Numerical solution of surface PDEs with Radial Basis Functions

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Localized Kernel-Based Meshless Methods for Partial Differential Equations

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Outline

1 Motivation

2 RBF-FD for surface-PDEs of reaction-diffusion-convection type

3 Outlook
Motivation: chemotaxis on a membrane

Figure: The membrane-cytoplasmic shuttling of Cdc42 (inactive form, blue; active, pink). Taken from [1].

System of ODEs

Figure: A 3D view on the surface of a yeast cell shows the distribution of the activated Cdc42. Taken from [1].

Motivation: chemotaxis in a porous media


\[ \partial_t u = \nabla_{\Gamma} \cdot (D_u \nabla_{\Gamma} u - \chi u \nabla_{\Gamma} \nu) \quad \text{on} \quad \Gamma, \]
\[ \partial_t \nu = \nabla \cdot (D_{\nu} \nabla \nu) + g \quad \text{in} \quad \Omega_0, \]

\( u \) is the density of the hematopoietic cells
\( \nu \) is a stromal cell-derived factor 1\( \alpha \).

Figure: Scanning electron micrographs of longitudinal sections of a porous mineralized collagen scaffold, seeded with osteoblast-like cells.
Motivation: $\Gamma$-applications for chemotaxis models

Charles M. Elliott, Björn Stinner and Chandrasekhar Venkataraman

Figure: Migration of cells.
Framework

PDEs in a domain

PDEs on a surface

evolution of a surface in time

coupled

coupled

coupled
\[ \frac{\partial^* \rho}{\partial t} + \nabla_{\Gamma(t)} \cdot (w \rho) = D \Delta_{\Gamma(t)} \rho + s(\cdot, \rho), \text{ on } \Gamma(t) \times T, \]
Framework

\[ \frac{\partial^* \rho}{\partial t} + \nabla \Gamma(t) \cdot (w \rho) = D \Delta \Gamma(t) \rho + s(\cdot, \rho), \quad \text{on } \Gamma(t) \times T, \]

where

\[ \frac{\partial^* \rho}{\partial t} = \partial_t \rho + v \cdot \nabla \rho + \rho \nabla \Gamma(t) \cdot v \]

and

\[ v = V n + v_S \]

is the velocity of the surface \( \Gamma(t) \).
\[
\frac{\partial^* \rho}{\partial t} + \nabla_{\Gamma(t)} \cdot (w \rho) = D\Delta_{\Gamma(t)} \rho + s(\cdot, \rho), \text{ on } \Gamma(t) \times T,
\]

where

\[
\partial^* \rho = \partial_t \rho + v \cdot \nabla \rho + \rho \nabla_{\Gamma(t)} \cdot v
\]

and

\[
v = V n + v_S
\]

is the velocity of the surface \( \Gamma(t) \).

analytical prescription of \( \Gamma = \Gamma(t) \).
\[ \Omega = \Omega_{\text{in}} \cup \Omega_{\text{out}} \]
Surface PDE (evolving $\Gamma$)

$$
\partial_t \rho + v \cdot \nabla \rho + \rho \nabla_{\Gamma(t)} \cdot v + \nabla_{\Gamma(t)} \cdot (w \rho) = D \Delta_{\Gamma(t)} \rho + s(\cdot, \rho)
$$
Surface PDE (evolving $\Gamma$)

\[
\partial_t \rho + v \cdot \nabla \rho + \rho \nabla_{\Gamma(t)} \cdot v + \nabla_{\Gamma(t)} \cdot (w \rho) = D\Delta_{\Gamma(t)} \rho + s(\cdot, \rho)
\]

The level-set function:

\[
\phi(x) = \begin{cases} 
< 0 & \text{if } x \text{ is inside } \Gamma \\
0 & \text{if } x \in \Gamma \\
> 0 & \text{if } x \text{ is outside } \Gamma
\end{cases}
\]
Surface PDE (evolving $\Gamma$)

