Multigrid techniques for a divergence-free finite element discretization

S. TUREK*

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Abstract — We derive basic properties for a class of discretely divergence-free finite elements which lead to a new proof of the smoothing property in a standard multigrid algorithm for solving the Stokes equations. Using appropriate grid transfer routines which are of second-order accuracy and interpolate in a divergence-free way, the ordinary multigrid convergence is obtained. The implementation of these operators is described in detail and the theoretical results are confirmed by numerical tests.

Keywords. Stokes equations, multigrid, discretely divergence-free finite elements.

1. INTRODUCTION

We consider multigrid techniques for solving the Stokes problem

\[-\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega \]

\[u = g \quad \text{on} \quad \partial \Omega \]

(1.1)

where the pair \(\{u, p\}\) represents the velocity and the pressure, respectively, of a viscous fluid contained in a bounded region \(\Omega \subset \mathbb{R}^2\), with prescribed boundary values on \(\partial \Omega\), and a given force \(f\). We assume for simplicity that \(\Omega\) is a convex polygon and the boundary values are homogeneous, \(g = 0\). Using a standard weak discrete formulation of (1.1) and discretely divergence-free finite element subspaces \(H^d_k \subset H_k\), we obtain a simplified positive definite scheme for the velocity only.

Find \(u^d_k \in H^d_k\), such that

\[a_k(u^d_k, v^d_k) = (f, v^d_k) \quad \forall v^d_k \in H^d_k.\]

(1.2)

Here, \(H_k\) and the bilinear form \(a_k(\cdot, \cdot)\) are discrete versions of \(H^1_0(\Omega)\) and \((\nabla \cdot, \nabla \cdot)_{L^2}\), respectively, and are defined more precisely in the following section, where examples of these spaces and some of their basic properties are also given.

There are obvious advantages of explicitly constructing the subspaces \(H^d_k\) (eliminating the pressure, reducing the number of unknowns, definite stiffness matrices). However, a disadvantage is a complicated implementation and a bad condition number of the system matrices, which can be overcome by an appropriate multigrid algorithm with convergence rates independent of the mesh size. The theoretical problem is to prove the corresponding smoothing and approximation properties for the smoothing and grid transfer routines used. The convergence proof of the approximation property is similar to that of Brenner [2], while the smoothing property can be shown by a new technique, mainly using the basic properties of the finite element spaces. On the other hand,

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*Institut für Angewandte Mathematik, University of Heidelberg, Germany
we have the practical problem of efficiently implementing these transfer operators. This can also be done by using a very precise and optimized code so that the final numerical effort of solving the Stokes equation is about the same as solving a scalar Poisson equation using the ordinary nonconforming spaces.

The subsequent numerical tests and the experience gathered in a fully developed nonsteady Navier–Stokes code for a wide range of Reynolds numbers [17] confirm the proposed theoretical results and show the high numerical flexibility of this class of finite elements.

$L^2(\Omega)$ and $H^m(\Omega)$ are the usual conventional Lebesgue and Sobolev spaces for the domain $\Omega \subset \mathbb{R}^2$ with the conventional norms $\| \cdot \|_0$ and $\| \cdot \|_m$. The inner product of $L^2(\Omega)$ is denoted by $\langle \cdot , \cdot \rangle$. The space $H^1_0(\Omega)$ is the completion in $H^1(\Omega)$ of the space of test functions $C_0^\infty(\Omega)$ and $H^{-1}(\Omega)$ is its dual space. By $L^2_0(\Omega)$, we denote the subspace of all $L^2(\Omega)$-functions over $\Omega$ having mean zero value. Vector valued functions and spaces and the corresponding norms and inner products are denoted as analogous scalar ones. We use a subscript $h$ for discrete mesh-dependent constructions like triangulations, discrete spaces, norms, inner products, etc. Normal vectors are denoted by $n$, and tangential vectors by $t$; $a \sim b$ implies that both expressions are equivalent, e.g. there are constants $c_1, c_2$ such that the relation $c_1 a \leq b \leq c_2 a$ is valid. Further special notation is introduced and described as needed.

2. THE SIMPLE NONCONFORMING FINITE ELEMENT SPACES

We consider the conventional steady Stokes problem (1.1) which reads with bilinear forms $a(u, v) := \langle \nabla u , \nabla v \rangle$ and $b(p, v) := -\langle p, \nabla \cdot v \rangle$:

Find a pair $\{ u, p \} \in H^1_0(\Omega) \times L^2_0(\Omega)$, such that

$$a(u, v) + b(p, v) + b(q, u) = ( f, v) \quad \forall \{ v, q \} \in H^1_0(\Omega) \times L^2_0(\Omega).$$

(2.1)

An equivalent 'shorter' formulation with $V(\Omega) = \{ v \in H^1_0(\Omega) : \nabla \cdot v = 0 \}$ is:

Find $u \in V(\Omega)$, such that

$$a(u, v) = ( f, v) \quad \forall v \in V(\Omega).$$

(2.2)

This problem has a unique solution for any force $f \in H^{-1}(\Omega)$ (see, e.g. [9]), which is a consequence of the familiar stability estimate

$$\sup_{v \in H^1_0(\Omega)} \frac{\langle q, \nabla \cdot v \rangle}{\| v \|_0} \geq \beta \| q \|_0 > 0 \quad \forall q \in L^2_0(\Omega), \quad q \neq 0.$$  

(2.3)

If $f \in L^2(\Omega)$, the solution is in $H^2(\Omega) \times H^1(\Omega)$ and satisfies the a priori estimate

$$\| u \|_2 + \| p \|_1 \leq c \| f \|_0.$$  

(2.4)

For discretization, let $T_h$ be a regular decomposition of the domain $\Omega$ into triangles or convex quadrilaterals denoted by $T$, where the mesh parameter $h > 0$ is the maximum diameter of the elements of $T_h$. By $\partial T_h$ we denote the set of all boundary edges of elements $T \in T_h$. Besides, the family $\{ T_h \}_h$ is assumed to satisfy the conventional uniform shape condition [6, 16]. The common edge between two elements $T_i, T_j \in T_h$ is denoted by $\Gamma_{ij}$ with corresponding midpoint $m_{ij}$. Analogously, we define the boundary edges $\Gamma_{i, 0} \subset (\partial T_h \cap \partial \Omega)$ with midpoints $m_{i, 0}$. To obtain the fine mesh $T_h$ from
a coarse mesh $T_{2h}$, we simply connect opposing midpoints (true domain boundaries are respected). In the new grid $T_h$ coarse midpoints become vertices.

To approximate the problem (V) by the finite element method we introduce discrete spaces $H_h \approx H^1_0(\Omega)$ and $L_h \approx L^2_0(\Omega)$. For the (parametric) quadrilateral case we use the reference element $\hat{T} = [-1, 1]^2$ and define for each $T \in T_h$ the corresponding one-to-one-transformation $\psi_T: \hat{T} \rightarrow T$. Then we set ("rotated bilinear elements" [14])

$$\hat{Q}_1(T) := \{ q \circ \psi_T^{-1} \mid q \in \text{span}\{x^2 - y^2, x, y, 1\}\}$$

while for the triangular case the linear functions from $P_1(T)$ are used. The degrees of freedom are determined by the nodal functionals $\{\Gamma|_T\}^\ast T_h \subset \partial T_h\}$ with

$$\begin{align*}
F^{(a)}_\Gamma(v) &:= |T|^{-1} \oint_T v \, d\gamma \quad \text{or} \\
F^{(b)}_\Gamma(v) &:= v(m_T).
\end{align*}$$

Either choice is unsolvable with $P_1(T)$ and $\hat{Q}_1(T)$, but for the quadrilateral case each of them leads to different finite element spaces since the applied midpoint rule is only correct for linear functions. Then, the corresponding (parametric) finite element spaces $H_h = H^{(a/b)}_h$ and $L_h$ are defined as

$$L_h := \{ q_h \in L^2_0(\Omega) \mid q_h|_T = \text{const} \ \forall T \in T_h \}, \quad H^{(a/b)}_h := S^{(a/b)}_h \times S^{(a/b)}_h$$

