mu_t - \lambda_t^r, u_t^r\rangle \leq 0, \tag{5}$$

for all $v \in V$, all $\mu_n \in \Lambda_n$, and all $\mu_t \in \Lambda_t$. If some smoothness assumptions are fulfilled, it holds $\lambda_n^r = -\sigma_{nn}^r$ and $\lambda_t^r = -\sigma_{nt}^r$. The existence and uniqueness of the solution w^r depends on the chosen friction law s^r and its parameters. For Tresca's and Coulomb's friction law with $\mathcal{F}$ small enough, there exists a unique weak solution. We refer to [21] for an overview on existence and uniqueness results.

We now reformulate the frictional contact conditions (5) in terms of two nonlinear equations. The geometrical contact conditions are equivalently expresed by

$$g - u_n^r \in H_+^{1/2}(\Gamma_C), \quad \lambda_n^r \in \Lambda_n, \quad \langle \lambda_n^r, u_n^r - g\rangle = 0. \tag{6}$$

Assuming $\lambda_n^r \in L^2(\Gamma_C)$, (6) simplifies to

$$g - u_n^r \in H_+^{1/2}(\Gamma_C), \quad \lambda_n^r \geq 0 \text{ a.e. on } \Gamma_C, \quad \lambda_n^r (u_n^r - g) = 0 \text{ a.e. on } \Gamma_C. \tag{7}$$

For geometrical contact problems, we have $\lambda_n^r \in L^2(\Gamma_C)$ for instance, if $\mathcal{C}_{ijkl} \in W^{1,\infty}(\Omega)$ and $g \in H^{5/2}(\Gamma_C)$, see [42]. The geometrical contact conditions (7) are equivalently reformulated as

$$\lambda_n^r - \max\{0, \lambda_n^r + u_n^r - g\} = 0 \text{ a.e. on } \Gamma_C \tag{8}$$

by an NCP function, cf. [29, Chapter 4]. Multiplying with a test function $\mu_n \in L^2(\Gamma_C)$ leads to

$$C(w^r)(\mu_n) := (\mu_n, \lambda_n^r - \max\{0, \lambda_n^r + u_n^r - g\})_{0,\Gamma_C} = 0.$$

However, the semilinear form C is not Fréchet differentiable in general.

If we require $\lambda_n^r \in L^2(\Gamma_C)$ and $\lambda_t^r \in \left(L^2(\Gamma_C)\right)^{d-1}$, we obtain $s^r \in S$, where $S := \left\{ s : L^2(\Gamma_C) \to L_+^2(\Gamma_C) \middle| s \text{ continuously differentiable} \right\}$ with $L_+^2(\Gamma_C) := \left\{ v \in L^2(\Gamma_C) \middle| v \geq 0 \text{ a.e.} \right\}$. Furthermore, Λ_t simplifies to

$$\Lambda_t(\lambda_n^r) = \left\{ \mu_t \in \left(L^2(\Gamma_C)\right)^{d-1} \middle| |\mu_t| \leq s^r(\lambda_n^r) \right\}.$$

The calculations in [29, Section 2.1] show that we obtain also the frictional conditions (3) a.e. Rewriting them with the help of a second NCP function, see [29, Chapter 5], gives us

$$\max\{s^r(\lambda_n^r), |\lambda_t^r + u_t^r|\}\lambda_t^r - s^r(\lambda_n^r)(\lambda_t^r + u_t^r) = 0 \text{ a.e. on } \Gamma_C.$$

The multiplication with a test function $\mu_t \in \left(L^2(\Gamma_C)\right)^{d-1}$ finally results in

$$D^r(w^r)(\mu_t) := (\mu_t, \max\{s^r(\lambda_n^r), |\lambda_t^r + u_t^r|\}\lambda_t^r - s^r(\lambda_n^r)(\lambda_t^r + u_t^r))_{0,\Gamma_C} = 0.$$

In general, the semilinear form D is not Fréchet differentiable, too.

We define the semilinear form

$$A^r(w^r)(\varphi) := a(u^r, v) + (\lambda_n^r, v_n)_{0,\Gamma_C} + (\lambda_t^r, v_t)_{0,\Gamma_C} - \langle \ell, v\rangle + C(w^r)(\mu_n) + D^r(w^r)(\mu_t)$$

with $\varphi = (v, \mu_n, \mu_t) \in W := V \times L^2(\Gamma_C) \times \left(L^2(\Gamma_C)\right)^{d-1}$. Then Signorini's problem (4-5) is to find $w^r \in W$ with

$$\forall \varphi \in W: \qquad A^r(w^r)(\varphi) = 0.$$

2.2 Model adaptive discretization

The model adaptive discretization is carried out in two steps. At first, we specify a model adaptive friction law. To this end, we define a friction model hierarchy $\mathcal{H} := \{s_0, s_1, \ldots, s_M\}$, where s_0 is the most inexact model and $s_M := s^r$ the most accurate or reference model. An example for such a hierarchy is given in Table 1. Finally, we compose a locally varying friction law $s \in S$ based on the different models in $\mathcal{H}$. Using the semilinear form

$$A(w)(\varphi) := a(u,v) + (\lambda_n, v_n)_{0,\Gamma_C} + (\lambda_t, v_t)_{0,\Gamma_C} - \langle \ell, v\rangle + C(w)(\mu_n) + D(w)(\mu_t)$$

with $w = (u, \lambda_n, \lambda_t) \in W$, $\varphi = (v, \mu_n, \mu_t) \in W$ and

$$D(w)(\mu_t) := (\mu_t, \max\{s(\lambda_n), |\lambda_t + u_t|\}\lambda_t - s(\lambda_n)(\lambda_t + u_t))_{0,\Gamma_C},$$

$w \in W$ is a solution of Signorini's problem with a model adaptive friction law, if

$$\forall \varphi \in W: \qquad A(w)(\varphi) = 0.$$

We directly obtain a generalized Galerkin orthogonality relation

$$a(u^r - u, v) + (\lambda_n^r - \lambda_n, v_n)_{0,\Gamma_C} + (\lambda_t^r - \lambda_t, v_t)_{0,\Gamma_C} = 0 \tag{9}$$

for all $v \in V$.

The second step consists in the specification of the general discretization requirements, which ensure that the presented a posteriori error analysis applies to a wide range of techniques to calculate approximations w_h to w. They have to fulfill the following assumption:

Assumption 1. *Let $W_h = V_h \times \Lambda_{n,h} \times \Lambda_{t,h}$ be a finite dimensional subspace of W, which contains the discrete solution w_h. Moreover, equation (4) has to hold for the discrete solution w_h, i.e.*

$$a(u_h, v_h) + (\lambda_{n,h}, v_{h,n})_{0,\Gamma_C} + (\lambda_{t,h}, v_{h,t})_{0,\Gamma_C} = \langle l, v_h\rangle \tag{10}$$

for all $v_h \in V_h$.

Remark 2. From (10), a generalized Galerkin orthogonality follows, i.e.

$$a(u - u_h, v_h) + (\lambda_n - \lambda_{n,h}, v_{h,n})_{0,\Gamma_C} + (\lambda_t - \lambda_{t,h}, v_{h,t})_{0,\Gamma_C} = 0 \tag{11}$$

for all $v_h \in V_h$.

Remark 3. In Assumption 1, we only prescribe $\Lambda_{n,h} \subseteq L^2(\Gamma_C)$ and $\Lambda_{t,h}(\lambda_{n,h}) \subseteq \left(L^2(\Gamma_C)\right)^{d-1}$. We do not require $\Lambda_{n,h} \subseteq \Lambda_n$ and $\Lambda_{t,h}(\lambda_{n,h}) \subseteq \Lambda_t(\lambda_{n,h})$. Thus, we consider also nonconforming approximations of the Lagrange multiplier.

Remark 4. Our analysis applies to different discretization schemes. It includes method, which are only based on the displacement. In this case, the Lagrange multipliers λ_n and λ_t have to be determined in a post processing step, cf., for instance, [15]. It also applies to mixed discretizations like the one presented in [24], which we use in our numerical experiments, or Mortar methods, see, e.g., [56].

3 A posteriori error analysis

This section focuses on the derivation of an a posteriori error estimate in a user defined quantity of interest, where we employ the DWR method. Specifically, the error is estimated in a possibly nonlinear quantity of interest $J : W \to \mathbb{R}$, i.e. J can include the displacement u as well as the Lagrange multipliers λ_n and λ_t representing the contact and the frictional stress respectively.

3.1 Model error estimation

Initially, the model error $J(w^r) - J(w)$ is treated. The first step in the derivation of the error identities is the definition of a suitable dual problem: Find $z^r = (y^r, \xi_n^r, \xi_t^r) \in W$ with

$$\begin{aligned} a(v, y^r) - b_r^c(\xi_n^r, v) + b_r^f(\xi_t^r, v) &= J_u'(w^r)(v), & (12)\\ (\mu_n, y_n^r)_{0,\Gamma_C} + c_r^c(\xi_n^r, \mu_n) + c_r^f(\xi_t^r, \mu_n) &= J_{\lambda_n}'(w^r)(\mu_n), & (13)\\ (\mu_t, y_t^r)_{0,\Gamma_C} + d_r^f(\xi_t^r, \mu_t) &= J_{\lambda_t}'(w^r)(\mu_t), & (14) \end{aligned}$$

for all $(v, \mu_n, \mu_t) \in W$. Here, the bilinear forms $b^c : L^2(\Gamma_C) \times V \to \mathbb{R}$ and $c : L^2(\Gamma_C) \times L^2(\Gamma_C) \to \mathbb{R}$ are given by

$$\begin{aligned} b_r^c(\omega_n, v) &:= \int_{\Gamma_C} \omega_n \chi_r^c v_n \, do,\\ c_r^c(\omega_n, \mu_n) &:= \int_{\Gamma_C} \omega_n \left[1 - \chi_r^c\right] \mu_n \, do, \end{aligned}$$

with

$$\chi_r^c(w^r) := \begin{cases} 1, & \text{if } \lambda_n^r + u_n^r - g > 0,\\ 0, & \text{if } \lambda_n^r + u_n^r - g \leq 0. \end{cases}$$

We also write $\chi_r^c = \chi_r^c(w^r)$. The bilinear forms b^c and c^c correspond to weighted L^2-scalar products on Γ_C, where the weight is given by the indicator function of the active and inactive geometrical contact set, respectively. Furthermore, we consider the bilinear forms $b_r^f : \left(L^2(\Gamma_C)\right)^{d-1} \times V \to \mathbb{R}$, $c_r^f : \left(L^2(\Gamma_C)\right)^{d-1} \times L^2(\Gamma_C) \to \mathbb{R}$, and $d_r^f : \left(L^2(\Gamma_C)\right)^{d-1} \times \left(L^2(\Gamma_C)\right)^{d-1} \to \mathbb{R}$ with

$$\begin{aligned} b_r^f(\omega_t, v) &:= \int_{\Gamma_C} \omega_t \left[\chi_r^f \lambda_t^r \left(n'(w^r)\right)^\top - s^r(\lambda_n^r) I\right] v_t \, do,\\ c_r^f(\omega_t, \mu_n) &:= -\int_{\Gamma_C} \omega_t (s^r)'(\lambda_n^r)(\mu_n) \left[\chi_r^f \lambda_t^r + u_t\right] do,\\ d_r^f(\omega_t, \mu_t) &:= \int_{\Gamma_C} \omega_t \left[\max\{s^r(\lambda_n^r), n(w^r)\} I - s(\lambda_n^r) I + \chi_r^f \lambda_t^r \left(n'(w^r)\right)^\top\right] \mu_t \, do. \end{aligned}$$

Here, we use the notation $n(w^r) := |\lambda_t^r + u_t^r|$ and

$$n'(w^r) = \begin{cases} \frac{\lambda_t^r + u_t^r}{n(w^r)}, & \text{if } w \neq 0,\\ 0, & \text{if } w = 0. \end{cases}$$

However, the case $w = 0$ does not occur because of the multiplication with the indicator function χ_r^f w.r.t. sliding and sticking, where

$$\chi_r^f(w^r) := \begin{cases} 1, & \text{if } s^r(\lambda_n^r) < n(w^r),\\ 0, & \text{if } s^r(\lambda_n^r) \geq n(w^r). \end{cases}$$

The shorter notation $\chi_r^f := \chi_r^f(w^r)$ is mostly used. We point out that, if the Fréchet derivative A' of A exists, the dual problem (12-14) matches $A'(w^r)(\varphi, z^r) = J'(w^r)(\varphi)$. Let us clearify the connection between $\max\{s^r(\lambda_n^r), n(w^r)\}$ and the indicator function χ_r^f:

$$\begin{aligned} D^r(w^r)(\mu_t) &= \left(\mu_t, \max\{s^r(\lambda_n^r), n(w^r)\} \lambda_t^r - s^r(\lambda_n^r)(\lambda_t^r + u_t^r)\right)_{0,\Gamma_C}\\ &= \left(\mu_t, \chi_r^f n(w^r) \lambda_t^r + \left(1 - \chi_r^f\right) s^r(\lambda_n^r) \lambda_t^r - s^r(\lambda_n^r)(\lambda_t^r + u_t^r)\right)_{0,\Gamma_C}\\ &= \left(\mu_t, \chi_r^f \left(n(w^r) \lambda_t^r - s^r(\lambda_n^r) \lambda_t^r\right) - s^r(\lambda_n^r) u_t^r\right)_{0,\Gamma_C} =: \bar{D}^r(w^r)(\mu_t). \end{aligned}$$

If we consider $\bar{D}^r$, we see χ_r^f as a fixed weighting function. Thus, the Fréchet differentiability of $\bar{D}^r$ depends only on the smoothness of s^r. In the following results, if not otherwise stated, we assume that $(\mu_n, s^r(\lambda_n^r))_{0,\Gamma_C}$ is three times Fréchet differentiable w.r.t. λ_n^r. A short calculation now shows

$$\begin{aligned}
&\left(\bar{D}^r\right)'(w^r)(\delta w, \mu_t) \\
&= \left(\bar{D}^r\right)'_u(w^r)(\delta u, \mu_t) + \left(\bar{D}^r\right)'_{\lambda_n}(w^r)(\delta\lambda_n, \mu_t) + \left(\bar{D}^r\right)'_{\lambda_t}(w^r)(\delta\lambda_t, \mu_t) \\
&= \left(\mu_t, \chi_r^f\left(\lambda_t^r\left(n'(w^r)\right)^\top \delta u_t\right) - s^r(\lambda_n^r)\,\delta u_t\right)_{0,\Gamma_C} \\
&\quad - \left(\mu_t, \chi_r^f (s^r)'(\lambda_n^r)(\delta\lambda_n)\,\lambda_t^r + (s^r)'(\lambda_n^r)(\delta\lambda_n)\,u_t^r\right)_{0,\Gamma_C} \\
&\quad + \left(\mu_t, \chi_r^f\left(\lambda_t^r\left(n'(w^r)\right)^\top \delta\lambda_t + n(w^r)\,\delta\lambda_t - s^r(\lambda_n^r)\,\delta\lambda_t\right)\right)_{0,\Gamma_C} \\
&= b_r^f(\mu_t, \delta u) + c_r^f(\mu_t, \delta\lambda_n) + d_r^f(\mu_t, \delta\lambda_t).
\end{aligned} \tag{15}$$

Analogously, we define the dual solution $z = (y, \xi_n, \xi_t) \in W$ w.r.t. the model adaptive friction law s using the bilinear forms b^c, c^c, b^f, c^f, and d^f as well as the indicator functions χ^c and χ^f. Furthermore, we set

$$\bar{D}(w)(\mu_t) := \left(\mu_t, \chi^f\left(n(w)\,\lambda_t - s(\lambda_n)\,\lambda_t\right) - s(\lambda_n)\,u_t\right)_{0,\Gamma_C} = D(w)(\mu_t)$$

and obtain

$$\bar{D}'(w)(\delta w, \mu_t) = b^f(\mu_t, \delta u) + c^f(\mu_t, \delta\lambda_n) + d^f(\mu_t, \delta\lambda_t). \tag{16}$$

We denote the model adaptive error w.r.t. the primal as well as to the dual solution by

$$\begin{aligned}
e_w^r := \left(e_u^r, e_{\lambda_n}^r, e_{\lambda_t}^r\right) &:= \left(u^r - u, \lambda_n^r - \lambda_n, \lambda_t^r - \lambda_t\right), \\
e_z^r := \left(e_y^r, e_{\xi_n}^r, e_{\xi_t}^r\right) &:= \left(y^r - y, \xi_n^r - \xi_n, \xi_t^r - \xi_t\right),
\end{aligned}$$

respectively. The error in the geometrical contact indicator function is $e_{\chi^c}^r := \chi_r^c(w^r) - \chi^c(w)$ and in the frictional indicator function $e_{\chi^f}^r := \chi_r^f(w^r) - \chi^f(w)$.