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla \Gamma(t) \cdot \mathbf{v} + \nabla \Gamma(t) \cdot (\mathbf{w} \rho) = D \Delta \Gamma(t) \rho + s(\cdot, \rho)$$

The level-set function:

$$\phi(x) = \begin{cases} < 0 & \text{if } x \text{ is inside } \Gamma \\ = 0 & \text{if } x \in \Gamma \\ > 0 & \text{if } x \text{ is outside } \Gamma \end{cases}$$

Then

$$P_\Gamma = I - \frac{\nabla \phi}{\| \nabla \phi \|} \otimes \frac{\nabla \phi}{\| \nabla \phi \|}$$

is a projection onto $T_x \Gamma$.

If $\phi$ is a signed distance, then $|\nabla \phi| = 1$. 
\[ \partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla_{\Gamma(t)} \cdot \mathbf{v} + \nabla_{\Gamma(t)} \cdot (\mathbf{w} \rho) = D \Delta_{\Gamma(t)} \rho + s(\cdot, \rho) \]

The level-set function:

\[ \phi(x) = \begin{cases} < 0 & \text{if } x \text{ is inside } \Gamma \\ = 0 & \text{if } x \in \Gamma \\ > 0 & \text{if } x \text{ is outside } \Gamma \end{cases} \]

Then

\[ P_\Gamma = I - \frac{\nabla \phi}{\| \nabla \phi \|} \otimes \frac{\nabla \phi}{\| \nabla \phi \|} \] is a projection onto \( \mathcal{T}_x \Gamma \).

If \( \phi \) is a signed distance, then \( |\nabla \phi| = 1 \).

\[ \Gamma_c(t) = \{ \mathbf{x} : \phi(t, \mathbf{x}) = c \} \]
Following Wright et al. 2012, 2014:

\[
\nabla \Gamma(t) \rho = \left( \begin{array}{c}
\mathbf{e}^x - n^x \mathbf{n} \\
\mathbf{e}^y - n^y \mathbf{n} \\
\mathbf{e}^z - n^z \mathbf{n}
\end{array} \right) \cdot \nabla \rho = \left( \begin{array}{c}
p^x \cdot \nabla \\
p^y \cdot \nabla \\
p^z \cdot \nabla
\end{array} \right) \rho = \left( \begin{array}{c}
G^x \\
G^y \\
G^z
\end{array} \right) \rho
\]

\[
(G^x I_\varphi \rho(x)) |_{x=x_i} = \sum_{j=1}^{N} c_j (G^x \varphi(r_j(x))) |_{x=x_i}
\]

\[
= \sum_{j=1}^{N} c_j \left[ ((1 - n^x_i n^x_i)(x_i - x_j) - n^y_i n^y_i (y_i - y_j) - n^z_i n^z_i (z_i - z_j)) \frac{\varphi'(r_j(x_i))}{r_j(x_i)} \right],
\]

where \( \varphi \) is a radial basis function.
Operator assembly: Laplace-Beltrami

\[
\partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla \Gamma(t) \cdot \mathbf{v} + \nabla \Gamma(t) \cdot (w \rho) = D \Delta \Gamma(t) \rho + s(\cdot, \rho)
\]

where \(\Xi = \{\varsigma, \varsigma_2, \varsigma_3, \ldots, \varsigma_K\}\) and \(\Sigma = \{\varsigma, \eta_2, \eta_3, \ldots, \eta_K\}\).
\[
\partial_t \rho + \nu \cdot \nabla \rho + \rho \nabla \Gamma(t) \cdot \nu + \nabla \Gamma(t) \cdot (w \rho) = D \Delta \Gamma(t) \rho + s(\cdot, \rho)
\]

\[
L(t, \Gamma(t)) \rho_h = (P_{\Gamma} \nabla_h \cdot (P_{\Gamma} \nabla_h) \rho_h
\]
Operator assembly: Laplace-Beltrami