Our definitions lead to piecewise constant pressure approximations and edge-oriented velocity approximations with midpoints or integral mean values as degrees of freedom. Since the spaces $H^{(a/b)}_h$ are nonconforming, i.e., $H^{(a/b)}_h \not\subset H^1_0(\Omega)$, we have to deal with elementwise defined discrete bilinear forms and corresponding energy norms. We set

$$a_h(u_h, v_h) := \sum_{T \in T_h} \int_T \nabla u_h \cdot \nabla v_h \, dz, \quad \| v_h \|_h := a_h(u_h, v_h)^{1/2}$$

and define $b_h(\cdot, \cdot)$, dependent on the choice of $H^{(a/b)}_h$,

$$\begin{align*}
b_h(q_h, v_h) &:= -\sum_{T \in T_h} q_h|_T \hat{Q}_1(v_h), \\
Q_T(v_h) &:= \sum_{c \in \partial T} |c| F^{(a/b)}_c(v_h) \cdot n_c
\end{align*}$$

which leads for $H^{(b)}_h$ to an additional $O(h^2)$ quadrature error. Furthermore, let $j_h: L^2_0(\Omega) \rightarrow L_h$ be the operator of piecewise constant interpolation (modified to retain the zero-mean value property) which satisfies for $q \in L^2_0(\Omega) \cap H^1(\Omega)$ [16]

$$\| q - j_h q \|_0 \leq ch \| q \|_1 \quad \forall q \in L^2_0(\Omega) \cap H^1(\Omega)$$

and let $i^{(a/b)}_h: H^1_0(\Omega) \rightarrow H^{(a/b)}_h$ be the global interpolation operator in $H^{(a/b)}_h$

$$F_\Gamma(i^{(a/b)}_h v) = F_\Gamma(v) \quad \forall \Gamma \subset \partial T_h.$$
Lemma 2.1. For the interpolation operators \( i_h = i_h^{(a/b)} \):
\[
\| v - i_h v \|_0 + h \| v - i_h v \|_h \leq ch^2 \| v \|_2 \quad \forall v \in H^0_0(\Omega) \cap H^2(\Omega).
\]

Analogously, we can show [14, 16] that for the element pairs \((H^{a/b}_h, L_h)\) the discrete version of the continuous estimate (2.3) is valid:
\[
\tilde{\beta} \| p_h \|_0 \leq \max_{v_h \in H^{(a/b)}_h} \frac{b_h(p_h, v_h)}{\| v_h \|_h}
\tag{2.13}
\]
and we can essentially end up with the asymptotic error estimate [14, 16]:

Lemma 2.2. Suppose that the preceding assumptions are valid. Then, the discrete Stokes problems have unique solutions \( \{ u_h, p_h \} \in H^{(a/b)}_h \times L_h \), and the inequality holds:
\[
\| u - u_h \|_0 + h \| u - u_h \|_h + h \| p - p_h \|_0 \leq ch^2 \{ \| u \|_2 + \| p \|_1 \}.
\tag{2.14}
\]

To explicitly construct the divergence-free subspaces \( H^d_h \subset H_h \) we give the definition.

Definition 2.1. A function \( v_h \in H_h \) is called a discretely divergence-free function, if
\[
b_h(q_h, v_h) = 0 \quad \forall q_h \in L_h.
\tag{2.15}
\]

Since only piecewise constant pressure approximations are used, an equivalent criterion is
\[
Q_T(v_h) = 0 \quad \forall T \in T_h.
\tag{2.16}
\]

With these modifications we can introduce subspaces \( H^d_h \subset H_h \), and our discrete problem for the velocity is reduced to:

Find \( u^d_h \in H^d_h \), such that
\[
a_h(u^d_h, v^d_h) = (f, v^d_h) \quad \forall v^d_h \in H^d_h.
\tag{2.17}
\]

Finally, the corresponding pressure \( p_h \in L_h \) is determined by the condition
\[
b_h(p_h, v^d_h) = (f, v^d_h) - a_h(u^d_h, v^d_h) \quad \forall v^d_h \in H^d_h
\tag{2.18}
\]
where the functions \( v^d_h \) span the curl-free part of the complete space \( H_h \). In our configuration this is done (Section 3) by a marching process from element to element not solving any linear system of equations. In the end we obtain the same error estimates for \( u^d_h \) and \( p_h \) as in Lemma 2.1.

3. THE DIVERGENCE-FREE SUBSPACES AND THEIR PROPERTIES

Divergence-free subspaces have originally been introduced and analyzed in [7] and [9]. Some helpful analytical results may be found in [8]. We show the construction process and some properties, which are important for a better understanding, and the multigrid proofs.

We consider a general quadrilateral \( T \in T_h \) (Fig. 1) with vertices \( a^j \), midpoints \( m^j \), edges \( \Gamma^j \), unit tangential vectors \( t^j \), and normal unit vectors \( n^j \). Let \( \varphi^j_h \in S^{(a/b)}_h \) be
ordinary nodal basis functions in the finite element space $S_h = S_h^{(a/h)}$, restricted to element $T$, satisfying $F_T(\varphi^j_h) = \delta_{ij}$, $i, j = 1, \ldots, 4$. Then, the first group of trial functions $\{v^i_h\}$ of $H^d_h$ corresponding to the edges of $T_h$ is given by the local definition

$$v^i_{h|T} \in \{\varphi^j_h, j = 1, \ldots, 4\}. \quad (3.1)$$

The second group $\{v^{i,j}_h\}$ corresponding to the vertices is locally determined by

$$v^{i,j}_h \in \left\{ \frac{\varphi^k_h n^k}{|\Gamma^k|} - \frac{\varphi^{j}_h n^j}{|\Gamma^j|}, \quad j = 1, \ldots, 4, \quad k = (j + 2) \mod 4 + 1 \right\}. \quad (3.2)$$

Thus, we get approximations for the tangential velocities at the edges, and for the stream function values at the nodes (see also [10]). The full space $H^d_h$ is the direct sum of these two subspaces. When defining the inner product $(\cdot, \cdot)_{d,h}$ on $H^d_h$ by

$$(u_h, v_h)_{d,h} := \frac{1}{4} \sum_{T \in T_h} |T| \sum_{\Gamma \in \partial T} F_{\Gamma}(u_h) \cdot F_{\Gamma}(v_h) \quad (3.3)$$

the induced norm $\|\cdot\|_{d,h}$ is equivalent to the $L^2$-norm (or even identical for the triangular case with weight $1/3$), and both groups of trial functions are orthogonal relative to this form. If we eliminate one of the functions $\{v^{i,j}_h\}$ by prescribing the stream function value at one (boundary) point or requiring that the mean value be zero, we get a basis for the discretely divergence-free subspace $H^d_h$, assuming that our problem has only one boundary component. This is a simple consequence of the familiar Euler–Poincaré characteristic.

**Lemma 3.1 (Euler–Poincaré characteristic).** Define for a triangulation $T_h$:

- $NVT := \#\text{vertices of } T_h$
- $NMT := \#\text{midpoints of } T_h( = \#\text{edges of } T_h)$
- $NEL := \#\text{elements of } T_h$
- $NBC := \#\text{boundary components of } \partial \Omega$

then the formula holds:

$$NVT + NEL + NBC = NMT + 2.$$

For only one boundary component, $NBC = 1$, the dimension of the subspace $H^d_h$ must be

$$2 \cdot NMT - NEL = NMT - NEL + NVT + NEL - 1 = NMT + NVT - 1.$$
By construction, we have $NMT + NVT$ trial functions, which implies that at least one of them has to be eliminated. It is easy to see that the tangential functions of (3.1) are orthogonal relative to $\| \cdot \|_{d,h}$. For the remaining functions $v^\psi_h = \sum \Psi^i v^i_h \psi$, we can show

$$
\| v^\psi_h \|_{d,h}^2 \sim \sum_{T \in \mathcal{T}_h} |T| \sum_{\Gamma \in \partial T} \frac{|\Psi^{j+1} - \Psi^j|^2}{|\Gamma|^2} \sim \sum_{T \in \mathcal{T}_h} \sum_{k=1}^4 \frac{|\Psi^{k+1} - \Psi^k|^2}{|\Gamma|^2}.
$$

(3.4)

This implies that the mass matrix is spectrally equivalent to the stiffness matrix corresponding to the discretization of the Poisson equation with natural boundary conditions for conforming linear or bilinear elements. Therefore, eliminating one of these functions leads to a basis in two space dimensions, at least for the case of one boundary component.