In preparation of the main result, we study the bilinear forms in the dual problem concerning the contact conditions:

Lemma 5. *We obtain that*

$$\begin{aligned}
&c_r^c\left(\xi_n^r, e_{\lambda_n}^r\right) - b_r^c\left(\xi_n^r, e_u^r\right) + c^c\left(\xi_n, e_{\lambda_n}^r\right) - b^c\left(\xi_n, e_u^r\right) \\
&= \int_{\Gamma_C} e_{\chi^c}^r\left[e_{\xi_n}^r\left(\lambda_n + u_n - g\right) - \xi_n\left(e_{\lambda_n}^r + e_{u,n}^r\right)\right] do =: 2\mathcal{R}_c^m
\end{aligned}$$

holds.

Remark 6. The term $\mathcal{R}_c^m$ is the product of the error in the indicator function of the contact conditions and the model error. Thus it is of higher order.

Proof. By the definition of the bilinear forms c_r^c, c^c, b_r^c, and b^c, we obtain using $C(w^r)(\mu_n) =$

$C(w)(\mu_n) = 0$ for all $\mu_n \in L^2(\Gamma_C)$

$$
\begin{aligned}
& c_r^c\left(\xi_n^r, e_{\lambda_n}^r\right) - b_r^c\left(\xi_n^r, e_u^r\right) + c^c\left(\xi_n, e_{\lambda_n}^r\right) - b^c\left(\xi_n, e_u^r\right) \\
= & \int_{\Gamma_C} \xi_n^r\left[1-\chi_r^c\right] e_{\lambda_n}^r \, do - \int_{\Gamma_C} \xi_n^r \chi_r^c e_{u,n}^r \, do + \int_{\Gamma_C} \xi_n\left[1-\chi^c\right] e_{\lambda_n}^r \, do - \int_{\Gamma_C} \xi_n \chi^c e_{u,n}^r \, do \\
= & \int_{\Gamma_C} \xi_n^r\left[e_{\lambda_n}^r - \chi_r^c\left(e_{\lambda_n}^r + e_{u,n}^r\right)\right] do + \int_{\Gamma_C} \xi_n\left[e_{\lambda_n}^r - \chi^c\left(e_{\lambda_n}^r + e_{u,n}^r\right)\right] do \\
= & \int_{\Gamma_C} \xi_n^r\left[\lambda_n^r - \chi_r^c\left(\lambda_n^r + u_n^r - g\right)\right] do - \int_{\Gamma_C} \xi_n^r\left[\lambda_n - \chi_r^c\left(\lambda_n + u_n - g\right)\right] do \\
& + \int_{\Gamma_C} \xi_n\left[\lambda_n^r - \chi^c\left(\lambda_n^r + u_n^r - g\right)\right] do - \int_{\Gamma_C} \xi_n\left[\lambda_n - \chi^c\left(\lambda_n + u_n - g\right)\right] do \\
= & -\int_{\Gamma_C} \xi_n^r\left[\lambda_n - \chi_r^c\left(\lambda_n + u_n - g\right)\right] do + \int_{\Gamma_C} \xi_n^r\left[\lambda_n - \chi^c\left(\lambda_n + u_n - g\right)\right] do \\
& -\int_{\Gamma_C} \xi_n\left[\lambda_n^r - \chi_r^c\left(\lambda_n^r + u_n^r - g\right)\right] do + \int_{\Gamma_C} \xi_n\left[\lambda_n^r - \chi^c\left(\lambda_n^r + u_n^r - g\right)\right] do \\
= & \int_{\Gamma_C} \xi_n^r e_{\chi^c}^r\left(\lambda_n + u_n - g\right) do - \int_{\Gamma_C} \xi_n e_{\chi^c}^r\left(\lambda_n^r + u_n^r - g\right) do \\
= & \int_{\Gamma_C} e_{\chi^c}^r\left[\xi_n^r\left(\lambda_n + u_n - g\right) - \xi_n\left(\lambda_n^r + u_n^r - g\right)\right] do \\
= & \int_{\Gamma_C} e_{\chi^c}^r\left[e_{\xi_n}^r\left(\lambda_n + u_n - g\right) - \xi_n\left(e_{\lambda_n}^r + e_{u,n}^r\right)\right] do = 2\mathcal{R}_c^m.
\end{aligned}
$$

□

We define the semilinear form $\Delta(w)(\mu_t) := D^r(w)(\mu_t) - D(w)(\mu_t)$ and obtain $A(w)(\varphi) + \Delta(w)(\mu_t) = A^r(w)(\varphi)$. Furthermore, we set $\bar{\Delta}(\omega,\varphi) := \left(\bar{D}^r\right)'(w)(\omega,\varphi)$. The second step is now to consider the bilinear form in the dual problems concerning the frictional conditions:

Lemma 7. *With the remainder term*

$$\mathcal{R}_f^m = \mathcal{R}_{\chi,1}^m + \mathcal{R}_{\chi,2}^m + \mathcal{R}_Q^m$$

including the frictional conditions, where

$$
\begin{aligned}
\mathcal{R}_{\chi,1}^m &= -\left(\xi_t^r, \left(\chi_r^f(w^r) - \chi_r^f(w)\right)\left(n(w)\lambda_t - s^r(\lambda_n)\lambda_t\right)\right)_{0,\Gamma_C}, \\
\mathcal{R}_{\chi,2}^m &= -\frac{1}{2}\left[\left(\bar{D}^r\right)'(w)\left(e_w^r, \xi_t\right) - \bar{D}'(w)\left(e_w^r, \xi_t\right)\right], \\
\mathcal{R}_Q^m &= \frac{1}{2}\int_0^1 \left(\bar{D}^r\right)'''\left(w + s e_w^r\right)\left(e_w^r, e_w^r, e_w^r, \xi_t^r\right) s(s-1)\, ds,
\end{aligned}
$$

it holds

$$
\begin{aligned}
& b_r^f\left(\xi_t^r, e_u^r\right) + c_r^f\left(\xi_t^r, e_{\lambda_n}^r\right) + d_r^f\left(\xi_t^r, e_{\lambda_t}^r\right) + b^f\left(\xi_t, e_u^r\right) + c^f\left(\xi_t, e_{\lambda_n}^r\right) + d^f\left(\xi_t, e_{\lambda_t}^r\right) \\
= & -2\Delta(w)(\xi_t) - 2\Delta(w)\left(e_{\xi_t}^r\right) - \bar{\Delta}\left(e_w^r, e_{\xi_t}^r\right) + 2\mathcal{R}_f^m.
\end{aligned}
$$

Remark 8. The remainder $\mathcal{R}_f^m$ is dominated by $\mathcal{R}_{\chi,1}^m$, which consists mainly of the error in the indicator function of the sticking and the sliping region. The other parts are of second and third order in the error, respectively.

Proof. At first, we notice using the definition of $\bar{D}^r$ and $D(w)(\mu_t) = 0$

$$\begin{aligned}
& \bar{D}^r(w)(\xi_t^r) \\
= & \left(\xi_t^r, \chi_r^f(n(w)\lambda_t - s^r(\lambda_n)\lambda_t) - s^r(\lambda_n)u_t\right)_{0,\Gamma_C} \\
= & \left(\xi_t^r, \left(\chi_r^f(w^r) - \chi_r^f(w)\right)(n(w)\lambda_t - s^r(\lambda_n)\lambda_t)\right)_{0,\Gamma_C} \\
& + \left(\xi_t^r, \chi_r^f(w)(n(w)\lambda_t - s^r(\lambda_n)\lambda_t) - s^r(\lambda_n)u_t\right)_{0,\Gamma_C} \\
= & -\mathcal{R}_{\chi,1}^m + D^r(w)(\xi_t^r) = D^r(w)(\xi_t^r - \xi_t) + D^r(w)(\xi_t) - \mathcal{R}_{\chi,1}^m \\
= & D^r(w)(\xi_t^r - \xi_t) - D(w)(\xi_t^r - \xi_t) + D^r(w)(\xi_t) - D(w)(\xi_t) - \mathcal{R}_{\chi,1}^m \\
= & \Delta(w)\left(e_{\xi_t}^r\right) + \Delta(w)(\xi_t) - \mathcal{R}_{\chi,1}^m.
\end{aligned}$$

The trapezoidal rule with its remainder term together with $\bar{D}^r(w^r)(\mu_t) = D^r(w^r)(\mu_t) = 0$ and the preceding calculations lead to

$$\begin{aligned}
& -\Delta(w)\left(e_{\xi_t}^r\right) - \Delta(w)(\xi_t) + \mathcal{R}_{\chi,1}^m \\
= & \bar{D}^r(w^r)(\xi_t^r) - \bar{D}^r(w)(\xi_t^r) \\
= & \int_0^1 \left(\bar{D}^r\right)'(w + s e_w^r)(e_w^r, \xi_t^r)\, ds \\
= & \frac{1}{2}\left(\bar{D}^r\right)'(w)(e_w^r, \xi_t^r) + \frac{1}{2}\left(\bar{D}^r\right)'(w^r)(e_w^r, \xi_t^r) - \mathcal{R}_Q^m.
\end{aligned}$$

From (15), (16) and

$$\begin{aligned}
& \left(\bar{D}^r\right)'(w)(e_w^r, \xi_t^r) \\
= & \left(\bar{D}^r\right)'(w)\left(e_w^r, e_{\xi_t}^r\right) + \left(\bar{D}^r\right)'(w)(e_w^r, \xi_t) - \bar{D}'(w)(e_w^r, \xi_t) + \bar{D}'(w)(e_w^r, \xi_t) \\
= & \bar{\Delta}\left(e_w^r, e_{\xi_t}^r\right) - 2\mathcal{R}_{\chi,2}^m + b^f(\xi_t, e_u^r) + c^f\left(\xi_t, e_{\lambda_n}^r\right) + d^f\left(\xi_t, e_{\lambda_t}^r\right),
\end{aligned}$$

we deduce the assertion by rearranging the single terms. □

Combining Lemma (5) and (7), we obtain the following Proposition concerning the model error:

Proposition 9. *Let the third Fréchet derivative of J, $J''' : W \to \mathcal{L}(W, \mathcal{L}(W, W^*))$, exist. Then, the error identity*

$$J(w^r) - J(w) = -\Delta(w)(z) - \Delta(w)(e_z^r) - \frac{1}{2}\bar{\Delta}(e_w^r, e_z^r) + \mathcal{R}_J^m + \mathcal{R}_c^m + \mathcal{R}_f^m$$

holds for the model error in the quantity of interest with the remainder terms

$$\mathcal{R}_J^m = \frac{1}{2}\int_0^1 J'''(w + s e_w^r)(e_w^r, e_w^r, e_w^r)\, s(s-1)\, ds$$

w.r.t. the quantity of interest J, $\mathcal{R}_c^m$ from Lemma 5 and $\mathcal{R}_f^m$ from Lemma 7.

Remark 10. The remainder $\mathcal{R}_J^m$ is of third order in the error. Consequently, the remainder terms are dominated by $\mathcal{R}_{\chi,1}^m$.

Proof. The trapezoidal quadrature rule with its remainder term leads to

$$J(w^r) - J(w) = \int_0^1 J'(w + s e_w^r)(e_w^r)\, ds = \frac{1}{2}J'(w)(e_w^r) + \frac{1}{2}J'(w^r)(e_w^r) + \mathcal{R}_J^m.$$

From the definition of the dual problems together with the generalized Galerkin orthogonality (9), we deduce

$$\begin{aligned} J'(w^r)(e^r_w) &= a(e^r_u, y^r) - b^c_r(\xi^r_n, e^r_u) + b^f_r(\xi^r_t, e^r_u) + \left(e^r_{\lambda_n}, y^r_n\right)_{0,\Gamma_C} \\ &\quad + c^c_r\left(\xi^r_n, e^r_{\lambda_n}\right) + c^f_r\left(\xi^r_t, e^r_{\lambda_n}\right) + \left(e^r_{\lambda_t}, y^r_t\right)_{0,\Gamma_C} + d^f_r\left(\xi^r_t, e^r_{\lambda_t}\right) \\ &= -b^c_r(\xi^r_n, e^r_u) + b^f_r(\xi^r_t, e^r_u) + c^c_r\left(\xi^r_n, e^r_{\lambda_n}\right) + c^f_r\left(\xi^r_t, e^r_{\lambda_n}\right) + d^f_r\left(\xi^r_t, e^r_{\lambda_t}\right) \end{aligned}$$

and

$$\begin{aligned} J'(w)(e^r_w) &= a(e^r_u, y) - b^c(\xi_n, e^r_u) + b^f(\xi_t, e^r_u) + \left(e^r_{\lambda_n}, y_n\right)_{0,\Gamma_C} \\ &\quad + c^c\left(\xi_n, e^r_{\lambda_n}\right) + c^f\left(\xi_t, e^r_{\lambda_n}\right) + \left(e^r_{\lambda_t}, y_t\right)_{0,\Gamma_C} + d^f\left(\xi_t, e^r_{\lambda_t}\right) \\ &= -b^c(\xi_n, e^r_u) + b^f(\xi_t, e^r_u) + c^c\left(\xi_n, e^r_{\lambda_n}\right) + c^f\left(\xi_t, e^r_{\lambda_n}\right) + d^f\left(\xi_t, e^r_{\lambda_t}\right). \end{aligned}$$

Lemma 5 and 7 together with the calculations above lead to

$$\begin{aligned} & J(w^r) - J(w) \\ =\; & \frac{1}{2} J'(w)(e^r_w) + \frac{1}{2} J'(w^r)(e^r_w) + \mathcal{R}^m_J \\ =\; & \frac{1}{2}\left[-b^c_r(\xi^r_n, e^r_u) + b^f_r(\xi^r_t, e^r_u) + c^c_r\left(\xi^r_n, e^r_{\lambda_n}\right) + c^f_r\left(\xi^r_t, e^r_{\lambda_n}\right) + d^f_r\left(\xi^r_t, e^r_{\lambda_t}\right)\right] \\ & + \frac{1}{2}\left[-b^c(\xi_n, e^r_u) + b^f(\xi_t, e^r_u) + c^c\left(\xi_n, e^r_{\lambda_n}\right) + c^f\left(\xi_t, e^r_{\lambda_n}\right) + d^f\left(\xi_t, e^r_{\lambda_t}\right)\right] + \mathcal{R}^m_J \\ =\; & -\Delta(w)(z) - \Delta(w)(e^r_z) - \frac{1}{2}\left(\bar{D}^r\right)'(w)\left(e^r_w, e^r_{\xi_t}\right) + \mathcal{R}^m_c + \mathcal{R}^m_f + \mathcal{R}^m_J, \end{aligned}$$

the assertion. □

3.2 Discretization error estimation

In this section, we consider the discretization error $J(w) - J(w_h)$ between the model adaptive solution w and its approximation w_h. To this end, we need a discrete approximation $z_h = (y_h, \xi_{n,h}, \xi_{t,h}) \in W_h$ to z, which does not have to fulfill any further assumptions. We denote by e_w and e_z the discretization error, i.e.

$$\begin{aligned} e_w = (e_u, e_{\lambda_n}, e_{\lambda_t}) &= (u - u_h, \lambda_n - \lambda_{n,h}, \lambda_t - \lambda_{t,h}), \\ e_z = (e_y, e_{\xi_n}, e_{\xi_t}) &= (y - y_h, \xi_n - \xi_{n,h}, \xi_t - \xi_{t,h}). \end{aligned}$$