\[
\partial_t \rho + \underbrace{\mathbf{v} \cdot \nabla \rho}_{\approx V(t, \Gamma(t)) \rho_h} + \underbrace{\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}}_{\approx G(t, \Gamma(t)) \rho_h} + \underbrace{\nabla_{\Gamma(t)} \cdot (\mathbf{w} \rho)}_{\approx K(t, \mathbf{w}, \Gamma(t)) \rho_h} = \underbrace{D \Delta_{\Gamma(t)} \rho + s(\cdot, \rho)}_{\approx L(t, \Gamma(t)) \rho_h}
\]

\[
L(t, \Gamma(t)) \rho_h = (P_{\Gamma} \nabla_h \cdot P_{\Gamma} \nabla_h) \rho_h
\]

where \( \Xi = \{\zeta, \zeta_2, \zeta_3, \ldots, \zeta_K\} \) and \( \Sigma = \{\zeta, \eta_2, \eta_3, \ldots, \eta_K\} \).
Operator assembly: convection

\[
\partial_t \rho + \nabla \cdot \rho \nabla \rho \approx \rho \nabla \Gamma(t) \cdot v + \nabla \Gamma(t) \cdot (w \rho) = D \Delta \Gamma(t) \rho + s(\cdot, \rho)
\]

\[
V(t, \Gamma(t)) \rho_h = (v \cdot \nabla_h) \rho_h
\]
Operator assembly: convection

\[
\partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla \Gamma(t) \cdot \mathbf{v} + \nabla \Gamma(t) \cdot (\mathbf{w} \rho) = D \Delta \Gamma(t) \rho + s(\cdot, \rho)
\]

\[
V(t, \Gamma(t)) \rho_h = (\mathbf{v} \cdot \nabla_h) \rho_h
\]

\[
(\mathbf{v} \cdot \nabla_h \rho_h)_i = \sum_{j \in \Xi} \sum_{p=1}^d v^p(\zeta_j) \omega_j \partial_p \rho_h(\zeta_j).
\]
Operator assembly: convection

\[ \partial_t \rho + v \cdot \nabla \rho \approx V(t, \Gamma(t)) \rho_h \]

\[ + \rho \nabla \Gamma(t) \cdot v + \nabla \Gamma(t) \cdot (w \rho) = D \Delta \rho + s(\cdot, \rho) \approx G(t, \Gamma(t)) \rho_h \]

\[ \approx K(t, w, \Gamma(t)) \rho_h \approx L(t, \Gamma(t)) \rho_h \]

\[ V(t, \Gamma(t)) \rho_h = (v \cdot \nabla \rho_h) \]

\[ (v \cdot \nabla \rho_h)_i = \sum_{j \in \Xi} \sum_{p=1}^{d} v^p(\zeta_j) \omega_j^p \rho_h(\zeta_j). \]

The operator \( K(t, w, \Gamma(t)) \approx w \cdot \nabla \Gamma(t) \rho \) is assembled in a similar way.
Operator assembly: \( \rho \nabla \Gamma(t) \cdot \nu \)

\[
\partial_t \rho + \nu \cdot \nabla \rho + \rho \nabla \Gamma(t) \cdot \nu + \nabla \Gamma(t) \cdot (w \rho) = D \Delta \Gamma(t) \rho + s(\cdot, \rho)
\]

\[
G(t, \Gamma(t)) \rho_h = \rho_h (P \nabla \cdot \nu_h)
\]
Operator assembly: \( \rho \nabla_{\Gamma(t)} \cdot \mathbf{v} \)

\[
\partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla_{\Gamma(t)} \cdot \mathbf{v} + \nabla_{\Gamma(t)} \cdot (w \rho) = D \Delta_{\Gamma(t)} \rho + s(\cdot, \rho)
\]

\[
G(t, \Gamma(t)) \rho_h = \rho_h (P_{\Gamma} \nabla_h \cdot \mathbf{v}_h)
\]

\[
(\rho_h P_{\Gamma} \nabla_h \cdot \mathbf{v}_h)_i = (\rho_h)_i (P_{\Gamma} \nabla_h \cdot \mathbf{v}_h)_i.
\]
Operator assembly: $\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}$