After introducing these new basis functions the size of our linear system for the quadrilateral case is reduced from about $5NVT$ unknowns for the ordinary formulation to approximately $3NVT$ for the divergence-free case, and from $8NVT$ to about $4NVT$ for the triangular case.

Let $v_h \in H^d_h$ be a discretely divergence-free function with the two different representations

$$
v_h = \sum \Psi^i v^i_h \psi + \sum_j U^j v^j_h = \sum_k U^k \varphi^k_h + \sum_l V^l \varphi^l_h
$$

(3.5)

where $\Psi$, $U$, are the coefficient vectors in a divergence-free and $U$, $V$ in a primitive representation. Then, we can construct [8] rectangular transfer matrices $R_p^d$ and $R_p^\psi$, such that

$$
R_p^d \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \Psi \\ U_t \end{bmatrix}, \quad R_p^\psi \begin{bmatrix} \Psi \\ U_t \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix}.
$$

(3.5)

Rewriting $v_h \in H^d_h$ as

$$
v_h = \sum_i X^d_i v^i_h = \sum_j \Psi^j (h \cdot v^j_h) + \sum_k U^k v^k_h = \sum_l X^p_l v^p_l = \sum_m U^m v^m_h \varphi^p_k + \sum_l V^l \varphi^p_k
$$

(3.7)

with coefficient vectors $X^d$ and $X^p$, we obtain

$$
R_p^d R_p^\psi X^d = X^d, \quad R_p^\psi R_p^d X^p = X^p.
$$

(3.8)

Furthermore, let $S_l^i$ be ordinary conforming linear or bilinear finite element spaces with nodal basis $\varphi^i_h$, satisfying $\varphi^i_h(a_i) = \delta_{ij}$ for all vertices $a_i$ of $T_h$. By $S_l$ we denote the corresponding positive definite stiffness matrix. Analogously, we assume that $S_p$ denotes the corresponding stiffness matrix for the scalar nonconforming basis functions,

$$
S^{(i,j)}_l = \sum_{T \in \mathcal{T}_h} \int_T \nabla \varphi^{i,j}_h \cdot \nabla \varphi^{i,j}_h \, dx, \quad S^{(i,j)}_p = \sum_{T \in \mathcal{T}_h} \int_T \nabla \varphi^{i,j}_h \cdot \nabla \varphi^{i,j}_h \, dx
$$

and define

$$
S_{lp} := \begin{bmatrix} S_l & 0 \\ 0 & S_p \end{bmatrix}, \quad Q_d := \begin{bmatrix} S_{l/2} & 0 \\ 0 & I_p \end{bmatrix}
$$

(3.9)
The corresponding details are straightforward but very technical, therefore, we omit them. Then, by the previous corollary we end up with

$$
\|\| S_p^{1/2} R_d^p Q_d^{-1} S_p^{-1/2} \|\|_0 \leq c.
$$

We can write the second expression as

$$
\|\| S_p^{1/2} Q_d X_d^d \|\|_0^2 = (X_d^d)^T \begin{bmatrix} S_i & S_t & 0 \\ 0 & S_p \end{bmatrix} X_d^d
$$

and obtain with the basic eigenvalue estimates for standard finite element discretizations

$$
\|\| v_h \|\|_1^2 \leq c(\Psi^T S_t S_i \Psi + U_t^T S_p U_t) \leq c(\Psi^T S_t S_i \Psi + U_t^T U_t).
$$

By definition we also know

$$
\|\| v_h \|\|_0^2 = \Psi^T \Psi + U_t^T U_t.
$$

Then, a first result in matrix–vector notation reads

$$
\|\| v_h \|\|_s^2 \leq c \begin{bmatrix} \Psi \\ U_t \end{bmatrix}^T \begin{bmatrix} S_i & S_t & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \Psi \\ U_t \end{bmatrix}
$$

and using interpolation arguments for norm scales (compare, for instance, with [13]), we obtain for $s = 1/2$

$$
\|\| v_h \|\|_1^2 \leq c(\Psi^T S_t \Psi + U_t^T U_t) = c \|\| Q_d X_d^d \|\|_0^2.
$$

By Corollary 3.1 we finally get:

$$
\|\| v_h \|\|_1 \leq c h^{-1} \|v_h\|_0.
$$

We also need estimates for the condition numbers of mass and stiffness matrices. As before we may rewrite $v_h \in H_d^s$ as

$$
v_h = \sum_i X_d^d v_h^i = \sum_j \Psi^j (h \cdot v_h^i) + \sum_k U_t^k v_h^k.
$$

Using the standard finite element estimates

$$
\|v_h\|_0 \leq c \|v_h\|_h \leq c h^{-1} \|v_h\|_0
$$

we obtain for the $L_2$- and energy norm

$$
\|\| v_h \|\|_1^2 = \|v_h\|_0^2 \leq \|h^{-2} v_h\|_0^2 \leq c \|Q_d X_d^d \|_E^2 = c(\Psi^T S_t \Psi + U_t^T U_t)
$$

$$
\leq c(\Psi^T \Psi + U_t^T U_t) = c \|\| v_h \|\|_0^2
$$

$$
\|\| v_h \|\|_0^2 = \Psi^T \Psi + U_t^T U_t \leq c(h^{-2} \Psi^T S_t \Psi + U_t^T U_t)
$$

$$
\leq ch^{-2}(\Psi^T S_t \Psi + U_t^T U_t) = ch^{-2} \|Q_d X_d^d \|_E^2
$$

$$
\leq ch^{-4} \|v_h\|_0^2 \leq ch^{-4} \|v_h\|_h^2 = ch^{-4} \|\| v_h \|\|_1^2.
$$
We summarize these results in the lemma.

**Lemma 3.3.** For functions $v_h \in H_h^1$, there holds:
(1) The condition number of the mass matrix $M_d$ is $O(h^{-2})$; that is
\[
ch^4 \|v_h\|_0 \leq \|v_h\|_0^2 \leq Ch^2 \|v_h\|_0^2.
\]
(2) The condition number of the stiffness matrix $S_d$ is $O(h^{-4})$; that is
\[
ch^4 \|v_h\|_1^2 \leq \|v_h\|_1^2 \leq C \|v_h\|_0^2.
\]

In the rest of this section we want to elucidate practical questions concerning the problem of several boundary components and the calculation of the pressure. We are mainly interested in flows associated with the boundary conditions
\[
g \cdot n_{\Gamma_i} = 0, \quad i = 2, 3, \ldots, n
\]
(3.23)
on each additional boundary component $\{\Gamma_i, i = 2, 3, \ldots, n\}$. For such boundary conditions we know that the streamfunction takes the fixed but unknown value $c_i$ on $\Gamma_i$.

In connection with iterative solution techniques for the finite elements used we apply a so-called projection method [8]. This implies that before and after each matrix-vector multiplication we have to correct the boundary values. We calculate the mean value of the streamfunction values on each boundary component $\Gamma_i$ and prescribe this value as a new guess to $c_i$. This method works for any number of components without modifying the code by explicitly constructing the corresponding nonlocal basis functions (see [15]).

In order to calculate $p_h \in L_h$ corresponding to the already known velocity $u_h^d \in H_h^1$, we use the remaining part of discretely curl-free trial functions $H_h^1 \subset H_h$:
\[
b_h(p_h, v_h^i) = (f, v_h^i) - a_h(u_h^d, v_h^i) \quad \forall v_h^i \in H_h^1.
\]
(3.24)

Analogous to our tangential trial functions $v_h^{t_i}$, we construct locally normal functions $v_h^{i,n}$,
\[
v_h^{i,n} \in \{\phi_h^{i,n}, j = 1, \ldots, 4\}
\]
(3.25)
which have support on only two elements called $T_1$ and $T_2$ with common edge $\Gamma$. Using them, equation (3.24) is reduced to
\[
p_{h|T_1} Q_{T_1}(v_h^{i,n}) + p_{h|T_2} Q_{T_2}(v_h^{i,n}) = (f, v_h^{i,n}) - a_h(u_h^d, v_h^{i,n})
\]
(3.26)
which can be further simplified
\[
p_{h|T_1} Q_{T_1}(v_h^{i,n}) + p_{h|T_2} Q_{T_2}(v_h^{i,n}) = p_{h|T_1} [\Gamma F_\Gamma(v_h^{i,n}) \cdot n_i - p_{h|T_2} [\Gamma F_\Gamma(v_h^{i,n}) \cdot n_i
\]
\[
= |\Gamma| (p_{h|T_1} - p_{h|T_2}).
\]
(3.27)
Fixing the pressure value in one element, the other values can be calculated by a marching process not solving any linear system of equations.