Furthermore, we define

$$\bar{D}_h(w_h)(\mu_{t,h}) := \left(\mu_{t,h}, \chi^f_h\left(n(w_h)\lambda_{t,h} - s(\lambda_{n,h})\lambda_{t,h}\right) - s(\lambda_{n,h})\, u_{h,t}\right)_{0,\Gamma_C} = D_h(w_h)(\mu_{t,h})$$

and notice

$$\bar{D}'_h(w_h)(\delta w, \mu_t) = b^f_h(\mu_t, \delta u) + c^f_h(\mu_t, \delta\lambda_n) + d^f_h(\mu_t, \delta\lambda_t) \tag{17}$$

with the bilinear forms $b^f_h : \Lambda_{t,h} \times V_h \to \mathbb{R}$, $c^f_h : \Lambda_{t,h} \times \Lambda_{n,h} \to \mathbb{R}$, and $d^f_h : \Lambda_{t,h} \times \Lambda_{t,h} \to \mathbb{R}$ concerning the frictional conditions,

$$\begin{aligned} b^f_h(\omega_t, v) &:= \int_{\Gamma_C} \omega_t \left[\chi^f_h \lambda_{t,h}\left(n(w_h)\right)^\top - s(w_h)\, I\right] v_t \, do, \\ c^f_h(\omega_t, \mu_n) &:= -\int_{\Gamma_C} \omega_t s'(\lambda_{n,h})(\mu_n)\left[\chi^f_h \lambda_{t,h} + u_{h,t}\right] do, \\ d^f_h(\omega_t, \mu_t) &:= \int_{\Gamma_C} \omega_t \left[\max\{s(\lambda_{n,h}), n(w_h)\} I - s(\lambda_{n,h})\, I + \chi^f_h \lambda_{t,h}\left(n'(w_h)\right)^\top\right] \mu_t \, do, \end{aligned}$$

and

$$\chi_h^f(w_h) := \begin{cases} 1, & \text{if } s(\lambda_{n,h}) < n(w_h), \\ 0, & \text{if } s(\lambda_{n,h}) \geq n(w_h). \end{cases}$$

The error in the frictional indicator function is $e_\chi^f = \chi^f - \chi_h^f$. First, we clarify the connection between the frictional part of the dual problem and the frictional conditions:

Lemma 11. *Let $(\mu_n, s(\lambda_n))_{0,\Gamma_C}$ for arbitrary $\mu_n \in L^2(\Gamma_C)$ be two times Fréchet differentiable w.r.t. λ_n. Then we have the identity*

$$b^f(\xi_t, e_u) + c^f(\xi_t, e_{\lambda_n}) + d^f(\xi_t, e_{\lambda_t}) = -D(w_h)(e_{\xi_t}) - D(w_h)(\xi_{t,h}) + \mathcal{R}_f^{(2)}.$$

The remainder term $\mathcal{R}_f^{(2)} = \mathcal{R}_\chi^{(2)} + \mathcal{R}_Q^{(2)}$ consists of

$$\mathcal{R}_\chi^{(2)} := -\left(\xi_t, e_\chi^f\left(n(w_h)\lambda_{t,h} - s(\lambda_{n,h})\lambda_{t,h}\right)\right)_{0,\Gamma_C}$$

and

$$\mathcal{R}_Q^{(2)} = \int_0^1 \bar{D}''(w_h + se_w)(e_w, e_w, \xi_t)\, s\, ds.$$

Proof. We obtain using the box quadrature rule with its remainder and (16)

$$\begin{aligned} \bar{D}(w)(\xi_t) - \bar{D}(w_h)(\xi_t) &= \int_0^1 \bar{D}'(w_h + se_w)(e_w, \xi_t)\; ds \\ &= \bar{D}'(w)(e_w, \xi_t) - \mathcal{R}_Q^{(2)} \\ &= b^f(\xi_t, e_u) + c^f(\xi_t, e_{\lambda_n}) + d^f(\xi_t, e_{\lambda_t}) - \mathcal{R}_Q^{(2)}. \end{aligned}$$

The equation $\bar{D}(w)(\xi_t) = D(w)(\xi_t) = 0$ leads to

$$\begin{aligned} &\bar{D}(w)(\xi_t) - \bar{D}(w_h)(\xi_t) = -\bar{D}(w_h)(\xi_t) \\ =\; & -\bar{D}(w_h)(\xi_t) - \bar{D}_h(w_h)(\xi_t) + \bar{D}_h(w_h)(\xi_t) \\ =\; & -\left(\xi_t, \chi^f\left(n(w_h)\lambda_{t,h} - s(\lambda_{n,h})\lambda_{t,h}\right) - s(\lambda_{n,h})u_{h,t}\right)_{0,\Gamma_C} \\ & + \left(\xi_t, \chi_h^f\left(n(w_h)\lambda_{t,h} - s(\lambda_{n,h})\lambda_{t,h}\right) - s(\lambda_{n,h})u_{h,t}\right)_{0,\Gamma_C} - D(w_h)(\xi_t) \\ =\; & -\left(\xi_t, e_\chi^f\left(n(w_h)\lambda_{t,h} - s(\lambda_{n,h})\lambda_{t,h}\right)\right)_{0,\Gamma_C} - D(w_h)(e_{\xi_t}) - D(w_h)(\xi_{t,h}) \\ =\; & \mathcal{R}_\chi^{(2)} - D(w_h)(e_{\xi_t}) - D(w_h)(\xi_{t,h}). \end{aligned}$$

Rearranging the terms finishes the proof. □

Using only the primal residual, we get the following error identity:

Proposition 12. *If the second Fréchet derivative of J, $J'' : W \to \mathcal{L}(W, W^*)$, exists as well as Assumption 1 and the assumptions of Lemma 11 hold, we obtain the error identity*

$$\begin{aligned} J(w) - J(w_h) =& \rho(w_h)(z - z_h) - C(w_h)(\xi_{n,h}) - D(w_h)(\xi_{t,h}) \\ &+ \mathcal{R}_J^{(2)} + \mathcal{R}_c^{(2)} + \mathcal{R}_f^{(2)}, \end{aligned} \tag{18}$$

with the primal residual

$$\rho(w_h)(\varphi) := -A(w_h)(\varphi).$$

The remainder term w.r.t. the quantity of interest, $\mathcal{R}_J^{(2)}$, is given by

$$\mathcal{R}_J^{(2)} = -\int_0^1 J''\left(w_h + se_w\right)\left(e_w, e_w\right) s\, ds.$$

For the contact conditions, we have the remainder

$$\mathcal{R}_c^{(2)} = \int_{\Gamma_C} \xi_n e_\chi^c \left[\lambda_{n,h} + u_{h,n} - g\right] do,$$

with $e_\chi^c = \chi^c - \chi_h^c$, and

$$\chi_h^c := \begin{cases} 1, & \text{if } \lambda_{n,h} + u_{h,n} - g > 0, \\ 0, & \text{if } \lambda_{n,h} + u_{h,n} - g \leq 0. \end{cases}$$

Remark 13. By $C\left(w_h\right)\left(\xi_{n,h}\right)$, we measure the violation of the geometrical contact conditions (2). The term $D\left(w_h\right)\left(\xi_{t,h}\right)$ represents the error of the discrete solution concerning the frictional conditions (3).

Remark 14. The term $\mathcal{R}_J^{(2)}$ corresponds to the usual remainder term of the DWR method for linear problems with nonlinear quantities of interest, cf. [3, Proposition 6.6]. It vanishes for linear quantities of interest J.

Remark 15. The remainder $\mathcal{R}_c^{(2)}$ w.r.t. the geometrical contact conditions becomes zero, if the analytic active set equals the discrete one. The frictional remainder term $\mathcal{R}_f^{(2)}$ has a higher order part of the same order as $\mathcal{R}_J^{(2)}$ and one in the indicator function of friction. The second part vanishes, if the sliding and sticking regions are exactly resolved. The remainder terms will be discussed in more detail in Section 4.

Proof. We use the box quadrature rule with its remainder term to obtain

$$\begin{aligned} J\left(w\right) - J\left(w_h\right) &= \int_0^1 J'\left(w_h + se_w\right)\left(e_w\right)\, ds = J'\left(w\right)\left(e_w\right) + \mathcal{R}_J^{(2)} \\ &= J'_u\left(w\right)\left(e_u\right) + J'_{\lambda_n}\left(w\right)\left(e_{\lambda_n}\right) + J'_{\lambda_t}\left(w\right)\left(e_{\lambda_t}\right) + \mathcal{R}_J^{(2)}. \end{aligned}$$

From the definition of the continuous dual problem, cf. (12-14), we conclude

$$\begin{aligned} & J'_u\left(w\right)\left(e_u\right) + J'_\lambda\left(w\right)\left(e_{\lambda_n}\right) + J'_{\lambda_t}\left(w\right)\left(e_{\lambda_t}\right) \\ = \; & a\left(e_u, y\right) - b^c\left(\xi_n, e_u\right) + b^f\left(\xi_t, e_u\right) + \left(e_{\lambda_n}, y_n\right)_{0,\Gamma_C} + c^c\left(\xi_n, e_{\lambda_n}\right) + c^f\left(\xi_t, e_{\lambda_n}\right) \\ & + \left(e_{\lambda_t}, y_t\right)_{0,\Gamma_C} + d^f\left(\xi_t, e_{\lambda_t}\right). \end{aligned}$$

The Galerkin orthogonality (11) leads to

$$\begin{aligned} & a\left(e_u, y\right) + \left(e_{\lambda_n}, y_n\right)_{0,\Gamma_C} + \left(e_{\lambda_t}, y_t\right)_{0,\Gamma_C} = a\left(e_u, e_y\right) + \left(e_{\lambda_n}, e_{y,n}\right)_{0,\Gamma_C} + \left(e_{\lambda_t}, e_{y,t}\right)_{0,\Gamma_C} \\ & = \langle l, e_y\rangle - a\left(u_h, e_y\right) - \left(\lambda_{n,h}, e_{y,n}\right)_{0,\Gamma_C} - \left(\lambda_{t,h}, e_{y,t}\right)_{0,\Gamma_C}. \end{aligned}$$

From the proof of Proposition 4 in [42], we know

$$c^c\left(\xi_n, e_{\lambda_n}\right) - b^c\left(\xi_n, e_u\right) = -C\left(w_h\right)\left(\xi_n - \xi_{n,h}\right) - C\left(w_h\right)\left(\xi_{n,h}\right) + \mathcal{R}_c^{(2)}$$

and from Lemma 11

$$b^f\left(\xi_t, e_u\right) + c^f\left(\xi_t, e_{\lambda_n}\right) + d^f\left(\xi_t, e_{\lambda_t}\right) = -D\left(w_h\right)\left(\xi_t - \xi_{t,h}\right) - D\left(w_h\right)\left(\xi_{t,h}\right) + \mathcal{R}_f^{(2)}.$$

In summary, we obtain

$$\begin{aligned}
& J(w) - J(w_h) \\
= \; & a(e_u, y) - b^c(\xi_n, e_u) + b^f(\xi_t, e_u) + (e_{\lambda_n}, y_n)_{0,\Gamma_C} + c^c(\xi_n, e_{\lambda_n}) + c^f(\xi_t, e_{\lambda_n}) \\
& + (e_{\lambda_t}, y_t)_{0,\Gamma_C} + d^f(\xi_t, e_{\lambda_t}) + \mathcal{R}_J^{(2)} \\
= \; & \langle l, e_y \rangle - a(u_h, e_y) - (\lambda_{n,h}, e_{y,n})_{0,\Gamma_C} - (\lambda_{t,h}, e_{y,t})_{0,\Gamma_C} - C(w_h)(\xi_n - \xi_{n,h}) \\
& - D(w_h)(\xi_t - \xi_{t,h}) - C(w_h)(\xi_{n,h}) - D(w_h)(\xi_{t,h}) + \mathcal{R}_J^{(2)} + \mathcal{R}_c^{(2)} + \mathcal{R}_f^{(2)} \\
= \; & \rho(w_h)(z - z_h) - C(w_h)(\xi_{n,h}) - D(w_h)(\xi_{t,h}) + \mathcal{R}_J^{(2)} + \mathcal{R}_c^{(2)} + \mathcal{R}_f^{(2)},
\end{aligned}$$

which is the assertion. □

Now, we study the error identity involving the dual residual. To this end, we need to apply the following lemma:

Lemma 16. *Under the general assumptions of this section, we obtain the identity*

$$\begin{aligned}
& b^f(\xi_t, e_u) + c^f(\xi_t, e_{\lambda_n}) + d^f(\xi_t, e_{\lambda_t}) + b_h^f(\xi_{t,h}, e_u) + c_h^f(\xi_{t,h}, e_{\lambda_n}) + d_h^f(\xi_{t,h}, e_{\lambda_t}) \\
= \; & -D(w_h)(e_{\xi_t}) - 2D(w_h)(\xi_{t,h}) + 2\mathcal{R}_f^{(3)}.
\end{aligned}$$

The remainder term $\mathcal{R}_f^{(3)} = \mathcal{R}_\chi^{(2)} + \mathcal{R}_\chi^{(3)} + \mathcal{R}_D^{(3)} + \mathcal{R}_Q^{(3)}$ *is given by a remainder in the frictional indicator function* $\mathcal{R}_\chi^{(3)} = \mathcal{R}_{\chi,1}^{(3)} + \mathcal{R}_{\chi,2}^{(3)}$ *with*

$$\begin{aligned}
\mathcal{R}_{\chi,1}^{(3)} &= -\frac{1}{2}\left[\bar{D}'(w_h)(e_w, \xi_{t,h}) - \bar{D}_h'(w_h)(e_w, \xi_{t,h})\right], \\
\mathcal{R}_{\chi,2}^{(3)} &= \frac{1}{2}\left[\bar{D}(w_h)(e_{\xi_t}) - \bar{D}_h(w_h)(e_{\xi_t})\right],
\end{aligned}$$

a cubic remainder

$$\mathcal{R}_D^{(3)} = -\frac{1}{2}\int_0^1 \bar{D}''(w_h + s e_w)(e_w, e_w, e_{\xi_t})\, s\, ds$$

in e_w *and* e_{ξ_t}*, as well as a quadrature remainder*

$$\mathcal{R}_Q^{(3)} = \frac{1}{2}\int_0^1 \bar{D}'''(w_h + s e_w)(e_w, e_w, e_w, \xi_t)\, s(s-1)\, ds,$$

which is of third order in the error e_w*.*

Remark 17. The remainder term $\mathcal{R}_f^{(3)}$ is dominated by the remainder $\mathcal{R}_\chi^{(2)}$, all other parts are of higher order in the error.