\[
\partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla_{\Gamma(t)} \cdot \mathbf{v} + \nabla_{\Gamma(t)} \cdot (w \rho) \approx V(t, \Gamma(t)) \rho_h + G(t, \Gamma(t)) \rho_h + D \Delta_{\Gamma(t)} \rho + s(\cdot, \rho) \\
\approx K(t, w, \Gamma(t)) \rho_h
\]

\[
G(t, \Gamma(t)) \rho_h = \rho_h (P_{\Gamma} \nabla_h \cdot \mathbf{v}_h)
\]

\[
(\rho_h P_{\Gamma} \nabla_h \cdot \mathbf{v}_h)_i = (\rho_h)_i (P_{\Gamma} \nabla_h \cdot \mathbf{v}_h)_i.
\]

This discrete operator leads to a diagonal matrix.
Surface PDE (evolving $\Gamma$): scheme

$$\partial_t \rho + \underbrace{v \cdot \nabla \rho}_{\approx V(t, \Gamma(t)) \rho_h} + \underbrace{\rho \nabla_{\Gamma(t)} \cdot v}_{\approx G(t, \Gamma(t)) \rho_h} + \underbrace{\nabla_{\Gamma(t)} \cdot (w \rho)}_{\approx K(t, w, \Gamma(t)) \rho_h} = \underbrace{D \Delta_{\Gamma(t)} \rho}_{\approx L(t, \Gamma(t)) \rho_h} + s(\cdot, \rho)$$
Surface PDE (evolving $\Gamma$): scheme

\[
\partial_t \rho + \underbrace{\mathbf{v} \cdot \nabla \rho}_{\approx V(t, \Gamma(t))\rho_h} + \underbrace{\rho \nabla \Gamma(t) \cdot \mathbf{v}}_{\approx G(t, \Gamma(t))\rho_h} + \underbrace{\nabla \Gamma(t) \cdot (\mathbf{w} \rho)}_{\approx K(t, w, \Gamma(t))\rho_h} = D \Delta \Gamma(t) \rho + s(\cdot, \rho)
\]

Given $\rho_h^n$ and $\Delta t = t_{n+1} - t_n$, solve for $\rho_h^{n+1}$

\[
\frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} + \theta \left( V^{n+1} + G^{n+1} + K^{n+1} - L^{n+1} \right) \rho_h^{n+1} = -(1 - \theta) \left( V^n + G^n + K^n - L^n \right) \rho_h^n + \theta s^{n+1} + (1 - \theta) s^n.
\]
Surface PDE (evolving $\Gamma$): scheme

\[ \partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla \Gamma(t) \cdot \mathbf{v} + \nabla \Gamma(t) \cdot (\mathbf{w} \rho) = D \Delta \Gamma(t) \rho + s(\cdot, \rho) \]

Given $\rho_h^n$ and $\Delta t = t_{n+1} - t_n$, solve for $\rho_h^{n+1}$

\[ \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} + \theta \left( V^{n+1} + G^{n+1} + K^{n+1} - L^{n+1} \right) \rho_h^{n+1} = -(1 - \theta) \left( V^n + G^n + K^n - L^n \right) \rho_h^n + \theta s^{n+1} + (1 - \theta) s^n. \]

\[ \theta = 1 \text{ -- Implicit - Euler} \]
\[ \theta = \frac{1}{2} \text{ -- Crank - Nicolson} \]
Numerical tests: example 1

Solve
\[ \frac{\partial^* \rho(x, t)}{\partial t} = D \Delta_{\Gamma(t)} \rho(x, t) + f(x, t) \] on \( \Gamma(t) \),

where \( \Gamma(t) \) is the zero level-set of

\[ \phi(x, t) = |x| - 1.0 + \sin(4t)(|x| - 0.5)(1.5 - |x|). \]