To conclude, we would like to state the advantages of the discretely divergence-free finite element spaces. The pressure is decoupled from the velocity, which results in a reduction of the unknowns and leading to definite systems of equations. The pressure can be obtained from the velocity not solving a linear system. Another advantage is that for graphical postprocessing the streamfunction values are directly available. Of course,
this requires a greater computational effort. As we pointed out, this is mainly due to constructing the basis functions and increasing the condition numbers. A remedy for this trouble is the multigrid algorithm.

4. THE MULTIGRID ALGORITHM AND ITS ANALYSIS

The following algorithm and its analysis follow the ideas of Brenner [2–4]. Any differences from Brenner’s approach are pointed out in the text.

Let \( \{ T_h \}_{h \in H} \) be a family of regular subdivisions which are obtained by the refinement process discussed in Section 2. The discrete Stokes problem at level \( k \) reads:

Find \( u^d_k \in H^d_k \), such that

\[
a_k(u^d_k, v^d_k) = (f, v^d_k) \quad \forall v^d_k \in H^d_k.
\] *(4.1)*

As before, we write \( v_k \in H^d_k \) as

\[
v_k = \sum_i X^d_i v^d_i = \sum_j \Psi^j(h_k \cdot v^i_k) + \sum_i U^i_l v^i_l.
\] *(4.2)*

and introduce the discrete scalar product \((\cdot, \cdot)_k\)

\[
(v_k, w_k)_k := \sum_i \Psi^i_v \Psi^i_w + \sum_j U^i_l U^j_l.
\] *(4.3)*

This corresponds to the Euclidean scalar product \(<\cdot, \cdot>_E\), since

\[
(v_k, w_k)_k = <X^d_v, X^d_w>_E.
\] *(4.4)*

The prolongation operator \( I_{k-1}^k : H^d_{k-1} \rightarrow H^d_k \) and its adjoint restriction operator \( I_{k-1}^k : H^d_k \rightarrow H^d_{k-1} \) are defined by

\[
(I_{k-1}^k w_k, v_{k-1})_{k-1} = (w_k, I_{k-1}^k v_{k-1})_k \quad \forall v_{k-1} \in H^d_{k-1}, \quad \forall w_k \in H^d_k.
\] *(4.5)*

We further define the positive definite discrete operator \( A_k : H^d_k \rightarrow H^d_k \)

\[
(A_k v_k, w_k)_k = a_k(v_k, w_k) \quad \forall v_k, w_k \in H^d_k
\] *(4.6)*

so that the eigenvalues \( \lambda^i_k \) of \( A_k \) satisfy (compare with Lemma 3.3)

\[
O(h^d_k) \leq \lambda^i_k \leq \ldots \leq \lambda^{\text{max}}_k \leq c
\] *(4.7)*

where \( c \) is a constant independent of \( h_k \). Finally, we introduce the operator \( P_{k-1}^k : H^d_k \rightarrow H^d_{k-1} \), which is the adjoint of \( I_{k-1}^k \) relative to \( a_k(\cdot, \cdot) \),

\[
a_{k-1}(P_{k-1}^k w_k, v_{k-1}) = a_k(w_k, I_{k-1}^k v_{k-1}) \quad \forall v_{k-1} \in H^d_{k-1}, \quad \forall w_k \in H^d_k.
\] *(4.8)*

**Corollary 4.1.** With the above definitions the inequality holds: \( P_{k-1}^k = A_{k-1}^{-1} I_{k-1}^{k-1} A_k \).

Again we introduce the mesh-dependent norm scale \( \|\| \cdot \|\|_{s,k} \) on \( H^d_k \)

\[
\|\|v_k\|\|_{s,k} := (A_k^s v_k, v_k)_k^{1/2}
\] *(4.9)*
and repeat the estimates from the preceding section

$$\|\|v_k\|\|^2 = ||v_k||^2, \quad ||v_k\|\|_E^2 = \|X_d^k\|^2, \quad ||v_k\|\|_{L_2}^2 \leq c h_k^{-2} \|v_k\|_0^2. \quad (4.10)$$

Our $k$-level multigrid algorithm $MG(k, \cdot, \cdot)$ for solving the problem $(V^d_k)$ reads:

The $k$-level iteration $MG(k, u^0_k, g_k)$

The $k$-level iteration with initial guess $u^0_k$ yields an approximation to $u_k$, the solution to the problem

$$A_k u_k = g_k. \quad (4.11)$$

One step can be described in the following way:

For $k = 1$, $MG(1, u^0_1, g_1)$ is the exact solution: $MG(1, u^0_1, g_1) = A_1^{-1} g_1$.

For $k > 1$, there are four steps:

1. $m$-Presmoothing steps.

   Apply $m$ smoothing steps to $u^0_k$ to obtain $u^m_k$.

   For the damped Jacobi method this procedure reads: Let $u^l_k, l = 1, \ldots, m$, be defined recursively by the equations

   $$u^l_k = u^{l-1}_k + \omega_k (g_k - A_k u^{l-1}_k) \quad (4.12)$$

   where $\omega_k$ is a damping parameter which has to be smaller than the inverse of the largest eigenvalue $\lambda_k^{\max}$.

2. A correction step.

   Calculate the restricted defect

   $$g_{k-1} = I^{k-1}_k (g_k - A_k u^m_k) \quad (4.13)$$

   and let $u^{i}_{k-1} \in H^d_{k-1}, 1 \leq i \leq p, p \geq 2$, be defined recursively by

   $$u^{i}_{k-1} = MG(k - 1, u^{i-1}_{k-1}, g_{k-1}), \quad 1 \leq i \leq p, \quad u^0_{k-1} = 0. \quad (4.14)$$

3. Step size control.

   Calculate $u^{m+1}_k$ by

   $$u^{m+1}_k = u^m_k + \alpha_k I^{p}_{k-1} u^p_{k-1} \quad (4.15)$$

   where the parameter $\alpha_k$ may be a fixed or chosen adaptively chosen value so as to minimize the error $u^{m+1}_k - u_k$ in the energy norm, that is

   $$\alpha_k = \frac{(g_k - A_k u^m_k, I^{k-1}_k u^p_{k-1})_k}{(A_k I^{p}_{k-1} u^p_{k-1}, I^{k-1}_k u^p_{k-1})_k}. \quad (4.16)$$

4. $m$-Postsmoothing steps.

   Analogous to step (1), apply $m$ smoothing steps to $u^{m+1}_k$ to obtain $u^{2m+1}_k$.

   One iteration step $MG(k, \cdot, \cdot)$ yields, given an initial $u^0_k$, the new approximate $u^{2m+1}_k$, which may be written as

   $$MG(k, u^0_k, g_k) = u^{2m+1}_k. \quad (4.17)$$

The full multigrid algorithm with optimal efficiency $[O(n_k)$ arithmetic operations] consists of a nested iteration of this scheme. For practical applications other smoothing
schemes like Gauß-Seidel or ILU may be used and the number of pre- and postsmoothing steps may vary. Also, other cycle-types like the V-cycle \((p = 1)\) or our favourite the F-cycle \([16]\) may be taken.

For the convergence analysis, we restrict ourselves to the case of a two-level method \((k = 2)\) without postsmoothing and step length control and show the ordinary smoothing and approximation property for the damped Jacobi-method. The essential new approach is that the smoothing property may be shown using only the properties of the finite element spaces. This is in contrast to the work of Brenner \([2]\), where a strong relation between triangular divergence-free finite elements and the nonconforming Morley element was used.