Proof. Using Lemma 11 and the trapezoidal rule with its remainder term, we obtain

$$\begin{aligned}
& -2D(w_h)(e_{\xi_t}) - 2D(w_h)(\xi_{t,h}) + 2\mathcal{R}_\chi^{(2)} \\
= \; & 2\left[\bar{D}(w)(\xi_t) - \bar{D}(w_h)(\xi_t)\right] = 2\int_0^1 \bar{D}'(w_h + s e_w)(e_w, \xi_t)\, ds \\
= \; & \bar{D}'(w)(e_w, \xi_t) + \bar{D}'(w_h)(e_w, \xi_t) - 2\mathcal{R}_Q^{(3)} \\
= \; & b^f(\xi_t, e_u) + c^f(\xi_t, e_{\lambda_n}) + d^f(\xi_t, e_{\lambda_t}) + \bar{D}'(w_h)(e_w, \xi_t) - 2\mathcal{R}_Q^{(3)}.
\end{aligned}$$

Studying $\bar{D}'(w_h)(e_w,\xi_t)$ in more detail lead to

$$\begin{aligned} & \bar{D}'(w_h)(e_w,\xi_t) \\ = \; & \bar{D}'(w_h)(e_w,e_{\xi_t}) + \bar{D}'(w_h)(e_w,\xi_{t,h}) \\ = \; & \bar{D}'(w_h)(e_w,e_{\xi_t}) + \bar{D}'(w_h)(e_w,\xi_{t,h}) - \bar{D}'_h(w_h)(e_w,\xi_{t,h}) + \bar{D}'_h(w_h)(e_w,\xi_{t,h}) \\ = \; & \bar{D}'(w_h)(e_w,e_{\xi_t}) + 2\mathcal{R}^{(3)}_{\chi,1} + b^f_h(\xi_{t,h},e_u) + c^f_h(\xi_{t,h},e_{\lambda_n}) + d^f_h(\xi_{t,h},e_{\lambda_t}). \end{aligned}$$

To finish the proof, we use the box quadrature rule and obtain

$$\begin{aligned} & \bar{D}'(w_h)(e_w,e_{\xi_t}) \\ = \; & \int_0^1 \bar{D}'(w_h+se_w)(e_w,e_{\xi_t})\,ds + 2\mathcal{R}^{(2)}_D \\ = \; & \bar{D}(w)(e_{\xi_t}) - \bar{D}(w_h)(e_{\xi_t}) + 2\mathcal{R}^{(2)}_D \\ = \; & -\bar{D}(w_h)(e_{\xi_t}) + \bar{D}_h(w_h)(e_{\xi_t}) - \bar{D}_h(w_h)(e_{\xi_t}) + 2\mathcal{R}^{(2)}_D \\ = \; & -D(w_h)(e_{\xi_t}) + 2\mathcal{R}^{(3)}_{\chi,2} + 2\mathcal{R}^{(3)}_D, \end{aligned}$$

the assertion. □

The bilinear forms $b^c_h : \Lambda_{n,h} \times V_h \to \mathbb{R}$ and $c^c_h : \Lambda_{n,h} \times \Lambda_{n,h} \to \mathbb{R}$ w.r.t. the geometrical contact conditions are given by

$$\begin{aligned} b^c_h(\omega_n,v) &:= \int_{\Gamma_C} \omega_n \chi^c_h v_n \,do, \\ c^c_h(\omega_n,\mu_n) &:= \int_{\Gamma_C} \omega_n \left[1-\chi^c_h\right] \mu_n \,do. \end{aligned}$$

Using the presented lemma above, we obtain the error representation:

Proposition 18. *We assume that the third Fréchet derivative of J, $J''' : W \to \mathcal{L}(W,\mathcal{L}(W,W^*))$ exists and that Assumption 1 hold. Then the error representation*

$$\begin{aligned} J(w) - J(w_h) = & \frac{1}{2}\rho(w_h)(e_z) + \frac{1}{2}\rho^*(w_h,z_h)(e_w) - C(w_h)(\xi_{n,h}) - D(w_h)(\xi_{t,h}) \\ & + \mathcal{R}^{(3)}_J + \mathcal{R}^{(3)}_c + \mathcal{R}^{(3)}_f \end{aligned} \tag{19}$$

is valid. Here, the dual residual ρ^* is defined as

$$\begin{aligned} \rho^*(w_h,z_h)(\varphi) := \; & J'(w_h)(\varphi) - a(v,y_h) + b^c_h(\xi_{n,h},v) - b^f_h(\xi_{t,h},v) - (\mu_n,y_{h,n})_{0,\Gamma_C} \\ & - c^c_h(\xi_{n,h},\mu_n) - c^f_h(\xi_{t,h},\mu_n) - (\mu_t,y_{h,t})_{0,\Gamma_C} - d^f_h(\xi_{t,h},\mu_t). \end{aligned}$$

For the remainder $\mathcal{R}^{(3)}_J$ w.r.t. the quantity of interest, it holds

$$\mathcal{R}^{(3)}_J = \frac{1}{2}\int_0^1 J'''(w_h+se_w)(e_w,e_w,e_w)\,s(s-1)\,ds$$

and for the remainder $\mathcal{R}^{(3)}_c$ concerning the geometrical contact conditions

$$\mathcal{R}^{(3)}_c = \frac{1}{2}\int_{\Gamma_C} e^c_\chi \left\{\xi_n\left[\lambda_{n,h}+u_{h,n}-g\right] + \xi_{n,h}\left[\lambda+u_n-g\right]\right\} do.$$

Remark 19. The remainder $\mathcal{R}^{(3)}_J$ is also obtained, if the DWR method is applied on other types of problems, see [3, Proposition 6.2] and compare Remark 14. It vanishes, if J is linear or quadratic in w.

Remark 20. The remainder terms $\mathcal{R}_c^{(3)}$ and $\mathcal{R}_f^{(3)}$ are of the same order in the error e_χ^c and e_χ^f as $\mathcal{R}_c^{(2)}$ and $\mathcal{R}_f^{(2)}$ because of the nonsmoothness of C and D.

PROOF. By applying the Trapezoidal quadrature rule with its remainder term, we obtain

$$\begin{aligned} J(w) - J(w_h) &= \int_0^1 J'(w_h + se_w)(e_w)\, ds = \frac{1}{2}J'(w_h)(e_w) + \frac{1}{2}J'(w)(e_w) + \mathcal{R}_J^{(3)} \\ &= \frac{1}{2}J_u'(w)(e_u) + \frac{1}{2}J_{\lambda_n}'(w)(e_{\lambda_n}) + \frac{1}{2}J_{\lambda_t}'(w)(e_{\lambda_t}) + \frac{1}{2}J_u'(w_h)(e_u) \\ &\quad + \frac{1}{2}J_{\lambda_n}'(w_h)(e_{\lambda_n}) + \frac{1}{2}J_{\lambda_t}'(w_h)(e_{\lambda_t}) + \mathcal{R}_J^{(3)}. \end{aligned}$$

We know from the proofs of Proposition 4 and 8 in [42] that

$$\begin{aligned} & c^c(\xi_n, e_{\lambda_n}) - b^c(\xi_n, e_u) + c_h^c(\xi_{n,h}, e_{\lambda_n}) - b_h^c(\xi_{n,h}, e_u) \\ = \; & -C(w_h)(\xi_n - \xi_{n,h}) - 2C(w_h)(\xi_{n,h}) + 2\mathcal{R}_c^{(3)}. \end{aligned}$$

From the proof of Proposition 12 together with Lemma 16 and the preceding equations, we deduce

$$\begin{aligned} & J_u'(w)(e_u) + J_\lambda'(w)(e_{\lambda_n}) + J_{\lambda_t}'(w)(e_{\lambda_t}) \\ = \; & \langle l, e_y\rangle - a(u_h, e_y) - (\lambda_{n,h}, e_{y,n})_{0,\Gamma_C} - (\lambda_{t,h}, e_{y,t})_{0,\Gamma_C} - b^c(\xi_n, e_u) + b^f(\xi_t, e_u) \\ & + c^c(\xi_n, e_{\lambda_n}) + c^f(\xi_t, e_{\lambda_n}) + d^f(\xi_t, e_{\lambda_t}) \\ = \; & \langle l, e_y\rangle - a(u_h, e_y) - (\lambda_{n,h}, e_{y,n})_{0,\Gamma_C} - (\lambda_{t,h}, e_{y,t})_{0,\Gamma_C} - c_h^c(\xi_{n,h}, e_{\lambda_n}) + b_h^c(\xi_{n,h}, e_u) \\ & - C(w_h)(\xi_n - \xi_{n,h}) - 2C(w_h)(\xi_{n,h}) + 2\mathcal{R}_c^{(3)} - b_h^f(\xi_{t,h}, e_u) - c_h^f(\xi_{t,h}, e_{\lambda_n}) - d_h^f(\xi_{t,h}, e_{\lambda_t}) \\ & - D(w_h)(\xi_t - \xi_{t,h}) - 2D(w_h)(\xi_{t,h}) + 2\mathcal{R}_f^{(3)} \\ = \; & \rho(w_h)(z - z_h) - c_h^c(\xi_{n,h}, e_{\lambda_n}) + b_h^c(\xi_{n,h}, e_u) - b_h^f(\xi_{t,h}, e_u) - c_h^f(\xi_{t,h}, e_{\lambda_n}) - d_h^f(\xi_{t,h}, e_{\lambda_t}) \\ & - 2C(w_h)(\xi_{n,h}) - 2D(w_h)(\xi_{t,h}) + 2\mathcal{R}_c^{(3)} + 2\mathcal{R}_f^{(3)}. \end{aligned}$$

Inserting the Galerkin orthogonality (11) and the definition of the dual residual ρ^* leads to

$$\begin{aligned} & J(w) - J(w_h) \\ = \; & \frac{1}{2}\rho(w_h)(z - z_h) + \frac{1}{2}J_u'(w_h)(e_u) + \frac{1}{2}J_{\lambda_n}'(w_h)(e_{\lambda_n}) + \frac{1}{2}J_{\lambda_t}'(w_h)(e_{\lambda_t}) \\ & - \frac{1}{2}a(e_u, y_h) - \frac{1}{2}(e_{\lambda_n}, y_{h,n})_{0,\Gamma_C} - \frac{1}{2}(e_{\lambda_t}, y_{h,t})_{0,\Gamma_C} - \frac{1}{2}c_h^c(\xi_{n,h}, e_{\lambda_n}) \\ & + \frac{1}{2}b_h^c(\xi_{n,h}, e_u) - \frac{1}{2}b_h^f(\xi_{t,h}, e_u) - \frac{1}{2}c_h^f(\xi_{t,h}, e_{\lambda_n}) - \frac{1}{2}d_h^f(\xi_{t,h}, e_{\lambda_t}) \\ & - C(w_h)(\xi_{n,h}) - D(w_h)(\xi_{t,h}) + \mathcal{R}_c^{(3)} + \mathcal{R}_f^{(3)} + \mathcal{R}_J^{(3)} \\ = \; & \frac{1}{2}\rho(w_h)(z - z_h) + \frac{1}{2}\rho^*(w_h, z_h)(w - w_h) - C(w_h)(\xi_{n,h}) - D(w_h)(\xi_{t,h}) \\ & + \mathcal{R}_c^{(3)} + \mathcal{R}_f^{(3)} + \mathcal{R}_J^{(3)}. \end{aligned}$$

□

The comparison of primal and dual residual leads to

Proposition 21. *If the second Fréchet derivative of J, $J'' : W \to \mathcal{L}(W, W^*)$, exists and Assumption 1 holds, we obtain for the difference between the primal residual ρ and the dual residual $\rho^\star$*

$$\rho^\star(w_h, z_h)(w - w_h) = \rho(w_h)(z - z_h) + \Delta J + \Delta C + \Delta D,$$

where

$$
\begin{aligned}
\Delta J &= -\int_0^1 J''(w_h + se_w)(e_w, e_w)\, ds, \\
\Delta C &= \int_{\Gamma_C} e_\chi^c \left\{ e_{\xi_n}\left[\lambda_n + u_n - g\right] - \xi_n\left[e_{\lambda_n} + e_{u,n}\right]\right\} do, \\
\Delta D &= \sum_{i=1}^4 \Delta D_i, \\
\Delta D_1 &= \int_0^1 \bar{D}''(w_h + se_w)(e_w, e_w, \xi_t)\, ds, \\
\Delta D_2 &= \int_0^1 \bar{D}''(w_h + se_w)(e_w, e_w, e_{\xi_t})\, s\, ds, \\
\Delta D_3 &= D(w_h)(e_{\xi_t}) - \bar{D}(w_h)(e_{\xi_t}), \\
\Delta D_4 &= \bar{D}'(w_h)(e_w, \xi_{t,h}) - \bar{D}'_h(w_h)(e_w, \xi_{t,h}).
\end{aligned}
$$

Remark 22. From Proposition 21 we learn that the difference between the primal and the dual residual is of higher order in the error than the remainder terms $\mathcal{R}_c^{(2)}$, $\mathcal{R}_f^{(2)}$, $\mathcal{R}_c^{(3)}$, and $\mathcal{R}_f^{(3)}$. Thus, the difference between the primal and dual residual is no estimate for the remainders $\mathcal{R}_c^{(2)}$ and $\mathcal{R}_f^{(2)}$ in contrast to smooth nonlinear problems, cf. [3, Proposition 6.6 and Remark 6.7].

Remark 23. The term ΔJ equals zero, if the quantity of interest J is linear in w.

Proof. The definition of the dual residual ρ^*, the continuous dual problem, and the definition of the primal residual lead to

$$
\begin{aligned}
& \rho^\star(w_h, z_h)(e_w) \\
= \; & J'(w_h)(e_w) - a(e_u, y_h) + b_h^c(\xi_{n,h}, e_u) - b_h^f(\xi_{t,h}, e_u) - (e_{\lambda_n}, y_{h,n})_{0,\Gamma_C} - c_h^c(\xi_{n,h}, e_{\lambda_n}) \\
& - c_h^f(\xi_{t,h}, e_{\lambda_n}) - (e_{\lambda_t}, y_{h,t})_{0,\Gamma_C} - d_h^f(\xi_{t,h}, e_{\lambda_t}) \\
= \; & J'(w_h)(e_w) - a(e_u, y_h) + b_h^c(\xi_{n,h}, e_u) - b_h^f(\xi_{t,h}, e_u) - (e_{\lambda_n}, y_{h,n})_{0,\Gamma_C} - c_h^c(\xi_{n,h}, e_{\lambda_n}) \\
& - c_h^f(\xi_{t,h}, e_{\lambda_n}) - (e_{\lambda_t}, y_{h,t})_{0,\Gamma_C} - d_h^f(\xi_{t,h}, e_{\lambda_t}) - J'(w)(e_w) + a(e_u, y) - b^c(\xi_n, e_u) \\
& + b^f(\xi_t, e_u) + (e_{\lambda_n}, y_n)_{0,\Gamma_C} + c^c(\xi_n, e_{\lambda_n}) + c^f(\xi_t, e_{\lambda_n}) + (e_{\lambda_t}, y_t)_{0,\Gamma_C} + d^f(\xi_t, e_{\lambda_t}) \\
= \; & -\int_0^1 J''(w_h + se_w)(e_w, e_w)\, ds + a(e_u, e_y) + (e_{\lambda_n}, e_{y,n})_{0,\Gamma_C} + (e_{\lambda_t}, e_{y,t})_{0,\Gamma_C} \\
& + b_h^c(\xi_{n,h}, e_u) - b^c(\xi_n, e_u) - c_h^c(\xi_{n,h}, e_{\lambda_n}) + c^c(\xi_n, e_{\lambda_n}) - b_h^f(\xi_{t,h}, e_u) + b^f(\xi_t, e_u) \\
& - c_h^f(\xi_{t,h}, e_{\lambda_n}) + c^f(\xi_t, e_{\lambda_n}) - d_h^f(\xi_{t,h}, e_{\lambda_t}) + d^f(\xi_t, e_{\lambda_t}) \\
= \; & \Delta J + \rho(w_h)(z - z_h) + C(w_h)(e_{\xi_n}) + D(w_h)(e_{\xi_t}) - \left[c_h^c(\xi_{n,h}, e_{\lambda_n}) - b_h^c(\xi_{n,h}, e_u)\right] \\
& + c^c(\xi_n, e_{\lambda_n}) - b^c(\xi_n, e_u) - \left[b_h^f(\xi_{t,h}, e_u) + c_h^f(\xi_{t,h}, e_{\lambda_t}) + d_h^f(\xi_{t,h}, e_{\lambda_t})\right] \\
& + b^f(\xi_t, e_u) + c^f(\xi_t, e_{\lambda_n}) + d^f(\xi_t, e_{\lambda_t}).
\end{aligned}
$$

From Proposition 11 in [42] we know

$$
C(w_h)(e_{\xi_n}) - \left[c_h^c(\xi_{n,h}, e_{\lambda_n}) - b_h^c(\xi_{n,h}, e_u)\right] + c^c(\xi_n, e_{\lambda_n}) - b^c(\xi_n, e_u) = \Delta C.
$$

The equations (16) and (17) imply

$$\begin{aligned}
& D\left(w_h\right)\left(e_{\xi_t}\right)+b^f\left(\xi_t, e_u\right)+c^f\left(\xi_t, e_{\lambda_n}\right)+d^f\left(\xi_t, e_{\lambda_t}\right) \\
& -\left[b_h^f\left(\xi_{t,h}, e_u\right)+c_h^f\left(\xi_{t,h}, e_{\lambda_t}\right)+d_h^f\left(\xi_{t,h}, e_{\lambda_t}\right)\right] \\
= & D\left(w_h\right)\left(e_{\xi_t}\right)+\bar{D}^{\prime}(w)\left(e_w, \xi_t\right)-\bar{D}_h^{\prime}\left(w_h\right)\left(e_w, \xi_{t,h}\right) \\
= & D\left(w_h\right)\left(e_{\xi_t}\right)+\bar{D}^{\prime}(w)\left(e_w, \xi_t\right)-\bar{D}^{\prime}\left(w_h\right)\left(e_w, \xi_t\right)+\bar{D}^{\prime}\left(w_h\right)\left(e_w, \xi_t\right)-\bar{D}_h^{\prime}\left(w_h\right)\left(e_w, \xi_{t,h}\right) \\
= & D\left(w_h\right)\left(e_{\xi_t}\right)+\int_0^1 \bar{D}^{\prime\prime}\left(w_h+s e_w\right)\left(e_w, e_w, \xi_t\right) d s+\bar{D}^{\prime}\left(w_h\right)\left(e_w, \xi_t\right)-\bar{D}_h^{\prime}\left(w_h\right)\left(e_w, \xi_{t,h}\right) \\
= & D\left(w_h\right)\left(e_{\xi_t}\right)+\Delta D_1+\bar{D}^{\prime}\left(w_h\right)\left(e_w, \xi_t\right)-\bar{D}_h^{\prime}\left(w_h\right)\left(e_w, \xi_{t,h}\right).
\end{aligned}$$