Analytical solution is

\[ \rho(x, t) = e^{-t/|x|^2} \frac{x_1}{|x|}. \]
Numerical tests: example 1

Solve

\[ \frac{\partial^* \rho(x, t)}{\partial t} = D \Delta_{\Gamma(t)} \rho(x, t) + f(x, t) \text{ on } \Gamma(t), \]

where \( \Gamma(t) \) is the zero level-set of

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\[
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\]

where \(\Gamma(t)\) is a zero level-set of

\[
\phi(\boldsymbol{x}, t) = |\boldsymbol{x}| - 1.0 + \sin(4t)(|\boldsymbol{x}| - 0.5)(1.5 - |\boldsymbol{x}|).
\]

Analytical solution is

\[
\rho(\boldsymbol{x}, t) = e^{-t/|\boldsymbol{x}|^2} \frac{x_1}{|\boldsymbol{x}|}.
\]

Choose the time interval \([0, T = 0.1]\),

\(\Delta t \approx h^2\) (for IE) and \(\Delta t \approx h\) (for CN).
Solve

\[ \frac{\partial^* \rho(x, t)}{\partial t} = D \Delta_{\Gamma(t)} \rho(x, t) + f(x, t) \quad \text{on} \quad \Gamma(t), \]

where \( \Gamma(t) \) is a zero level-set of

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\[ \rho(x, t) = e^{-t/|x|^2} \frac{x_1}{|x|}. \]

Choose the time interval \([0, T = 0.1], \)

\( \Delta t \approx h^2 \) (for IE) and \( \Delta t \approx h \) (for CN).

\[ l_2(\Omega)-\text{error} = \left( \frac{1}{N} \sum_{i=1}^{N} |u_{\text{analyt}}(x_i, T) - u_{\text{num}}(x_i, T)|^2 \right)^{\frac{1}{2}} \]
Numerical tests: example 1

(a) analyt. solution at level lev 4

(b) num. solution at $T = 0.1$, level 4
### Table: Convergence of the Implicit-Euler and Crank-Nicolson schemes.

<table>
<thead>
<tr>
<th>lev.</th>
<th>d.o.f</th>
<th>num. of time steps</th>
<th>$l_2(\Omega)$-error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td><strong>Implicit-Euler scheme</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>3</td>
<td>0.035854</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>10</td>
<td>0.009567</td>
<td>1.905</td>
</tr>
<tr>
<td>3</td>
<td>360</td>
<td>40</td>
<td>0.002602</td>
<td>1.878</td>
</tr>
<tr>
<td>4</td>
<td>1360</td>
<td>160</td>
<td>0.000748</td>
<td>1.798</td>
</tr>
<tr>
<td>5</td>
<td>5280</td>
<td>640</td>
<td>0.000213</td>
<td>1.812</td>
</tr>
<tr>
<td></td>
<td></td>
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</tr>
<tr>
<td><strong>Crank-Nicolson</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>5</td>
<td>0.040218</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>10</td>
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<td>2.127</td>
</tr>
<tr>
<td>3</td>
<td>360</td>
<td>20</td>
<td>0.002367</td>
<td>1.959</td>
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<tr>
<td>4</td>
<td>1360</td>
<td>40</td>
<td>0.000673</td>
<td>1.814</td>
</tr>
<tr>
<td>5</td>
<td>5280</td>
<td>80</td>
<td>0.000192</td>
<td>1.809</td>
</tr>
</tbody>
</table>
Solve
\[
\frac{\partial^* \rho(x, t)}{\partial t} = D \Delta_{\Gamma(t)} \rho(x, t) + f(x, t) \quad \text{on} \quad \Gamma(t),
\]
where \( \Gamma(t) \) is a zero level-set of
\[
\phi(x, t) = |x| - 1.0.
\]