**Lemma 4.1 (Smoothing property).** For the error \(e_h^m := u_h - u_h^m\) the relations hold for \(\rho(m) = m^{-1/4}\):

(a) \[ |||e_h^m|||_1 \leq c\rho(m)h^{-1}|||e_h^0|||_0 \]

(b) \[ |||e_h^m|||_{3/2} \leq c\rho(m)^2h^{-1}|||e_h^0|||_0. \]

**Proof.** Applying \(m\) damped Jacobi-steps to \(e_h^0\) yields

\[ e_h^m = (I_h - \omega_h A_h)^m e_h^0. \]  

(4.18)

Furthermore,

\[ A_h^{1/2}e_h^m = A_h^{1/2}(I_h - \omega_h A_h)^m e_h^0 \]

\[ = \omega_h^{-1/4}A_h^{1/4}\omega_h^{1/4}A_h^{1/4}(I_h - \omega_h A_h)^m A_h^{-1/4}A_h^{1/4}e_h^0 \]

and hence

\[ |||A_h^{1/2}e_h^m|||_0^2 \leq \omega_h^{-1/2}|||A_h^{1/4}\omega_h^{1/4}A_h^{1/4}(I_h - \omega_h A_h)^m A_h^{-1/4}|||_0^2 |||A_h^{1/4}e_h^0|||_0^2 \]

\[ \leq c|||(\omega_h A_h)^{1/4}(I_h - \omega_h A_h)^m|||_0^2 |||A_h^{1/4}e_h^0|||_0^2 \]

respectively,

\[ |||e_h^m|||_1^2 \leq c|||(\omega_h A_h)^{1/4}(I_h - \omega_h A_h)^m|||_0^2 |||e_h^0|||_{3/2}^2. \]  

(4.19)

By standard arguments for positive definite operators (see, e.g. \([1, 11]\))

\[ |||(\omega_h A_h)^{1/4}(I_h - \omega_h A_h)^m|||_0 \leq cm^{-1/4} = c\rho(m) \]  

(4.20)

and, therefore, with the help of Lemma 3.2,

\[ |||e_h^m|||_1 \leq c\rho(m)|||e_h^0|||_{1/2} \leq c\rho(m)h^{-1}|||e_h^0|||_0. \]

Analogously, for \(s = 3/2\), we have

\[ A_h^{3/4}e_h^m = A_h^{3/4}(I_h - \omega_h A_h)^m e_h^0 = \omega_h^{-1/2}A_h^{1/2}\omega_h^{1/2}A_h^{1/2}(I_h - \omega_h A_h)^m A_h^{-1/4}A_h^{1/4}e_h^0 \]  

(4.21)

resulting in:

\[ |||e_h^m|||_{3/2} \leq c\rho(m)^2h^{-1}|||e_h^0|||_0. \]  

(4.22)

The main work was done in the preceding section by proving Lemma 3.2. We now show the appropriate approximation property following the ideas of Brenner \([2-4]\). The
main difference is that the quadrilateral case is evaluated without using explicitly the fact that the finite elements are divergence-free in a pointwise sense.

Since in the 2-level iteration the correction equation is solved exactly, the coarse grid solution \( u_{2h} := u_{2h}^p \) satisfies:

\[
u_{2h} = A_{2h}^{-1} g_{2h} = A_{2h}^{-1} J_{h}^{2h}(g_h - A_h u^m_h) = A_{2h}^{-1} J_{h}^{2h} A_h e^m_h = P_{h}^{2h} e^m_h.\] (4.23)

We make the following assumptions concerning the prolongation operator:

**Condition I.**

1. \( \| J_{h}^{2h} u_{2h} - v_{2h} \|_0 \leq c h \| v_{2h} \|_{2h} \quad \forall v_{2h} \in H^{2h}_{2h}; \)
2. \( \exists \Pi_{h/2h} : V(\Omega) \cap H^2(\Omega) \rightarrow H^{h/2h}_{h/2h} \), such that
   \[
   \| v - \Pi_{h/2h} v \|_0 + h \| v - \Pi_{h/2h} v \|_{h/2h} \leq c_1 h^2 \| v \|_2 \quad \forall v \in V(\Omega) \cap H^2(\Omega)
   \]
   \[
   \| \Pi_{h} v - J_{h}^{2h} \Pi_{2h} v \|_0 + h \| \Pi_{h} v - J_{h}^{2h} \Pi_{2h} v \|_h \leq c h^2 \| v \|_2 \quad \forall v \in V(\Omega) \cap H^2(\Omega)
   \]

**Condition II.**

1. \( (J_{h}^{2h}, u_{2h} - u_{2h}, v_h) = 0 \quad \forall v_{2h} \in H^2_{2h} \quad \forall v_h \in H^d_h; \)
2. \( ||| J_{h}^{2h} A_{h}^{1/4} u_h |||_0 \leq c ||| v_h |||_0 \quad \forall v_h \in H^d_h. \)

Condition II is satisfied by the \( L_2 \)-projection only. Furthermore, condition I and the inverse estimate for finite elements imply the relations:

**Corollary 4.2.** For \( v_{2h} \in H^2_{2h}; \)
\[
\| J_{h}^{2h} v_{2h} \|_h \leq c \| v_{2h} \|_{2h}; \| J_{h}^{2h} v_{2h} \|_0 \leq c \| v_{2h} \|_0.
\]

**Corollary 4.3.** For \( v_h \in H^d_h; \)
\[
\| P_{h}^{2h} v_h \|_{2h} \leq c \| v_h \|_h.
\]

We can now show the main result for the approximation property.

**Lemma 4.2 (Approximation property).** For \( v_h \in H^d_h \) and \( \hat{v}_h = (I_h - I_{2h} P_{h}^{2h}) v_h \in H^d_h; \)
\[
\| \hat{v}_h \|_0 \leq c h \| v_h \|_1. \]
If Condition II is valid, then: \( \| \hat{v}_h \|_0 \leq c h \| v_h \|_{1.5}. \)