Furthermore, we find using $D(w)\left(e_{\xi_t}\right)=\bar{D}(w)\left(e_{\xi_t}\right)=0$ and the box quadrature rule with its remainder that

$$\begin{aligned}
& D\left(w_h\right)\left(e_{\xi_t}\right)+\bar{D}^{\prime}\left(w_h\right)\left(e_w, \xi_t\right)-\bar{D}_h^{\prime}\left(w_h\right)\left(e_w, \xi_{t,h}\right)+\Delta D_1 \\
= & D\left(w_h\right)\left(e_{\xi_t}\right)+\bar{D}^{\prime}\left(w_h\right)\left(e_w, e_{\xi_t}\right)+\bar{D}^{\prime}\left(w_h\right)\left(e_w, \xi_{t,h}\right)-\bar{D}_h^{\prime}\left(w_h\right)\left(e_w, \xi_{t,h}\right)+\Delta D_1 \\
= & \bar{D}\left(w_h\right)\left(e_{\xi_t}\right)-\bar{D}(w)\left(e_{\xi_t}\right)+\bar{D}^{\prime}\left(w_h\right)\left(e_w, e_{\xi_t}\right)-\bar{D}\left(w_h\right)\left(e_{\xi_t}\right)+D\left(w_h\right)\left(e_{\xi_t}\right) \\
& +\Delta D_1+\Delta D_4 \\
= & -\int_0^1 \bar{D}^{\prime}\left(w_h+s e_w\right)\left(e_w, e_{\xi_t}\right) d s+\bar{D}^{\prime}\left(w_h\right)\left(e_w, e_{\xi_t}\right)+\Delta D_1+\Delta D_3+\Delta D_4 \\
= & \int_0^1 \bar{D}^{\prime\prime}\left(w_h+s e_w\right)\left(e_w, e_w, e_{\xi_t}\right) s\, d s+\Delta D_1+\Delta D_3+\Delta D_4 \\
= & \Delta D_1+\Delta D_2+\Delta D_3+\Delta D_4 .
\end{aligned}$$

Combining the different parts, we get the assertion with $\Delta D=\Delta D_1+\Delta D_2+\Delta D_3+\Delta D_4$. □

3.3 Estimation of model and discretization error

As last result in this section, we estimate the error $J\left(w^r\right)-J\left(w_h\right)$ including the modeling as well as the discretization error in the quantity of interest. We define

$$\begin{aligned}
e_w^{r,h}=\left(e_u^{r,h}, e_{\lambda_n}^{r,h}, e_{\lambda_t}^{r,h}\right) & =\left(u^r-u_h, \lambda_n^r-\lambda_{n,h}, \lambda_t^r-\lambda_{t,h}\right), \\
e_z^{r,h}=\left(e_y^{r,h}, e_{\xi_n}^{r,h}, e_{\xi_t}^{r,h}\right) & =\left(y^r-y_h, \xi_n^r-\xi_{n,h}, \xi_t^r-\xi_{t,h}\right).
\end{aligned}$$

Furthermore, we set $e_{\chi^c}^{r,h}:=\chi_r^c-\chi_h^c$ and $e_{\chi^f}^{r,h}:=\chi_r^f-\chi_h^f$. In addition, we need an analogous result to Lemma 7:

Lemma 24. *It holds*

$$\begin{aligned}
& b_h^f\left(\xi_{t,h}, e_u^{r,h}\right)+c_h^f\left(\xi_{t,h}, e_{\lambda_n}^{r,h}\right)+d_h^f\left(\xi_{t,h}, e_{\lambda_t}^{r,h}\right) \\
& +b_r^f\left(\xi_t^r, e_u^{r,h}\right)+c_r^f\left(\xi_t^r, e_{\lambda_n}^{r,h}\right)+d_r^f\left(\xi_t^r, e_{\lambda_t}^{r,h}\right)+D\left(w_h\right)\left(e_{\xi_t}^{r,h}\right) \\
= & -2 D\left(w_h\right)\left(\xi_{t,h}\right)-2 \Delta\left(w_h\right)\left(\xi_{t,h}\right)-2 \Delta\left(w_h\right)\left(e_z^{r,h}\right)-\bar{\Delta}\left(e_w^{r,h}, e_z^{r,h}\right)+2 \mathcal{R}_f^{m,h}
\end{aligned}$$

with the remainder term $\mathcal{R}_f^{m,h}=\mathcal{R}_{\chi,1}^{m,h}+\mathcal{R}_{\chi,2}^{m,h}+\mathcal{R}_Q^{m,h}$,

$$\begin{aligned}
\mathcal{R}_{\chi,1}^{m,h} & =-\left(\xi_t^r,\left(\chi_r^f\left(w^r\right)-\chi_r^f\left(w_h\right)\right)\left(n\left(w_h\right) \lambda_{t,h}-s^r\left(\lambda_{n,h}\right) \lambda_{t,h}\right)\right)_{0,\Gamma_C}, \\
\mathcal{R}_{\chi,2}^{m,h} & =-\frac{1}{2}\left[\left(\bar{D}^r\right)^{\prime}\left(w_h\right)\left(e_w^{r,h}, \xi_{t,h}\right)-\bar{D}_h^{\prime}\left(w_h\right)\left(e_w^{r,h}, \xi_{t,h}\right)\right] \\
\mathcal{R}_Q^{m,h} & =\frac{1}{2} \int_0^1\left(\bar{D}^r\right)^{\prime\prime\prime}\left(w+s e_w^{r,h}\right)\left(e_w^{r,h}, e_w^{r,h}, e_w^{r,h}, \xi_t^r\right) s(s-1)\, d s,
\end{aligned}$$

Proof. The definition of $\bar{D}^r$ leads to

$$
\begin{aligned}
& \bar{D}^r(w_h)(\xi_t^r) \\
= & \left(\xi_t^r, \chi_r^f (n(w_h)\lambda_{t,h} - s^r(\lambda_{n,h})\lambda_{t,h}) - s^r(\lambda_{n,h}) u_{h,t}\right)_{0,\Gamma_C} \\
= & \left(\xi_t^r, \left(\chi_r^f(w^r) - \chi_r^f(w_h)\right)(n(w_h)\lambda_{t,h} - s^r(\lambda_{n,h})\lambda_{t,h})\right)_{0,\Gamma_C} \\
& + \left(\xi_t^r, \chi_r^f(w_h)(n(w_h)\lambda_{t,h} - s^r(\lambda_{n,h})\lambda_{t,h}) - s^r(\lambda_{n,h}) u_{h,t}\right)_{0,\Gamma_C} \\
= & -\mathcal{R}_{\chi,1}^{m,h} + D^r(w_h)(\xi_t^r) = D^r(w_h)\left(e_{\xi_t}^{r,h}\right) + D^r(w_h)(\xi_{t,h}) - \mathcal{R}_{\chi,1}^{m,h} \\
= & D^r(w_h)\left(e_{\xi_t}^{r,h}\right) - D(w_h)\left(e_{\xi_t}^{r,h}\right) + D^r(w_h)(\xi_{t,h}) - D(w_h)(\xi_{t,h}) + D(w_h)\left(e_{\xi_t}^{r,h}\right) \\
& + D(w_h)(\xi_{t,h}) - \mathcal{R}_{\chi,1}^{m,h} \\
= & \Delta(w_h)\left(e_{\xi_t}^{r,h}\right) + \Delta(w_h)(\xi_{t,h}) + D(w_h)\left(e_{\xi_t}^{r,h}\right) + D(w_h)(\xi_{t,h}) - \mathcal{R}_{\chi,1}^{m,h}.
\end{aligned}
$$

The trapezoidal rule with its remainder term together with $\bar{D}^r(w^r)(\mu_t) = D^r(w^r)(\mu_t) = 0$ and the preceding calculations lead to

$$
\begin{aligned}
& -\Delta(w_h)\left(e_{\xi_t}^{r,h}\right) - \Delta(w_h)(\xi_{t,h}) - D(w_h)\left(e_{\xi_t}^{r,h}\right) - D(w_h)(\xi_{t,h}) + \mathcal{R}_{\chi,1}^{m,h} \\
= & \bar{D}^r(w^r)(\xi_t^r) - \bar{D}^r(w_h)(\xi_t^r) \\
= & \int_0^1 (\bar{D}^r)'\left(w_h + s e_w^{r,h}\right)\left(e_w^{r,h}, \xi_t^r\right) ds \\
= & \frac{1}{2}(\bar{D}^r)'(w_h)\left(e_w^{r,h}, \xi_t^r\right) + \frac{1}{2}(\bar{D}^r)'(w^r)\left(e_w^{r,h}, \xi_t^r\right) - \mathcal{R}_Q^{m,h}.
\end{aligned}
$$

We use equation (15) and

$$
\begin{aligned}
& (\bar{D}^r)'(w_h)\left(e_w^{r,h}, \xi_t^r\right) \\
= & (\bar{D}^r)'(w_h)\left(e_w^{r,h}, e_{\xi_t}^{r,h}\right) + (\bar{D}^r)'(w_h)\left(e_w^{r,h}, \xi_{t,h}\right) - \bar{D}_h'(w_h)\left(e_w^{r,h}, \xi_{t,h}\right) \\
& + \bar{D}_h'(w_h)\left(e_w^{r,h}, \xi_{t,h}\right) \\
= & \bar{\Delta}\left(e_w^{r,h}, e_{\xi_t}^{r,h}\right) - 2\mathcal{R}_{\chi,2}^{m,h} + b_h^f\left(\xi_{t,h}, e_u^{r,h}\right) + c_h^f\left(\xi_{t,h}, e_{\lambda_n}^{r,h}\right) + d_h^f\left(\xi_{t,h}, e_{\lambda_t}^{r,h}\right),
\end{aligned}
$$

apply (17) to deduce the assertion by combining the single terms. □

Applying the above presented lemma, we obtain the following error identity for the error w.r.t. modeling and discretization:

Proposition 25. *We assume that the third Fréchet derivative of J, $J''' : W \to \mathcal{L}(W, \mathcal{L}(W, W^*))$, exist and that Assumption 1 holds. Then, the error identity*

$$
\begin{aligned}
& J(w^r) - J(w_h) \\
= & -\Delta(w_h)(z_h) + \frac{1}{2}\rho(w_h)(e_z) + \frac{1}{2}\rho^*(w_h, z_h)(e_w) - C(w_h)(\xi_{n,h}) - D(w_h)(\xi_{t,h}) \\
& - \Delta(w_h)\left(e_z^{r,h}\right) - \frac{1}{2}\bar{\Delta}\left(e_w^{r,h}, e_z^{r,h}\right) + \mathcal{R}_J^{m,h} + \mathcal{R}_c^{m,h} + \mathcal{R}_f^{m,h}
\end{aligned}
\tag{20}
$$

holds for the model and discretization error in the quantity of interest. Here, the remainder terms are given by

$$
\mathcal{R}_J^{m,h} = \frac{1}{2}\int_0^1 J'''\left(w_h + s e_w^{r,h}\right)\left(e_w^{r,h}, e_w^{r,h}, e_w^{r,h}\right) s(s-1)\, ds
$$

w.r.t. the quantity of interest J,

$$\mathcal{R}_c^{m,h} = \frac{1}{2}\int_{\Gamma_C} \xi_n^r e_\chi^{c,r}\left[\lambda_{n,h} + u_{h,n} - g\right] + \xi_{n,h} e_\chi^{c,r}\left[\lambda_n^r + u_h^r - g\right] do$$

and $\mathcal{R}_f^m$ from Lemma 7.

Remark 26. The remainder $\mathcal{R}_J^{m,h}$ is of third order in the error $e_w^{r,h}$ and equals mainly the remainders $\mathcal{R}_J^m$ from Propositon 9 and $\mathcal{R}_J^{(3)}$ from Proposition 18. The remainder $\mathcal{R}_c^{m,h}$ is of the same structure as $\mathcal{R}_c^{(3)}$ in Proposition 18.

Proof. The starting point is again the application of the trapezoidal rule with its remainder leading to

$$\begin{aligned} J(w^r) - J(w_h) &= \int_0^1 J'\left(w_h + s e_w^{r,h}\right)\left(e_w^{r,h}\right) ds \\ &= \frac{1}{2} J'(w_h)\left(e_w^{r,h}\right) + \frac{1}{2} J'(w^r)\left(e_w^{r,h}\right) + \mathcal{R}_J^{m,h}. \end{aligned}$$

We now proceed as in the proof of Proposition 9. However, we have to take into account that (11) holds instead of (9). Thus, we obtain by (11)

$$\begin{aligned} & a\left(e_u^{r,h}, y^r\right) + \left(e_{\lambda_n}^{r,h}, y_n^r\right)_{0,\Gamma_C} + \left(e_{\lambda_t}^{r,h}, y_t^r\right)_{0,\Gamma_C} \\ = {} & a\left(e_u^{r,h}, e_y^{r,h}\right) + \left(e_{\lambda_n}^{r,h}, e_{y,n}^{r,h}\right)_{0,\Gamma_C} + \left(e_{\lambda_t}^{r,h}, e_{y,t}^{r,h}\right)_{0,\Gamma_C} \\ = {} & \rho(w_h)\left(e_z^{r,h}\right) + C(w_h)\left(e_{\xi_n}^r\right) + D(w_h)\left(e_{\xi_t}^r\right). \end{aligned}$$

The defintions of the continuous dual problem (12-14) and the preceding calculation imply

$$\begin{aligned} J'(w^r)\left(e_w^{r,h}\right) &= a\left(e_u^{r,h}, y^r\right) - b_r^c\left(\xi_n^r, e_u^{r,h}\right) + b_r^f\left(\xi_t^r, e_u^{r,h}\right) + \left(e_{\lambda_n}^{r,h}, y_n^r\right)_{0,\Gamma_C} \\ &\quad + c_r^c\left(\xi_n^r, e_{\lambda_n}^{r,h}\right) + c_r^f\left(\xi_t^r, e_{\lambda_n}^{r,h}\right) + \left(e_{\lambda_t}^{r,h}, y_t^r\right)_{0,\Gamma_C} + d_r^f\left(\xi_t^r, e_{\lambda_t}^{r,h}\right) \\ &= \rho(w_h)\left(e_z^{r,h}\right) + C(w_h)\left(e_{\xi_n}^{r,h}\right) + D(w_h)\left(e_{\xi_t}^{r,h}\right) \\ &\quad - b_r^c\left(\xi_n^r, e_u^{r,h}\right) + b_r^f\left(\xi_t^r, e_u^{r,h}\right) + c_r^c\left(\xi_n^r, e_{\lambda_n}^{r,h}\right) + c_r^f\left(\xi_t^r, e_{\lambda_n}^{r,h}\right) + d_r^f\left(\xi_t^r, e_{\lambda_t}^{r,h}\right). \end{aligned}$$

By (11) and the definition of the dual residual ρ^*, we deduce

$$\begin{aligned} & J'(w^r)\left(e_w^{r,h}\right) \\ = {} & J'(w^r)\left(e_w^{r,h}\right) - a\left(e_u^{r,h}, y_h\right) - \left(e_{\lambda_n}^{r,h}, y_{h,n}\right)_{0,\Gamma_C} - \left(e_{\lambda_t}^{r,h}, y_{h,t}\right)_{0,\Gamma_C} \\ & + b_h^c\left(\xi_{n,h}, e_u^{r,h}\right) - b_h^f\left(\xi_{t,h}, e_u^{r,h}\right) - c_h^c\left(\xi_{n,h}, e_{\lambda_n}^{r,h}\right) - c_h^f\left(\xi_{t,h}, e_{\lambda_n}^{r,h}\right) - d_h^f\left(\xi_{t,h}, e_{\lambda_t}^{r,h}\right) \\ & - b_h^c\left(\xi_{n,h}, e_u^{r,h}\right) + b_h^f\left(\xi_{t,h}, e_u^{r,h}\right) + c_h^c\left(\xi_{n,h}, e_{\lambda_n}^{r,h}\right) + c_h^f\left(\xi_{t,h}, e_{\lambda_n}^{r,h}\right) + d_h^f\left(\xi_{t,h}, e_{\lambda_t}^{r,h}\right) \\ = {} & - b_h^c\left(\xi_{n,h}, e_u^{r,h}\right) + b_h^f\left(\xi_{t,h}, e_u^{r,h}\right) + c_h^c\left(\xi_{n,h}, e_{\lambda_n}^{r,h}\right) + c_h^f\left(\xi_{t,h}, e_{\lambda_n}^{r,h}\right) + d_h^f\left(\xi_{t,h}, e_{\lambda_t}^{r,h}\right) \\ & + \rho^*(w_h, z_h)\left(e_w^{r,h}\right) \end{aligned}$$