Initial condition is
\[
\rho_0(x, t) = \sin(4 \gamma), \quad \gamma \in [0, 2\pi].
\]

\[
atan2(x_2, x_1) = \begin{cases} 
\arctan\left(\frac{x_2}{x_1}\right), & x_1 > 0 \\
\arctan\left(\frac{x_2}{x_1}\right) + \pi, & x_2 \geq 0, \ x_1 < 0 \\
\arctan\left(\frac{x_2}{x_1}\right) - \pi, & x_2 < 0, \ x_1 < 0 \\
+\frac{\pi}{2}, & x_2 > 0, \ x_1 = 0
\end{cases}
\]
Numerical tests: example 2

Figure: Solution at various time instances, $\Delta t = 0.0001$
Numerical tests: example 2

(a) at $t = 0.05$

(b) at $t = 0.1$

Figure: Solution at various time instances, $\Delta t = 0.0001$

Vanishing of $\rho_0$ occurs at a rate which depends on the radius of the circle.
Numerical tests: example 3

Solve

\[ \frac{\partial^* \rho(x, t)}{\partial t} = D \Delta_{\Gamma(t)} \rho(x, t) + f(x, t) \quad \text{on } \Gamma(t), \]

where \( \Gamma(t) \) is a zero level-set of

\[ \phi(x, t) = |x| - 1.0. \]

Initial condition is

\[ \rho_0(x, t) = \sin(4 \gamma), \quad \gamma \in [0, 2\pi], \]

and

\[ \mathbf{v} := \mathbf{v}_S = \left( -\phi_{x_2}, \phi_{x_1} \right)^T \frac{1}{|\nabla \phi|}. \]
Numerical tests: example 3

(a) at $t = 0.0$

(b) at $t = 0.002$

Figure: Solution at various time instances, $\Delta t = 0.0001$
Numerical tests: example 3

Figure: Solution at various time instances, $\Delta t = 0.0001$

(a) at $t = 0.05$

(b) at $t = 0.1$
Numerical tests: example 3

Figure: Solution at various time instances, $\Delta t = 0.0001$

(a) at $t = 0.05$

(b) at $t = 0.1$

PDE on a surface which 'evolves' in the tangential direction.
Numerical tests: anisotropic diffusion

**Anisotropic diffusion:**

\[-\nabla (A \nabla u(x)) = f(x) \quad \text{in} \quad \Omega = [0, 1]^2,
\]

where

\[A = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.\]
Anisotropic diffusion:

$$-\nabla (A \nabla u(x)) = f(x) \quad \text{in} \quad \Omega = [0, 1]^2,$$

where

$$A = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.$$

**Figure:** Numerical solution, h=1/20.
Numerical tests: anisotropic diffusion

Anisotropic diffusion:

\[-\nabla \left( A \nabla u(x) \right) = f(x) \quad \text{in} \quad \Omega = [0, 1]^2,\]

where

\[A = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.\]

\[
\begin{array}{c|c|c|c|c}
& E_{l_2} & EOC(l_2) & E_{\text{max}} & EOC(\text{max}) \\
\hline
\text{Stencil}=9, \varepsilon = 10^{-6}, \phi = \pi/6 \\
h=1/5 & 4895 & - & 10783 & - \\
h=1/10 & 1882 & 1.379 & 4025 & 1.421 \\
h=1/20 & 560 & 1.748 & 1150 & 1.807 \\
\hline
h \downarrow & ?? & ?? & ?? & ??
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
& E_{l_2} & EOC(l_2) & E_{\text{max}} & EOC(\text{max}) \\
\hline
\text{Stencil}=25, \varepsilon = 10^{-6}, \phi = \pi/6 \\
h=1/5 & 609 & - & 1529 & - \\
h=1/10 & 47 & 3.695 & 85 & 4.168 \\
h=1/20 & 3 & 3.969 & 12 & 2.824 \\
\hline
h \downarrow & ?? & ?? & ?? & ??
\end{array}
\]

Figure: Numerical solution, h=1/20.
Numerical tests: anisotropic diffusion

**Anisotropic diffusion:**

\[-\nabla (A \nabla u(x)) = f(x) \quad \text{in} \quad \Omega = [0, 1]^2,\]

where

\[
A = \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & \varepsilon
\end{pmatrix}
\begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}.
\]

**Special treatment of the RBF-FD stencil is required!!**

**Figure:** Numerical solution, h=1/20.
Numerical tests: anisotropic diffusion

Anisotropic diffusion:

$$-\nabla (A \nabla u(x)) = f(x) \quad \text{in} \quad \Omega = [0, 1]^2,$$

where

$$A = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.$$

Special treatment of the RBF-FD stencil is required!!