**Proof.** Let \( v_h \in H^d_h \) and \( \hat{v}_h = (I_h - I_{2h} P_{h}^{2h}) v_h \in H^d_h \) be given. We consider the auxiliary problem:
Find \( r \in V(\Omega) \), such that
\[
a(r, w) = \langle \hat{v}_h, w \rangle \quad \forall w \in V(\Omega). \]
\[
(4.24)
\]
We already know from the \textit{a priori} estimate (2.4) that
\[
\| r \|_2 \leq c \| \hat{v}_h \|_0. \]
\[
(4.25)
\]
Let \( r_h \in H^d_h \) and \( r_{2h} \in H^d_{2h} \) be the corresponding discrete solutions which satisfy the error estimates
\[
\| r - r_h \|_0 + h \| r - r_h \|_h \leq c h^2 \| \hat{v}_h \|_0
\]
\[
(4.26)
\]
\[
\| r - r_{2h} \|_0 + h \| r - r_{2h} \|_{2h} \leq C h^2 \| \hat{v}_h \|_0.
\]
Introducing the term \( z_{2h} = P_{h}^{2k} v_h \), we can write
\[
(\hat{v}_h, \hat{v}_h) = (\hat{v}_h, v_h - I_{2h}^h z_{2h})
\]
\[
= a_h(r_h, v_h) - (\hat{v}_h, I_{2h}^h z_{2h}) - (\hat{v}_h, z_{2h})
\]
\[
= a_h(r_h, v_h) - a_{2h}(r_{2h}, z_{2h})
\]
Further, letting \( \Pi_h : V(\Omega) \cap H^2(\Omega) \to H_h^2 \) and \( \Pi_{2h} : V(\Omega) \cap H^2(\Omega) \to H_{2h}^2 \) be interpolation operators as in Condition I, we have
\[
\|\hat{v}_h\|_0^2 = a_h(r_h - \Pi_h r_h, v_h) - a_{2h}(r_{2h} - \Pi_{2h} r_h, z_{2h})
\]
\[
+ a_h(\Pi_h r_h, v_h) - a_{2h}(\Pi_{2h} r_h, z_{2h}) - (\hat{v}_h, I_{2h}^h z_{2h}) - (\hat{v}_h, I_{2h}^h z_{2h})
\]
\[
= a_h(r_h - \Pi_h r_h, v_h) - a_{2h}(r_{2h} - \Pi_{2h} r_h, z_{2h})
\]
\[
+ a_h(\Pi_h r_h - I_{2h}^h \Pi_{2h} r_h, v_h) - (\hat{v}_h, I_{2h}^h z_{2h}) - (\hat{v}_h, I_{2h}^h z_{2h})
\]
\[
= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4.
\]
We now show that each term \( \Sigma_i \) can be estimated by
\[
|\Sigma_i| \leq c_h \|v_h\|_1 \|\hat{v}_h\|_0, \quad \text{respectively,} \quad |\Sigma_i| \leq c_h \|v_h\|_{3/2} \|\hat{v}_h\|_0
\]
which are valid because
\[
|\Sigma_1| = |a_h(r_h - \Pi_h r_h, v_h)| \leq \|r_h - \Pi_h r_h\|_h \|v_h\|_h \leq c_h \|\hat{v}_h\|_0 \|v_h\|_h
\]
\[
|\Sigma_2| = |a_{2h}(r_{2h} - \Pi_{2h} r_h, z_{2h})| \leq \|r_{2h} - \Pi_{2h} r_h\|_{2h} \|P_{h}^{2h} v_h\|_{2h} \leq c_h \|\hat{v}_h\|_0 \|v_h\|_h
\]
\[
|\Sigma_3| = |a_h(\Pi_h r_h - I_{2h}^h \Pi_{2h} r_h, v_h)| \leq \|\Pi_h r_h - I_{2h}^h \Pi_{2h} r_h\|_h \|v_h\|_h \leq c_h \|\hat{v}_h\|_0 \|v_h\|_h
\]
\[
|\Sigma_4| = |(\hat{v}_h, I_{2h}^h z_{2h})| \leq \|\hat{v}_h\|_0 \|I_{2h}^h z_{2h} - z_{2h}\|_0
\]
\[
\leq c_h \|\hat{v}_h\|_0 \|I_{2h}^h v_h\|_{2h} \leq c_h \|\hat{v}_h\|_0 \|v_h\|_h.
\]
As a result, we immediately obtain: \( \|\hat{v}_h\|_0 \leq c_h \|v_h\|_1 \).
For \( s = 3/2 \), we also require Condition II:
\[
|\Sigma_1| \leq \|r_h - \Pi_h r_h\|_{1/2} \|v_h\|_{3/2}
\]
\[
\leq c_h^{-1} \|r_h - \Pi_h r_h\|_0 \|v_h\|_{3/2} \leq c_h \|\hat{v}_h\|_0 \|v_h\|_{3/2}
\]
\[
|\Sigma_2| \leq \|r_{2h} - \Pi_{2h} r_h\|_{1/2} \|P_{h}^{2h} v_h\|_{3/2}
\]
\[
\leq c_h^{-1} \|r_{2h} - \Pi_{2h} r_h\|_0 \|A_{2h}^{3/4} A_{2h}^{-1/4} I_{2h}^h A_{2h} v_h\|_0
\]
\[
\leq c_h \|\hat{v}_h\|_0 \|A_{2h}^{-1/4} I_{2h}^h A_{2h}^{1/4} v_h\|_0 \leq c_h \|\hat{v}_h\|_0 \|v_h\|_{3/2}
\]
\[
|\Sigma_3| \leq \|\Pi_h r_h - I_{2h}^h \Pi_{2h} r_h\|_{1/2} \|v_h\|_{3/2} \leq c_h^{-1} \|\Pi_h r_h - I_{2h}^h \Pi_{2h} r_{2h}\|_0 \|v_h\|_{3/2}
\]
\[
\leq c_h \|\hat{v}_h\|_0 \|v_h\|_{3/2}
\]
\[
|\Sigma_4| = 0.
\]
This shows the second result: \( \|\hat{v}_h\|_0 \leq c_h \|v_h\|_{3/2} \).
We can summarize both lemmas in the following theorem.

Theorem 4.1 (Convergence of the 2-level scheme). Let $e^{m+1}_h$ be an error after one 2-level step with $m$ damped Jacobi-smoothing steps and initial error $e^0_h$. Using the grid transfer routines $l^{h}_{2h}$, and meeting Condition I, we get the error reduction

$$
\|e^{m+1}_h\|_0 \leq C\rho(m)\|e^0_h\|_0
$$

where $\rho(m) = m^{-1/4}$. In addition, if Condition II is valid, we obtain

$$
\|e^{m+1}_h\|_0 \leq C\rho(m)^2\|e^0_h\|_0.
$$

Hence, our method is convergent if the number $m$ of smoothing steps is large enough.

Using standard inductive arguments [1,2,17], we can verify the theorem.

Theorem 4.2 (Convergence of the $k$-level scheme with $p \geq 2$). Performing one step of $MG(k,u^0_k,g_k)$ with $m$ damped Jacobi-steps $t$ we have for Condition I and $m$ large enough

$$
\|u_k - MG(k,u^0_k,g_k)\|_0 \leq C\rho(m)\|u_k - u^0_k\|_0
$$

with $\rho(m) = m^{-1/4}$. If besides Condition II is valid, then

$$
\|u_k - MG(k,u^0_k,g_k)\|_0 \leq C\rho(m)^2\|u_k - u^0_k\|_0.
$$

We proved some results for the W-cycle with sufficiently many smoothing steps. It seems impossible to derive by this technique analogous estimates for the V-cycle or F-cycle, or any number of smoothing steps. However, we can easily show that the method is convergent.

Theorem 4.3. The $k$-level iteration is a convergent method, if the adaptive step-size control and at least one smoothing step are applied.

Since the stiffness matrices are positive definite, the inequality holds for the error $e^m_k$ after $m$ smoothing steps

$$
\|e^m_k\|_k \leq c_k\|e^0_k\|_k
$$

with a constant $c_k = c(h_k) < 1$. Because of the step size control we have for the correction

$$
\|e^{m+1}_k\|_k \leq \|e^m_k\|_k
$$

and, therefore,

$$
\|e^{m+1}_k\|_k \leq c_k\|e^0_k\|_k < \|e^0_k\|_k.
$$

Hence, an error reduction in each iteration step is ensured but the convergence rates are not independent of the step size $h_k$.

In the following, we construct transfer operators $l^{h}_{2h} \colon H^d_{2h} \to H^d_h$, satisfying Condition I, and, if need be, Condition II. The natural choice analogous to scalar nonconforming finite elements [5,18] is the $L^2$-projection $l^{h,P}_{2h}$ from $H^d_{2h}$ to $H^d_h$,

$$
(l^{h,P}_{2h} u_{2h}, v_h) = (u_{2h}, v_h) \quad \forall u_{2h} \in H^d_{2h}, \, \forall v_h \in H^d_h.
$$

(4.27)
In matrix-vector notation with coefficient vectors $X_{2h}$ and $X_h = I_{2h}^{hP}X_{2h}$, we can rewrite

$$M_{h,h}X_h = N_{h,2h}X_{2h},$$
respectivey,

$$X_h = M_{h,h}^{-1}N_{h,2h}X_{2h}$$

where $M_{h,h}$ is the mass matrix at level $h$, and $N_{h,2h}$ is the transfer matrix with coefficients

$$M_{h,h}^{(i,j)} = (u_h^{i,d}, v_h^{j,d}), \quad N_{h,2h}^{(i,j)} = (u_{2h}^{i,d}, v_{2h}^{j,d}).$$

**Lemma 4.3.** The transfer operator $I_{2h}^h = I_{2h}^{hP}$ satisfies Conditions I and II.

**Proof.** It is clear that the definition of an $L^2$-projection implies

$$\|I_{2h}^hv_{2h} - v_{2h}\|_0 \leq ch\|v_{2h}\|_2 \quad \forall v_{2h} \in H_{2h}^d.$$

Let $i_h$ (and analogously $i_{2h}$) be the standard interpolation operator $i_h: V(\Omega) \cap H^2(\Omega) \rightarrow H_{h}^d$ (2.12), which satisfies the estimate

$$\|v - i_h v\|_0 + h\|v - i_h v\|_h \leq ch^2\|v\|_2 \quad \forall v \in V(\Omega) \cap H^2(\Omega).$$

Then, letting $\Pi_h = i_h$ and $\Pi_{2h} = i_{2h}$,

$$\|\Pi_h v - I_{2h}^h\Pi_{2h}v\|_0^2 = \langle \Pi_h v - I_{2h}^h\Pi_{2h}v, \Pi_h v - I_{2h}^h\Pi_{2h}v \rangle$$

$$\leq \|\Pi_h v - I_{2h}^h\Pi_{2h}v\|_0 (\|v - \Pi_h v\|_0 + \|v - \Pi_{2h}v\|_0)$$

and, consequently, $\|\Pi_h v - I_{2h}^h\Pi_{2h}v\|_0 \leq ch^2\|v\|_2$.