By the same technique as in the proofs of Proposition 4 and 8 in [42], we obtain

$$\begin{aligned}
& c_r^c\left(\xi_n^r, e_{\lambda_n}^{r,h}\right) - b_r^c\left(\xi_n^r, e_u^{r,h}\right) + c_h^c\left(\xi_{n,h}, e_{\lambda_n}^{r,h}\right) - b_h^c\left(\xi_{n,h}, e_u^{r,h}\right) + C\left(w_h\right)\left(e_{\xi_n}^{r,h}\right) \\
= \; & c_r^c\left(\xi_n^r, e_{\lambda_n}^{r,h}\right) - b_r^c\left(\xi_n^r, e_u^{r,h}\right) + C\left(w_h\right)\left(\xi_n^r\right) \\
& + c_h^c\left(\xi_{n,h}, e_{\lambda_n}^{r,h}\right) - b_h^c\left(\xi_{n,h}, e_u^{r,h}\right) + C\left(w_h\right)\left(\xi_{n,h}\right) - 2C\left(w_h\right)\left(\xi_{n,h}\right) \\
= \; & \int_{\Gamma_C} \xi_n^r e_{\chi^c}^{r,h}\left[\lambda_{n,h} + u_{h,n} - g\right] + \xi_{n,h} e_{\chi^c}^{r,h}\left[\lambda_n^r + u_h^r - g\right] do - 2C\left(w_h\right)\left(\xi_{n,h}\right) \\
= \; & 2\mathcal{R}_c^{m,h} - 2C\left(w_h\right)\left(\xi_{n,h}\right).
\end{aligned}$$

All in all, we deduce applying Lemma 24

$$\begin{aligned}
& 2\left(J\left(w^r\right) - J\left(w_h\right)\right) \\
= \; & J'\left(w_h\right)\left(e_w^{r,h}\right) + J'\left(w^r\right)\left(e_w^{r,h}\right) + 2\mathcal{R}_J^{m,h} \\
= \; & \rho\left(w_h\right)\left(e_z^{r,h}\right) + \rho^*\left(w_h, z_h\right)\left(e_w^{r,h}\right) + 2\mathcal{R}_J^{m,h} \\
& + c_r^c\left(\xi_n^r, e_{\lambda_n}^{r,h}\right) - b_r^c\left(\xi_n^r, e_u^{r,h}\right) + c_h^c\left(\xi_{n,h}, e_{\lambda_n}^{r,h}\right) - b_h^c\left(\xi_{n,h}, e_u^{r,h}\right) + C\left(w_h\right)\left(e_{\xi_n}^{r,h}\right) \\
& + b_h^f\left(\xi_{t,h}, e_u^{r,h}\right) + c_h^f\left(\xi_{t,h}, e_{\lambda_n}^{r,h}\right) + d_h^f\left(\xi_{t,h}, e_{\lambda_t}^{r,h}\right) \\
& + b_r^f\left(\xi_t^r, e_u^{r,h}\right) + c_r^f\left(\xi_t^r, e_{\lambda_n}^{r,h}\right) + d_r^f\left(\xi_t^r, e_{\lambda_t}^{r,h}\right) + D\left(w_h\right)\left(e_{\xi_t}^{r,h}\right) \\
= \; & \rho\left(w_h\right)\left(e_z^{r,h}\right) + \rho^*\left(w_h, z_h\right)\left(e_w^{r,h}\right) + 2\mathcal{R}_J^{m,h} + 2\mathcal{R}_c^{m,h} - 2C\left(w_h\right)\left(\xi_{n,h}\right) \\
& - 2D\left(w_h\right)\left(\xi_{t,h}\right) - 2\Delta\left(w_h\right)\left(\xi_{t,h}\right) - 2\Delta\left(w_h\right)\left(e_z^{r,h}\right) - \bar{\Delta}\left(e_w^{r,h}, e_z^{r,h}\right) + 2\mathcal{R}_f^{m,h}.
\end{aligned}$$

Division by 2 then gives the assertion. □

4 Numerical evaluation of the error identities

The error identities (18), (19), and (20) from Proposition 12, 18 and 25 cannot be evaluated numerically, because they involve the analytic solutions w and z as well as the unknown remainder terms. The remainder terms $\mathcal{R}_J^{(2)}$, $\mathcal{R}_J^{(3)}$, and $\mathcal{R}_J^{m,h}$ are of second and third order in the error, respectively, which implies that they are of higher order and negligible. The remainder terms with respect to the geometrical contact conditions, $\mathcal{R}_c^{(2)}$, $\mathcal{R}_c^{(3)}$ and $\mathcal{R}_c^{m,h}$, are of first order in the error of the active set. Numerical examples substantiate that they are decreasing fast. However, a strict analysis of there convergence properties is missing and strongly depends on the chosen discretization. The remainder terms $\mathcal{R}_f^{(2)}$, $\mathcal{R}_f^{(3)}$ and $\mathcal{R}_f^{m,h}$ with respect to the friction conditions consist of terms which are of first order in the error of the frictional active set and ones which are of higher order in the error. While it is clear that the second ones can be neglected, the same is not true for the first ones. However, the remarks for the remainder terms $\mathcal{R}_c^{(2)}$, $\mathcal{R}_c^{(3)}$ and $\mathcal{R}_c^{m,h}$ also hold here. The remaining terms $\Delta\left(w_h\right)\left(e_z^{r,h}\right)$ and $\frac{1}{2}\bar{\Delta}\left(e_w^{r,h}, e_z^{r,h}\right)$, which arise in the estimation of the model error, are of second as well as third order in the error and are neglected. The numerical results in Section 5 substantiate that neglecting the remainder terms is feasible. Beside the remainder terms, the error identities also include the analytic primal and dual solution, which have to be numerically approximated. The corresponding discretization dependent operator is denoted by $\mathcal{A}$. We refer to [3, Section 4.1 and Section 5.2] for an overview of possible choices and their mathematical justification under strong smoothness assumptions.

All in all, we obtain the primal error estimator

$$J(w) - J(w_h) \quad \approx \quad \eta_p := \rho(w_h)(\mathcal{A}(z_h) - z_h) - C(w_h)(\xi_{n,h}) - D(w_h)(\xi_{t,h}),$$

the primal dual one

$$\begin{aligned} J(w) - J(w_h) \quad \approx \eta := \quad & \frac{1}{2}\rho(w_h)(\mathcal{A}(z_h) - z_h) + \frac{1}{2}\rho^*(w_h, z_h)(\mathcal{A}(w_h) - w_h) \\ & - C(w_h)(\xi_{n,h}) - D(w_h)(\xi_{t,h}), \end{aligned}$$

and the model as well as discretization error estimator

$$J(w^r) - J(w_h) \approx -\Delta(w_h)(z_h) + \eta = \eta_m + \eta.$$

Up to this point, we have not specified any further assumptions on the discretization. Henceforth, we carry out further steps to obtain concrete error estimators for a mixed discretization. It was first proposed for geometrical contact problems in [23] and extended to frictional contact problems in [24, 25] as well as higher order methods in [48]. In the aforementioned references, a Schur-complement ansatz is used to solve the discrete problems. Here, we use a primal-dual-active-set-strategy, which was developed for this discretization approach in [9]. We outline the discretization in more detail here: Let $\mathcal{T}_h$ be a finite element mesh of Ω with mesh size h and let $\mathcal{E}_C$ be a finite element mesh of Γ_C with mesh size H, respectively. The number of mesh elements in $\mathcal{T}_h$ is denoted by M_Ω and in $\mathcal{E}_C$ by M_C. We use line segments, quadrangles or hexahedrons to define $\mathcal{T}_h$ or $\mathcal{E}_C$. But this is not a restriction, triangles and tetrahedrons are also possible. Furthermore, let $\Psi_T : [-1,1]^d \to T \in \mathcal{T}_h$ and $\Phi_E : [-1,1]^{d-1} \to E \in \mathcal{E}_C$ be affine and d-linear transformations. We define

$$\begin{aligned} V_h &:= \left\{ v \in V \,\middle|\, \forall T \in \mathcal{T}_h : \; v_{i|T} \circ \Psi_T \in Q_1 \right\}, \\ \Lambda_H &:= \left\{ \mu \in L^2(\Gamma_C) \middle| \; \forall E \in \mathcal{E}_C : \; \mu_{|E} \circ \Phi_E \in \mathbb{P}_0 \right\}, \\ \Lambda_{n,H} &:= \left\{ \mu_n \in \Lambda_H \middle| \; \forall E \in \mathcal{E}_C : \; \mu_{n|E} \geq 0 \right\}, \\ \Lambda_{t,H}(\lambda_{n,H}) &:= \left\{ \mu_t \in \Lambda_H^{d-1} \middle| \; \forall E \in \mathcal{E}_C : \; \mu_{t|E} \leq \frac{1}{|E|} \int_E s(\lambda_{n,H}) \, do \right\}, \end{aligned}$$

where Q_1 is the set of d-linear functions on $[-1,1]^d$ and $\mathbb{P}_0$ the set of piecewise constant basis functions for the Lagrange Multiplier on $[-1,1]^{d-1}$. The discrete saddle point problem is to find $(u_h, \lambda_{n,H}, \lambda_{t,H}) \in V_h \times \Lambda_{n,H} \times \Lambda_{t,H}$ such that

$$a(u_h, v_h) + (\lambda_{n,H}, v_{h,n})_{0,\Gamma_C} + (\lambda_{t,H}, v_{h,t})_{0,\Gamma_C} \;=\; \langle l, v_h \rangle, \tag{21}$$

$$(\mu_{n,H} - \lambda_{n,H}, u_{h,n} - g)_{0,\Gamma_C} + (\mu_{t,H} - \lambda_{t,H}, u_{h,t})_{0,\Gamma_C} \;\leq\; 0, \tag{22}$$

holds for all $v_h \in V_h$, all $\mu_{n,H} \in \Lambda_{n,H}$, and all $\mu_{t,H} \in \Lambda_{t,H}$. It is well-known that we obtain a stable discretization if a discrete inf-sup condition is fulfilled. In the case of quasi-uniform meshes the discrete inf-sup condition holds if the quotient of the mesh sizes h/H is sufficiently small, cf., for instance, [48]. If different mesh sizes h and H are used, the Lagrange multiplier has to be defined on a coarser mesh leading to a higher implementational complexity than using a surface mesh $\mathcal{E}_C$ inherited from the interior mesh $\mathcal{T}_h$. In our numerical experiments, we observe oscillating Lagrange multipliers for $h = H$ and stable schmes for $H = 2h$, which corresponds to the results in the mentioned reference. Consequently, the numerical experiments in Section 5 are based on meshes with $H = 2h$.

Our definition of the discrete dual solution is motivated by the primal-dual-active-set-strategy to solve the discrete problem (21-22) outlined in [9, Section 5.4]. There, the active and inactive sets are based on the surface mesh $\mathcal{E}_C$. Consequently, we define the following discrete indicator functions for $E \in \mathcal{E}_C$:

$$\chi_E^c(w_h) := \begin{cases} 1, & \text{if } \int_E \lambda_{n,H} + u_{h,n} - g\, do > 0, \\ 0, & \text{if } \int_E \lambda_{n,H} + u_{h,n} - g\, do \leq 0, \end{cases}$$

$$\chi_E^f(w_h) := \begin{cases} 1, & \text{if } \int_E s(\lambda_{n,H}) - ||E|\, \lambda_{t,H} + u_{h,t}|\, do < 0, \\ 0, & \text{if } \int_E s(\lambda_{n,H}) - ||E|\, \lambda_{t,H} + u_{h,t}|\, do \geq 0. \end{cases}$$

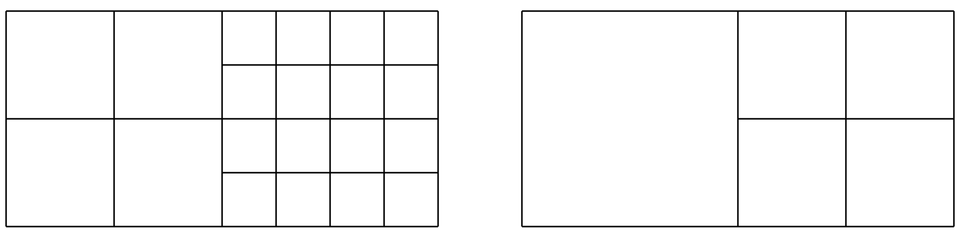
The discrete bilinear forms are then given by

$$\begin{aligned} \bar{b}_h^c(\omega_n, v) &:= \sum_{E\in\mathcal{E}_C} \int_E \omega_n \chi_E^c v_n\, do, \\ \bar{c}_h^c(\omega_n, \mu_n) &:= \sum_{E\in\mathcal{E}_C} \int_{\Gamma_C} \omega_n [1-\chi_E^c]\, \mu_n\, do, \\ \bar{b}_h^f(\omega_t, v) &:= \sum_{E\in\mathcal{E}_C} \int_E \omega_t \left[\chi_E^f \lambda_{t,H} \left(n'(w_h)\right)^\top - s(\lambda_{n,H})\, I\right] v_t\, do, \\ \bar{c}_h^f(\omega_t, \mu_n) &:= -\sum_{E\in\mathcal{E}_C} \int_E \omega_t\, (s)'(\lambda_{n,H})\, (\mu_n) \left[\chi_E^f \lambda_{t,h} + u_{h,t}\right] do, \\ \bar{d}_h^f(\omega_t, \mu_t) &:= \sum_{E\in\mathcal{E}_C} \int_E \omega_t \left[\max\{s(\lambda_{n,H}), n(w_h)\}\, I - s(\lambda_{n,H})\, I\right] \mu_t\, do \\ &\quad + \sum_{E\in\mathcal{E}_C} \int_E \omega_t \chi_E^f \lambda_{t,H} \left(n'(w_h)\right)^\top \mu_t\, do. \end{aligned}$$

The discrete dual problem is to find a dual solution $z_h = (y_h, \xi_{n,H}, \xi_{t,H}) \in V_h \times \Lambda_H \times \Lambda_H^{d-1}$ with

$$\begin{aligned} a(v_h, y_h) - \bar{b}_h^c(\xi_{n,H}, v_h) + \bar{b}_h^f(\xi_{t,H}, v_h) &= J_u'(w_h)(v_h), \\ (\mu_{n,H}, y_{h,n})_{0,\Gamma_C} + \bar{c}_h^c(\xi_{n,H}, \mu_{n,H}) + \bar{c}_h^f(\xi_{t,H}, \mu_{n,H}) &= J_{\lambda_n}'(w_h)(\mu_{n,H}), \\ (\mu_{t,H}, y_{h,t})_{0,\Gamma_C} + \bar{d}_h^f(\xi_{t,H}, \mu_{t,H}) &= J_{\lambda_t}'(w_h)(\mu_{t,H}), \end{aligned}$$

for all $(y_h, \mu_{n,H}, \mu_{t,H}) \in V_h \times \Lambda_H \times \Lambda_H^{d-1}$. We should remark that we use the bilinear forms $\bar{b}_h^c$, $\bar{b}_h^f$, $\bar{c}_h^c$, $\bar{c}_h^f$, and $\bar{d}_h^h$ instead of b_h^c, b_h^f, c_h^c, c_h^f, and d_h^h, since the dual problem using b_h^c, b_h^f, c_h^c, c_h^f, and d_h^h is not necessaryly well posed.