Figure: Numerical solution, $h=1/20$. 
The phase-field method:

\[ \rho_t - \nabla \cdot (\nabla \Gamma \rho(x)) = \rho(x) + f \quad \text{on} \quad \Gamma \subset \mathbb{R}^d, \]
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\[ B(\phi) \rho_t - \nabla \cdot (B(\phi) \nabla \rho(x)) = B(\phi) (\rho(x) + f) \quad \text{in} \quad \Omega_\varepsilon \subset \mathbb{R}^d, \]

where

\[ \phi(x) = \frac{1}{2} \left( 1.0 - \tanh \left( \frac{2}{5h} (|x| - 0.3) \right) \right), \]

and

\[ B(\phi) = 36 \phi^2 (1 - \phi^2). \]

\[ \Gamma = \{ x : |x - (0.5, 0.5)^T| = 0.3 \} \]
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Transport equation (the solid body rotation):

\[ \rho_t + \mathbf{v} \cdot \nabla \rho = 0 \quad \text{in} \quad \Omega = [0, 1]^2. \]
Numerical tests: transport equation

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... or ???
Numerical tests: evolution along a curve

Solve

\[
\frac{\partial^* \rho}{\partial t} + \alpha \rho = D \Delta_{\Gamma(t)} \rho \quad \text{on} \quad \Gamma(t) \subset \Omega = \{ x \in \mathbb{R}^2 : 0.5 \leq |x| \leq 1.5 \}.
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\]

The level set function:

\[
\phi(\mathbf{x}, t) = |\mathbf{x}| - (1.0 + b t \sin(5\gamma)).
\]

The initial condition:

\[
\rho_0 = \begin{cases} 
0.75 & \text{if } 0.65 \leq |\mathbf{x}| \leq 0.85, \\
0.0 & \text{otherwise}.
\end{cases}
\]
Numerical tests: evolution along a curve

Solve

\[ \frac{\partial^* \rho}{\partial t} + \alpha \rho = D \Delta_{\Gamma(t)} \rho \quad \text{on} \quad \Gamma(t) \subset \Omega = \{ \mathbf{x} \in \mathbb{R}^2 : 0.5 \leq |\mathbf{x}| \leq 1.5 \}. \]

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0.75 & \text{if } 0.65 \leq |\mathbf{x}| \leq 0.85, \\
0.0 & \text{otherwise.} 
\end{cases} \]

We set

\[ \alpha = 0.2, \quad b = 10 \quad \text{and} \quad \gamma = atan2(x_2, x_1). \]
Numerical tests: evolution along a curve

(a) level set $t = 0.0$  (b) level set, $t = 0.02$  (c) level set, $t = 0.04$

(d) initial solution  (e) $t = 0.001$  (f) $t = 0.002$
Conclusions

1. It is possible to treat PDEs of time-dependent surfaces which evolve both in normal and in tangential directions.

2. The method is accurate and robust.

3. The RBF-FD nature of the method allows sufficient flexibility while working with meshes.
Conclusions

1. A special treatment of RBF-FD generated stencils is required.

2. Stabilization of convection-like terms is necessary.

3. Coupling of surface PDEs with equation(s) which describe corresponding evolution of the surface is necessary.

4. The flexibility of our approach regarding ’meshes’ should be further exploited.

5. The method should be implemented in an HPC-fashion code.
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Weights are combined in a form

$$\text{trace} (A B) = \text{sum} \left( \text{sum} \left( A^T \cdot B \right) \right).$$