Furthermore, using a standard inverse inequality implies

$$\|\Pi_h v - I_{2h}^h\Pi_{2h}v\|_h \leq c^{-1}\|\Pi_h v - I_{2h}^h\Pi_{2h}v\|_0 \leq ch\|v\|_2.$$

Condition II is straightforward, since equation (1) in Condition II is just the definition, and estimate (2) is a direct consequence for $L^2$-projections.

We have thus developed a grid transfer operator which is easy to analyze, and the convergence rates are also excellent. However, in each transfer step a mass matrix problem

$$M_{h,h}X_h = Y_h$$

where $Y_h = N_{h,2h}X_{2h}$, must be solved. The mass matrix corresponds to a second-order problem, and the part for the streamfunction values is equivalent to a conformingly discretized Laplacian operator [compare with (3.4)]. Consequently, we need fast Poisson solvers for each prolongation and restriction, which may be a second (standard) multigrid algorithm. However, the numerical effort is enormous. Thus we are looking for simpler transfers with about the same convergence rates. For this we present two operators which act on macroelements at level $2h$, interpolating directly into the divergence-free subspace at level $h$. Then, the problem is to show that the approximation properties are good enough. We consider the macrotriangle and quadrilateral with streamfunction values $\Psi$, and tangential components $U_{ij}$ (Fig.2). One regular refinement leads to Fig.3. We have to define three, respectively, five new streamfunction values at the vertices, and tangential components at all new edges. At vertices belonging to the macrolevel the values are retained. Then our elementwise macrointerpolation can be defined as follows:
The elementwise macrointerpolation algorithm

1. Transfer the divergence-free coefficient vector \((\Psi_{2h}, Ut_{2h})\) to the primitive coefficient vector \((U_{2h}, V_{2h})\).
2. Interpolate 'fully' on the macro elements to get \((U_h, V_h)\).
3. Compute the tangential and normal components \(Ut_h\) and \(Un_h\) at all fine grid edges.
4. Set \(\Psi_h = \Psi_{2h}\) at the macro nodes and calculate at the new vertices the values for \(\Psi_h\) by integrating \(Un_h\).
5. Take the average for \(\Psi_h\) and \(Ut_h\), which lie at macroedges.

Figure 2. Macro element.

Figure 3. Refined elements.

Figure 4. Configuration for interpolation.
In the following, we denote the full interpolation of the primitive nonconforming finite elements by \( I_{2h} \). The problem to be analyzed is that the values for the inner normal velocities are only implicitly given while the tangential ones are directly defined. Let us start using the ‘full’ interpolation for \( I_{2h} \), which implies linear or rotated bilinear interpolation on each macroelement. Then we get the prescriptions for the function values (Fig. 4) at the new edges [18]. For the linear case we calculate
\[
U_4 = u_1 - \frac{1}{2} u_2 + \frac{1}{2} u_3, \quad U_5 = \frac{1}{2} (u_1 + u_3)
\]  
for the trial space \( \hat{H}_h^{(a)} \)
\[
U_5 = u_1 + \frac{1}{8} u_2 + \frac{1}{2} u_4, \quad U_6 = \frac{5}{8} u_1 + \frac{1}{8} (u_2 + u_3 + u_4)
\]  
and for \( \hat{H}_h^{(b)} \)
\[
U_5 = \frac{15}{16} u_1 - \frac{3}{16} u_2 - \frac{1}{16} u_3 + \frac{5}{16} u_4, \quad U_6 = \frac{9}{16} u_1 + \frac{3}{16} u_2 + \frac{1}{16} u_3 + \frac{3}{16} u_4.
\]

This choice for \( I_{2h} \) is denoted by \( I_{2h}^L \). Another possibility is to use a constant interpolation \( I_{2h} \) on each macro element, which results for the linear elements in
\[
U_4 = u_1, \quad U_5 = u_2
\]  
and for the quadrilateral spaces \( \hat{H}_h^{(a/b)} \)
\[
U_5 = u_1, \quad U_6 = u_1.
\]

Before starting with the analysis, we would like to make some remarks concerning an efficient implementation. The described procedure looks very complicated, following step (1)-(5). The main idea is to rewrite this procedure using local matrices, resulting in \( 15 \times 6 \), respectively, \( 21 \times 8 \) elementwise defined matrices. This has to be done very carefully but we obtain discretely divergence-free interpolation operators comparable to corresponding operators for scalar Poisson equations [16] with quadratic elements in terms of computational effort required.

For the analysis we start with the operator \( I_{2h}^L = I_{2h}^{L,L} \) which is identical to the operator proposed by Brenner [2], only formulated differently. Since our quadrilateral elements are not pointwise divergence-free, we have to introduce slight modifications. However, we demonstrate this technique for the linear elements, since the analysis of the quadrilateral elements is the same but with more technical details.

In Section 3 we introduced on \( \hat{H}_h^d \oplus \hat{H}_h^d \) the inner product \( (\cdot, \cdot)_{d,h} \)
\[
(u_h, v_h)_{d,h} := \frac{1}{3} \sum_{T \in T_h} |T| \sum_{T \in \partial T} F_T(u_h) \cdot F_T(v_h)
\]
and the induced norm \( \| \cdot \|_{d,h} \) which in our configuration is identical to the \( L^2 \)-norm. Analogously, we can show that the elementwise defined functional \( \Theta_T(v_h) \), with
\[
\Theta_T(v_h) := \sum_{T, T_1 \in \partial T} (F_{T_1}(v_h) - F_{T_1}(v_h))^2
\]
defines a norm \( \| \cdot \|_{\theta,A} \) which is equivalent to the energy norm,

\[
\|v_h\|_{\theta,A} := \left( \sum_{T \in T_h} \theta_T(v_h) \right)^{1/2}.
\]

(4.35)

Then, the relations are valid

\[
\begin{align*}
\|v_h\|_0^2 & \sim ch^2 \sum_{T \in T_h} \sum_{r \in \partial T} |v_h(m_r)|^2 \sim ch^2 \sum_{r \in \partial T_h} |v_h(m_r)|^2 \\
\|v_h\|_A^2 & \sim \sum_{T \in T_h} \sum_{r_i, r_j \in \partial T} |v_h(m_{r_i}) - v_h(m_{r_j})|^2.
\end{align*}
\]

(4.36)

Lemma 4.4. For functions \( v_{2h} \in H^d_{2h} \), \( \| I_{2h}^h v_{2h} - v_{2h} \|_0 \leq ch \| v_{2h} \|_{2h} \).

Proof. We can first write

\[
\begin{align*}
\| I_{2h}^h v_{2h} - v_{2h} \|_0^2 & \leq ch^2 \sum_{T \in T_h} \sum_{r \in \partial T} |(I_{2h}^h v_{2h} - v_{2h})(m_r)|^2 \\
& = ch^2 (S_1 + S_2 + S_3 + S_4).
\end{align*}
\]

The expressions \( S_i \) are defined as

\[
S_1 := \sum_{m} |(I_{2h}^h v_{2h} - v_{2h})(m)|^2 + |(I_{2h}^h v_{2h} - v_{2h})(m)|^2
\]

(4.37)

where \( m \) ranges over all the midpoints of \( T_h \) that belong to an inner edge in \( T_{2h} \). The elements \( T_1, T_2 \in T_{2h} \) contain \( m \) (Fig.5). We further have

\[
S_2 := \sum_{m} |(I_{2h}^h v_{2h} - v_{2h})(m)|^2
\]

(4.38)

where \( m \) ranges over all midpoints along \( T_h \) on \( \partial \Omega \), and \( T \in T_{2h} \) is the triangle that contains this midpoint (Fig.5). We also have

\[
S_3 := 2 \sum_{m} |(I_{2h}^h v_{2h} - v_{2h})(m) \cdot t_m|^2, \quad S_4 := 2 \sum_{m} |(I_{2h}^h v_{2h} - v_{2h})(m) \cdot n_m|^2
\]