(a) Mesh with patch structure (b) Corresponding patch mesh

Fig. 2: Illustration of the patch structure of the finite element mesh

In this article, we use higher order reconstructions of the discrete solutions for the approximation of w and z, because this procedure is computationally cheaper than the calculation

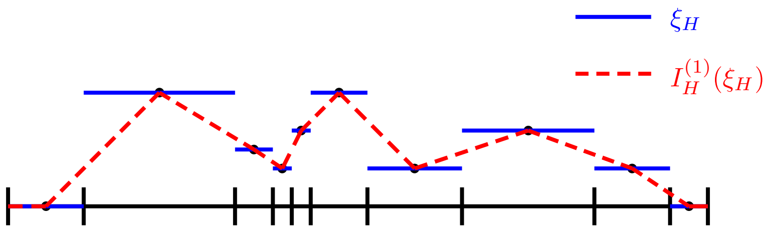


Fig. 3: Illustration of $i_{2H}^{(1)}$

of higher order solutions or extrapolation techniques. The primal and dual displacement u and y is approximated using patchwise d-quadratic reconstruction, cf., e.g., [3, Section 4.1] for this well known procedure. Let $i_{2h}^{(2)}$ be the corresponding interpolation operator. For the evaluation of $i_{2h}^{(2)}$, we require a special structure of the adaptively refined finite element mesh. This so-called patch-structure is obtained through the refinement of all children of a refined element, provided that one of these children is actually marked for refinement. This property is illustrated in Figure 2. For the higher order reconstruction of the Lagrange multipliers, we use a patchwise linear interpolation $i_{2H}^{(1)}$, it is illustrated in Figure 3. We define $\mathcal{A}^I\left((v_h, \mu_{n,H}, \mu_{t,H})\right) := \left(i_{2h}^{(2)} v_h, i_{2H}^{(1)} \mu_{n,H}, i_{2H}^{(1)} \mu_{t,H}\right)$ and obtain the error estimators

$$
\begin{aligned}
\eta_p &:= \rho\left(w_h\right)\left(\mathcal{A}^I\left(z_h\right) - z_h\right) - C\left(w_h\right)\left(\xi_{n,h}\right) - D\left(w_h\right)\left(\xi_{t,h}\right), \\
\eta &:= \frac{1}{2}\rho\left(w_h\right)\left(\mathcal{A}^I\left(z_h\right) - z_h\right) + \frac{1}{2}\rho^*\left(w_h, z_h\right)\left(\mathcal{A}^I\left(w_h\right) - w_h\right) \\
&\quad -C\left(w_h\right)\left(\xi_{n,h}\right) - D\left(w_h\right)\left(\xi_{t,h}\right).
\end{aligned}
$$

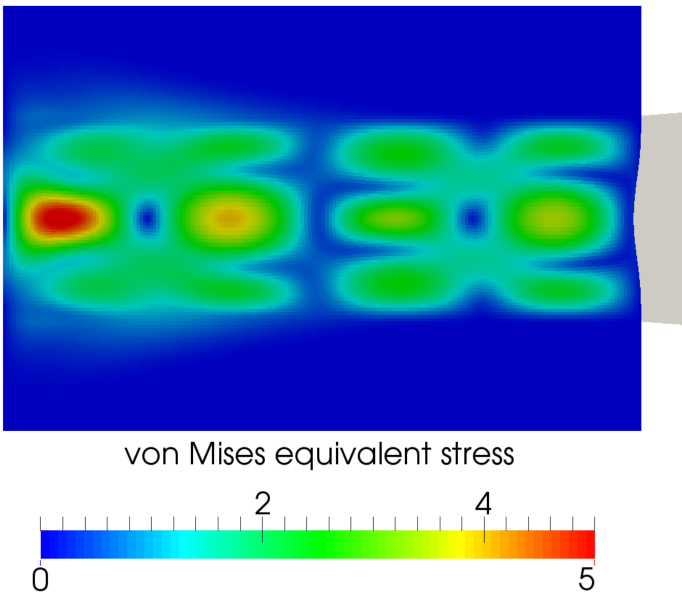
To utilize the error estimators η_p and η in an adaptive refinement strategy, we have to localize the error contributions given by the residuals with respect to the single mesh elements $T \in \mathcal{T}_h$ leading to local error indicators η_T . Here, the filtering technique developed in [13] is applied, which implies less implementational effort than the standard approach using integration by parts outlined for instance in [3]. An alternative localization method was recently proposed in [45]. The terms connected to C and D are added to the adjacent volume cells to the boundary cells.

5 Numerical results

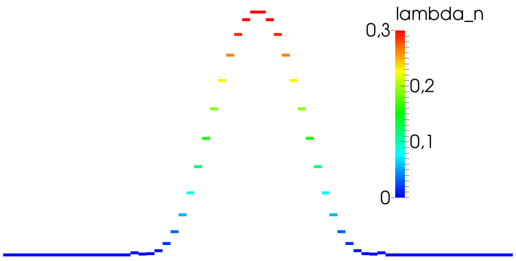
This section is devoted to numerical tests of the presented error estimator. At first, we consider an example with known analytical solution in order to check the accuracy. Afterwards, a more complex example is presented, where we apply a model adaptive algorithm. For results concerning 3D examples from sheet-bulk-metal-forming, we refer to [6, 46].

5.1 First example: Known analytical solution

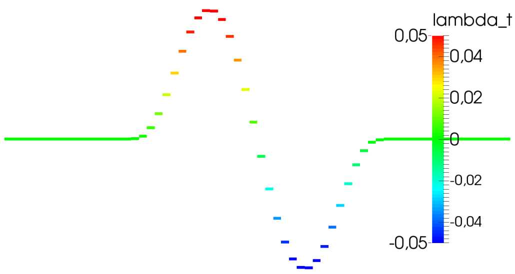
At first, we consider a 2D Signorini problem with Tresca friction, whose analytical solution is known. It is a modified version of an example used in [42, 43]. Let $\Omega := (-3,0)\times(-1,1)$ be the domain. We prescribe homogeneous Dirichlet boundary conditions on $\Gamma_D := \{-3\} \times [-1,1]$ and homogeneous Neumann boundary conditions on $\Gamma_N := (-3,0) \times \{-1,1\}$. The possible contact boundary is denoted by $\Gamma_C := \{0\} \times [-1,1]$. The material law is given by Hooke's law with Young's modulus $E := 10$ and Poisson number $\nu := 0.3$ using the plain strain assumption. By L the number of uniform refinements based on a coarse initial triangulation



(a) Plot of u_h in Ω and the obstacle



(b) $\lambda_{n,H}$



(c) $\lambda_{t,H}$

Fig. 4: Numerical solution of the first example for $M_\Omega = 24576$ and $M_C = 64$

M_Ω	L	$E_{\text{rel}}(J_{a,1})$	$I_{\text{eff}}(J_{a,1},\eta_p)$	$I_{\text{eff}}(J_{a,1},\eta)$
384	0	$1.28466 \cdot 10^{-2}$	-4.5051	-8.4898
1536	1	$1.28322 \cdot 10^{-2}$	1.25195	1.31199
6144	2	$3.48578 \cdot 10^{-3}$	1.01771	1.02354
24576	3	$8.89653 \cdot 10^{-4}$	1.00401	1.00517
98304	4	$2.22857 \cdot 10^{-4}$	1.00120	1.00179
393216	5	$5.57355 \cdot 10^{-5}$	1.00042	1.00068
1572864	6	$1.39339 \cdot 10^{-5}$	1.00017	1.00029

Tab. 2: Results of the presented error estimators for $J_{a,1}$

M_Ω	L	$E_{\text{rel}}(J_{a,2})$	$I_{\text{eff}}(J_{a,2},\eta_p)$	$I_{\text{eff}}(J_{a,2},\eta)$
384	0	$6.09378 \cdot 10^{-1}$	0.48802	0.58637
1536	1	$-6.2106 \cdot 10^{-2}$	-0.1867	-0.5573
6144	2	$8.29963 \cdot 10^{-3}$	0.19384	0.54481
24576	3	$4.43956 \cdot 10^{-3}$	0.44977	0.95398
98304	4	$1.09222 \cdot 10^{-3}$	0.45098	0.98520
393216	5	$2.71978 \cdot 10^{-4}$	0.45124	0.99497
1572864	6	$6.79232 \cdot 10^{-5}$	0.45128	0.99814

Tab. 3: Results of the presented error estimators for $J_{a,2}$

is denoted. The analytical solution is called $u(x,y) := (u_1(x,y), u_2(x,y))^\top$, where

$$\begin{aligned} u_1(x,y) &:= \begin{cases} -(x+3)^2(y - \frac{x^2}{18} - \frac{1}{2})^4(y + \frac{x^2}{18} + \frac{1}{2})^4, & |y| < \frac{x^2}{18} + \frac{1}{2}, \\ 0, & \text{else}, \end{cases} \\ u_2(x,y) &:= \begin{cases} \frac{24}{\pi} \sin\left(\frac{4\pi(x+3)}{3}\right) \left[(y-\frac{1}{2})^3(y+\frac{1}{2})^4 + (y-\frac{1}{2})^4(y+\frac{1}{2})^3\right], & |y| < \frac{1}{2}, \\ 0, & \text{else}. \end{cases} \end{aligned}$$

The volume force is then given by $f := -\text{div}(\sigma(u))$ and the obstacle by $g(y) := u_1(0,y)$. The friction law is Tresca with $s = 0.1$. The discrete solution w_h is illustrated in Figure 4.

We consider the quantities of interest

$$\begin{aligned} J_{a,1}(u) &:= \int_\Omega \omega(x)\,|u|^2\,dx, \\ J_{a,2}(\lambda_n) &:= \int_{-1}^{1} \left(0.5\tanh\left(20(0.25 - |y - 0.125|)\right) + 0.5\right) \lambda_t^2(y)\,dy, \end{aligned}$$

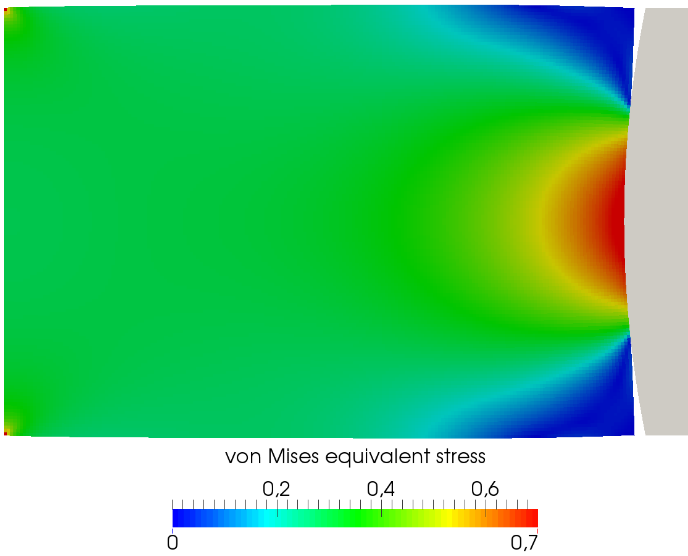
where $\omega(x) = 0.5\left(\tanh(20(d - |x - (-0.5,0)\,|)) + 1\right)$ is a cut off function w.r.t. the disc $B_{0.5}\left((-0.5,0)\right)$. The relative discretization error w.r.t. the quantity of interest is given by

$$E_{\text{rel}}(J) := \frac{J(u,\lambda_n) - J(u_h,\lambda_{n,H})}{J(u,\lambda_n)},$$

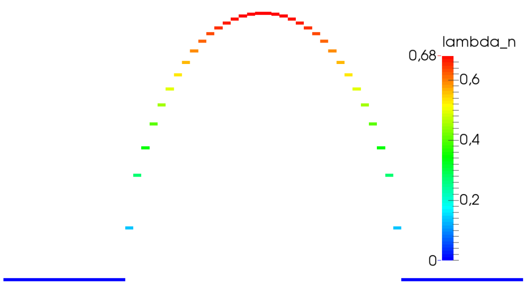
and the effectivity index by

$$I_{\text{eff}}(J,\tilde{\eta}) := \frac{J(u,\lambda_n) - J(u_h,\lambda_{n,H})}{\tilde{\eta}}.$$

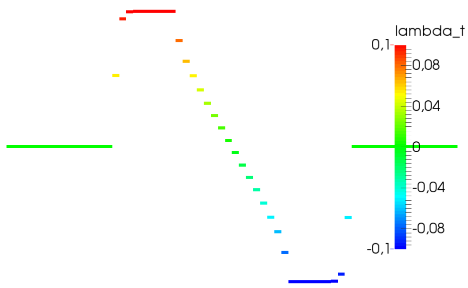
In Table 2, the results for the quantity of interest $J_{a,1}$ are listed. We found by analyzing the data that the effectivity indices seem to converge of order h^2 to 1 for η_p and η, which is almost optimal. When regarding $J_{a,2}$, see Table 3, we observe an almost constant effectivity index



(a) Plot of u_h in Ω and the obstacle



(b) $\lambda_{n,H}$



(c) $\lambda_{t,H}$

Fig. 5: Numerical solution of the second example for $M_\Omega = 24576$ and $M_C = 64$

of 0.45 for η_p. However, the effectivity index is very good. In contrast to η_p, the effectivity index for η seems to converges to 1 with order h^2. From the numerical experiments in [43], we know that $i_{2H}^{(1)} \lambda_{n,H}$ is not of higher order in the integral over Γ_C. But, in this approach, the contribution of the terms involving $i_{2H}^{(1)}$ is so small that we could not observe this behavior on the considered meshes. Consequently, we obtain an accurate but not asymptotically exact error estimator. It is one advantage of this approach that it is sufficient to work with the higher order reconstruction to obtain reasonable results.

5.2 Second example: Adaptivity

In the last section, we have examined the accuracy of the error estimator. Now, we address the adaptive techniques. We use the same domain, subdivision of the boundary, and material law as above. The volume force is set to zero. The gap function is given by $0.1\,(y-1)\,(y+1)$. We choose the friction law of Betten s_3 with the paramters $C_T = 0.1$, $\mathcal{F} = 0.4$, and $n = 3$, cf. Table 1. The solution is illustrated in Figure 5, where we show the von Mises equivalent stress

$$\sigma_{M,2}(\sigma, \sigma_e) := \frac{\sqrt{\sigma_{11}^2 + \sigma_{22}^2 + 3\sigma_{21}^2}}{\sigma_e}$$

with $\sigma_e = 1$. The regularity of the problem is distorted by three different sources: We observe stress peaks in the left corners of the domain, where the Dirichlet boundary conditions change

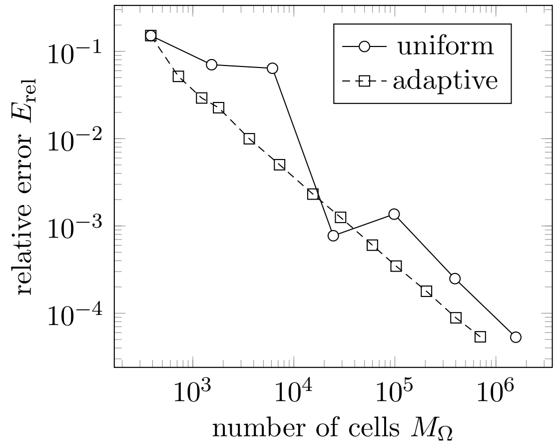


Fig. 6: Comparison of adaptive and uniform refinement for the quantity of interest J_1

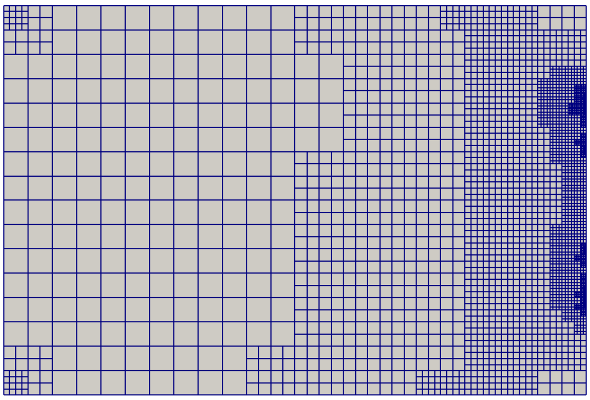
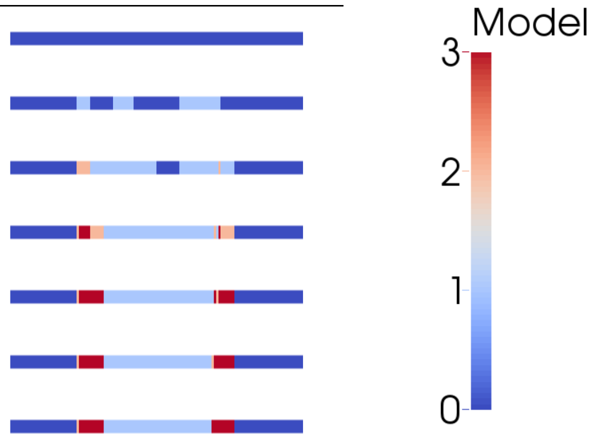


Fig. 7: Adaptive mesh in the 7^{th} iteration for the quantity of interest J_1

M_Ω	L	E_{rel}	I_{eff}
384	1	$1.51570 \cdot 10^{-1}$	-0.4748
456	2	$-2.5789 \cdot 10^{-2}$	0.10037
564	3	$-1.6315 \cdot 10^{-2}$	0.11956
720	4	$5.16983 \cdot 10^{-2}$	-1.2446
1224	5	$2.29215 \cdot 10^{-2}$	-4.3496
1800	6	$2.26209 \cdot 10^{-2}$	7.17555
3624	7	$9.97274 \cdot 10^{-3}$	3.98811
7224	8	$5.02568 \cdot 10^{-3}$	1.50262
15432	9	$2.31588 \cdot 10^{-3}$	1.72271
29016	10	$1.25329 \cdot 10^{-3}$	1.49387
59868	11	$6.02395 \cdot 10^{-4}$	1.63777
102840	12	$3.46647 \cdot 10^{-4}$	1.44213
205392	13	$1.78526 \cdot 10^{-4}$	1.44686
401856	14	$8.86439 \cdot 10^{-5}$	1.48696
699960	15	$5.36923 \cdot 10^{-5}$	1.46484

Tab. 4: Detailed results of the adaptive algorithm for the second example

L	E_{rel}	I_{eff}	Model
1	$1.00000 \cdot 10^{0}$	31.8617	
2	$-9.4015 \cdot 10^{-1}$	−0.7837	
3	$1.51606 \cdot 10^{-1}$	0.691113	
4	$4.50757 \cdot 10^{-1}$	−0.0350	
5	$6.50010 \cdot 10^{-2}$	−0.5200	
6	$8.74934 \cdot 10^{-2}$	−0.6447	
7	$7.10923 \cdot 10^{-8}$	−1.5836	



Tab. 5: Results of the model adaptive algorithm for $M_\Omega = 98304$

to Neumann boundary conditions. Furhtermore, the transition zones between contact to non-contact as well as between sticking and sliding are problematic.

In the first step, we test the mesh adaptive algorithm. We consider the quantity of interest

$$J_1(u, \lambda_t) = \int_{\Gamma_C} \lambda_t u_t \, do,$$

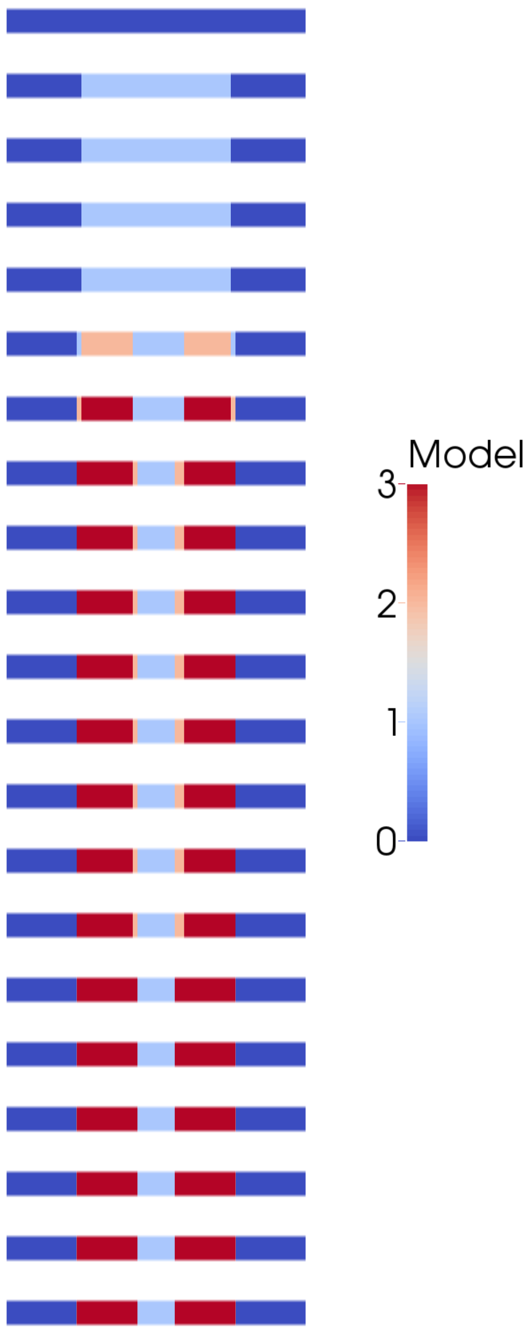
which corresponds to the dissipated energy in this example. We solve this problem based on a uniform mesh refinement and obtain a reference value $J_{1,\text{ref}} = 3.9999331493 \cdot 10^{-5}$ by extrapolation over all calculated values of J_1. We use $J_{1,\text{ref}}$ to determine the relative error E_{rel} and the effectivity index I_{eff} approximately. The error on the different meshes is plotted in Figure 6, where we observe a change of the sign of the error between the 4^{th} and 5^{th} iteration. We compare the uniform refinement with an adaptive algorithm based on η and an optimal mesh strategy, see [44]. We find a better convergence behavior of the adaptive algorithm. The adaptive mesh is outlined in Figure 7, where the left corners of the domain and the transition zones between contact to non-contact as well as between sticking and sliding are well resolved as expected. The efficiency indices of the error estimator are listed in Table 4 and are around 1.4.

5.3 Model adaptivity

We consider a model adaptive algorithm in this section. We test it with the example of the last section for a uniform mesh of $M_\Omega = 98304$ elements. In the initial configuration, we assume no friction on the complete contact boundary, i.e. $s \equiv s_0$ on Γ_C. We solve the problem and estimate the model error by the estimator η_m. Afterwars, we choose in the cells with the largest error a better model, i.e. increase the model index by 1. Here, a fraction of 25% is used. The results are outlined in detail in Table 5. We obtain in the middle of the contact zone Tresca friction and on the boundary of the contact zone the model of Betten. This correponds to the expactations, since λ_n is large in the middle of the contact zone and small at the boundary.

In a second step, we combine the model adaptive algorithm with the mesh adaptive one. Here, we use an equilibration strategy. If $|\eta_m| \geq C_e |\eta|$ with an equilibration constant $C_e \geq 1$, we conduct a model adaptive step with a refinement fraction of 50%. If $|\eta| \geq C_e |\eta_m|$, the mesh is adaptively refined. If $C_e |\eta| \geq |\eta_m| \geq C_e^{-1} |\eta|$, we improve the model first and adaptively refine the mesh afterwards. The detailed results of the algorithm for the example of the

L	M_Ω	E_{rel}	I_{eff}	Model
1	384	$1.00000 \cdot 10^{0}$	-12.334	
2	384	$1.51579 \cdot 10^{-1}$	0.54735	
3	480	$-3.1565 \cdot 10^{-2}$	-0.2684	
4	660	$-4.0661 \cdot 10^{-2}$	-3.4936	
5	876	$-1.0422 \cdot 10^{-2}$	-0.2474	
6	876	$6.9472 \cdot 10^{-1}$	-0.0742	
7	876	$4.01757 \cdot 10^{-2}$	-5.3661	
8	876	$4.00031 \cdot 10^{-2}$	-5.7265	
9	1308	$2.82515 \cdot 10^{-2}$	-31.910	
10	1908	$1.97301 \cdot 10^{-2}$	2.47728	
11	4116	$8.48886 \cdot 10^{-3}$	2.62066	
12	7116	$4.80574 \cdot 10^{-3}$	1.08973	
13	14112	$2.27704 \cdot 10^{-3}$	1.05157	
14	26736	$9.95986 \cdot 10^{-4}$	0.83356	
15	61104	$2.32142 \cdot 10^{-4}$	0.47734	
16	61104	$4.50727 \cdot 10^{-3}$	1.01859	
17	94260	$4.32043 \cdot 10^{-3}$	0.97587	
18	94260	$3.83807 \cdot 10^{-4}$	1.18358	
19	230868	$1.52591 \cdot 10^{-4}$	1.24280	
20	433512	$8.27891 \cdot 10^{-5}$	1.17654	
21	943560	$3.80197 \cdot 10^{-5}$	1.25396	



Tab. 6: Results of the mesh and model adaptive algorithm for the quantity of interest J_1

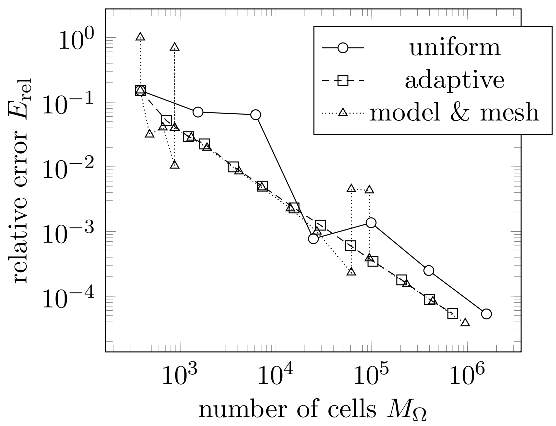


Fig. 8: Comparison of mesh adaptive, model and mesh adaptive, as well as uniform refinement for the quantity of interest J_1

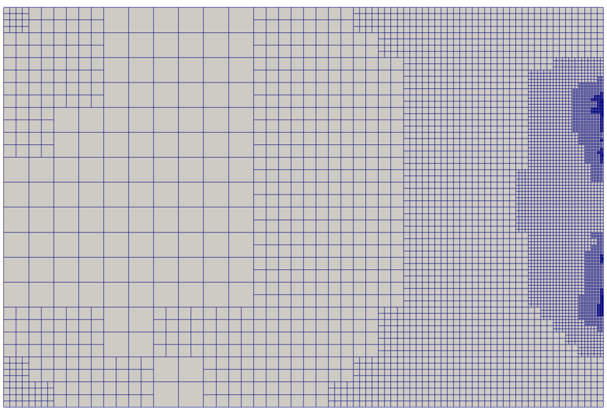
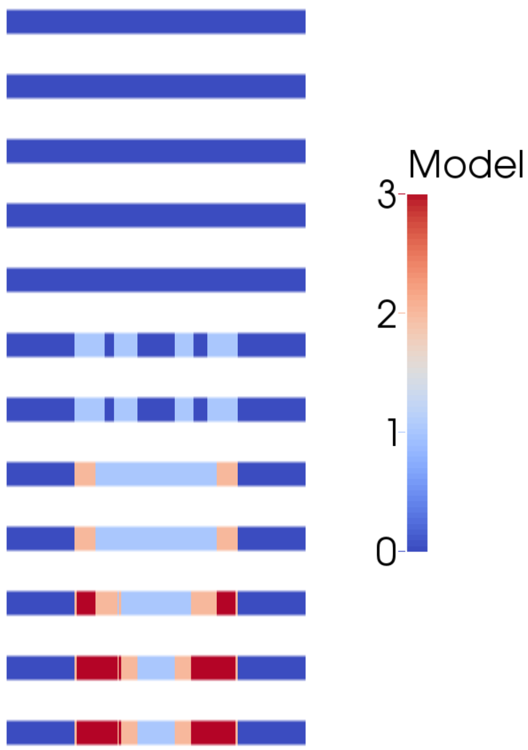


Fig. 9: Adaptive mesh in the 12th iteration of the model and mesh adaptive algorithm

L	M_Ω	E_{rel}	I_{eff}	Model
1	384	$-6.5283 \cdot 10^{-4}$	0.05562	
2	480	$4.15588 \cdot 10^{-3}$	3.88737	
3	552	$1.10060 \cdot 10^{-3}$	-1.4788	
4	1032	$3.80930 \cdot 10^{-4}$	4.00148	
5	3432	$-1.4927 \cdot 10^{-4}$	3.91706	
6	3432	$-9.5859 \cdot 10^{-4}$	3.28939	
7	8160	$-9.8240 \cdot 10^{-4}$	3.42586	
8	8160	$-1.6305 \cdot 10^{-4}$	0.72628	
9	30936	$-1.4366 \cdot 10^{-4}$	0.79535	
10	30936	$-2.8915 \cdot 10^{-5}$	-0.0136	
11	30936	$6.27223 \cdot 10^{-6}$	-1.0887	
12	119520	$1.81614 \cdot 10^{-6}$	0.67998	



Tab. 7: Results of the mesh and model adaptive algorithm for the quantity of interest J_2

last section are given in Table 6. We observe the same model distribution as for the model adaptive algorithm. In the first iterations, the model is roughly chosen and afterwards only small corrections at the boundary of the contact zone are conducted. We compare the mesh adaptive, the mesh and model adaptive, and the uniform approach in Figure 8. We see that the mesh as well as the mesh and model adaptive algorithm lead finally to similar results with a better accuracy than the uniform approach. This observation is substantiate by the comparison of the generated adaptive meshes, cf. Figure 7 and 9. They only show small deviations.

The quantity of interest J_1 is focused on the frictional forces and the tangential displacement on the contact boundary. Thus it is located on the contact boundary. To test a completely different setting, we consider the quantity of interest

$$J_2(u) := \int_\Omega \bar{\omega}(x)\, |u|^2 \, dx,$$

where $\bar{\omega}(x) = 0.5\,(\tanh(20(d - |x - (-2.5, 0)\,|)) + 1)$. Here, J_2 is located at the left end of Ω. In Table 7, the results of the model and mesh adpative algorithm are listed. In contrast to the results concerning J_1, the model is changed later and Coulomb's model is used more frequently. The adaptive mesh in the 9^{th} iteration is depicted in Figure 11. Here, more refinements in the middle and in the left corners of the domain are found. The results of the three different refinements approaches are compared in Figure 10. The mesh and model adaptive algorithm performs best.

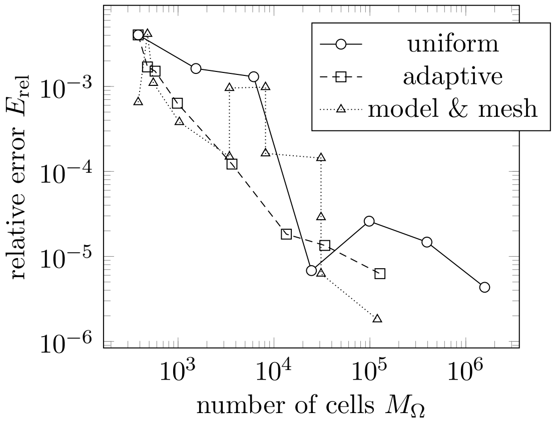


Fig. 10: Comparison of mesh adaptive, model and mesh adaptive, as well as uniform refinement for the quantity of interest J_2

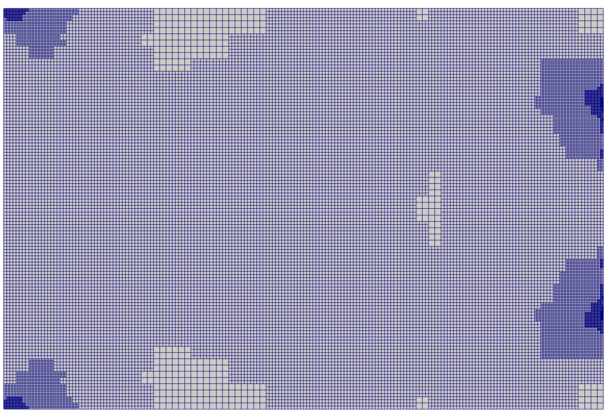


Fig. 11: Adaptive mesh in the 9^{th} iteration of the model and mesh adaptive algorithm for the quantity of interest J_2

6 Conclusions and outlook

We have derived goal oriented a posteriori error estimates with respect to the discretization as well as model error for discretizations of frictional contact problems in this article. The presented approach leads to an accurate estimates even using higher order reconstruction, although it is not asymptotically exact. Furthermore, it is based on a linear dual problem and directly measures the error in the frictional contact conditions, which is necessary for the estimation of the model error. However, it is not clear, whether the remainder terms are of higher order or not. Numerical results substantiate the assumption that they are of higher order. However, a precise analysis is a topic of further research. A further content is the extension to dynamic contact problems. Especially here, the precise consideration of the error in the contact conditions is needed to accurately resolve impact phenomena.

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