(4.39)

where \( m \) ranges over all the midpoints of \( T_h \) that are inside an element \( T \in T_{2h} \), and \( t_m \), respectively, \( n_m \) are the unit vectors tangential and normal to the edge \( \Gamma \), containing \( m \) (Fig.6). We complete the proof by showing that for \( I_{2h}^h = I_{2h}^{h,L} \) and 'fully' linear interpolation \( I_{2h}^h = I_{2h}^L \) there holds:

\[
S_i \leq c \| v_{2h} \|_{2h}^2.
\]

(4.40)

Figure 5. Figure for \( S_1 \) and \( S_2 \).
We begin with $S_1$ and $S_2$ and consider the following configuration (Fig. 7). The proof in this case is analogous to that of Brenner [2]. The definitions of $I^h_{2h}$ and $I_{2h}$ imply:

$$\left| (I^h_{2h} v_{2h} - v_{2h/2}) (m) \right|^2 + \left| (I_{2h}^h v_{2h} - v_{2h/2}) (m) \right|^2$$

$$= \left( (u_1 + \frac{1}{4} (u_2 - u_3 + u_5 - u_4)) - (u_1 + \frac{1}{2} (u_2 - u_3)) \right)^2$$

$$+ \left( (u_1 + \frac{1}{4} (u_2 - u_3 + u_5 - u_4)) - (u_1 + \frac{1}{2} (u_5 - u_4)) \right)^2$$

$$= \left[ \frac{1}{4} (u_3 - u_2) + \frac{1}{4} (u_5 - u_4) \right]^2 + \left[ \frac{1}{4} (u_2 - u_3) + \frac{1}{4} (u_4 - u_5) \right]^2.$$

Referring to the definition of $\Theta_{T_i} (v_{2h})$, we obtain

$$S_1 = \sum_m \left| (I^h_{2h} v_{2h} - v_{2h/2}) (m) \right|^2 + \left| (I_{2h}^h v_{2h} - v_{2h/2}) (m) \right|^2 \leq c \| v_{2h} \|_{2h}^2$$

$$S_2 = \sum_m \left| (I^h_{2h} v_{2h} - v_{2h/2}) (m) \right|^2 \leq c \| v_{2h} \|_{2h}^2.$$

For $S_3$ there is nothing to show since by definition: $I^h_{2h} v_{2h} (m) \cdot t_m = v_{2h} (m) \cdot t_m$, and therefore, $S_3 = 0$.

The last and most interesting term is $S_4$. Our problem is to determine the value for $I^h_{2h} v_{2h} (\cdot) \cdot n_m$ at the edge $S_m$ (Fig. 8). We know

$$v_{2h} (m) \cdot n_m = u_m \cdot n_m = \frac{1}{2} (u_1 + u_3) \cdot n_m, \quad I^h_{2h} v_{2h} (m) \cdot n_m = \frac{\psi_6 - \psi_4}{|S_m|}. \quad (4.41)$$
Furthermore,

\[ \psi_6 = \psi_1 + |S_3|u_5 \cdot n_3 = \psi_1 + |S_3|\left(u_3 + \frac{1}{4}(u_1 - u_2 + \tilde{u}_1 - \tilde{u}_2)\right) \cdot n_3 \]

and in the same way

\[ \psi_4 = \psi_1 + |S_1|\left(u_1 + \frac{1}{4}(u_3 - u_2 + \tilde{u}_3 - \tilde{u}_2)\right) \cdot n_1. \]

This implies

\[ \frac{\psi_6 - \psi_4}{|S_m|} = \frac{1}{|S_m|}(|S_3|u_3 \cdot n_3 - |S_1|u_1 \cdot n_1) + c(v_{2h}) \]

with

\[ c(v_{2h}) = \frac{|S_3|}{4|S_m|}|(u_1 - u_2) + (\tilde{u}_1 - \tilde{u}_2)| \cdot n_3 + \frac{|S_1|}{4|S_m|}|(u_2 - u_3) + (\tilde{u}_2 - \tilde{u}_3)| \cdot n_1. \]

Since \( v_{2h} \) is discretely divergence-free on \( T \), we obtain by definition

\[ |S_3|u_3 \cdot n_3 - |S_1|u_1 \cdot n_1 = |S_m|u_2 \cdot n_m \]

and by the definition of \( \Theta_T(v_{2h}) \), we easily see: \( c(v_{2h})^2 \leq c\|v_{2h}\|_{2h}^2 \).

Summarizing the last estimates, we have

\[ |I_{2h}^Hv_{2h}(m) \cdot n_m - v_{2h}(m) \cdot n_m|^2 \leq \left(u_2 \cdot n_m - \frac{1}{2}(u_1 + u_3) \cdot n_m\right)^2 + c(v_{2h})^2 \]

\[ \leq \frac{1}{4}|(u_2 - u_1) \cdot n_m|^2 + \frac{1}{4}|(u_2 - u_3) \cdot n_m|^2 + c(v_{2h})^2 \]

\[ \leq C\|v_{2h}\|_{2h}^2 \]

and finally:

\[ S_4 = 2 \sum_m |(I_{2h}^Hv_{2h} - v_{2h,T})(m) \cdot n_m|^2 \leq c\|v_{2h}\|_{2h}^2. \]
The second inequality in Condition I is proved by using homogeneity arguments. For this we need the lemma concerning estimates on the reference element. The proof can be found in [2, 17].

**Lemma 4.5.** Let $G$ be a union of two triangles $T_1, T_2$ with $\text{diam} \, G = 1$. Referring to Fig. 9, there exist constants $c$ and $C$, dependent only on the angles of $T_1, T_2$, such that for all functions $u \in H^2(G)$:

\[
\begin{align*}
(a) \quad & \left| \frac{1}{|S_1|} \int_{S_1} u \, d\sigma + \frac{1}{4} \left( \frac{1}{|S_3|} \int_{S_3} u \, d\sigma - \frac{1}{|S_2|} \int_{S_2} u \, d\sigma + \frac{1}{|S_4|} \int_{S_4} u \, d\sigma - \frac{1}{|S_5|} \int_{S_5} u \, d\sigma \right) \right| \\
& - \frac{1}{|S_1|} \int_{S_1} u \, d\sigma \leq c|u|_{H^2(G)}; \\
(b) \quad & \left| \frac{1}{2} \left( \frac{1}{|S_1|} \int_{S_1} u \, d\sigma + \frac{1}{|S_2|} \int_{S_2} u \, d\sigma \right) - \frac{1}{|S_2|} \int_{S_2} u \, d\sigma \right| \leq C|u|_{H^2(G)}. 
\end{align*}
\]

To show the second relation in Condition I we again need an interpolation operator $\Pi_h$: $V(\Omega) \cap H^2(\Omega) \to H^d_h$, satisfying

\[
||v - \Pi_h v ||_0 + h||v - \Pi_h v ||_h \leq ch^2 ||v ||_2 \quad \forall v \in V(\Omega) \cap H^2(\Omega).
\]

As before, let $\Pi_h = i_h$ be a standard interpolation operator.

**Lemma 4.6.** For $v \in V(\Omega) \cap H^2(\Omega)$: $||\Pi_h v - I_{2h}^h \Pi_{2h} v ||_0 \leq ch^2 ||v ||_2$.

**Proof.** As before we make the splitting

\[
||\Pi_h v - I_{2h}^h \Pi_{2h} v ||_0^2 \leq c h^2 \sum_{T \in T_h} \sum_{\gamma \in \partial T} ||\Pi_h v - I_{2h}^h \Pi_{2h} v||_{m_T}^2
\]

\[
= c h^2 (S_1 + S_2 + S_3 + S_4).
\]

Lemma 4.5 and standard estimates for finite elements (see, for instance, [6]) lead to

\[
S_1 + S_2 + S_3 \leq c h^2 ||v ||_2^2.
\]

Since $\Pi_h v - I_{2h}^h \Pi_{2h} v \in H^d_h$ is discretely divergence-free, we also have

\[
S_4 \leq c (|S_1| + |S_2|) \leq c h^2 ||v ||_2^2
\]

which implies the desired result. $\blacksquare$
With analogous conclusions, the proof of which requires more technical details due to a larger number of local degrees of freedom Condition I can be shown for the quadrilateral cases. Condition II could not be proved for this operator, only for the choice \( I_{2h}^h = I_{2h}^{h,P} \).

For the operator \( I_{2h}^h = I_{2a}^{h,K} \) which is a modification of \( I_{2a}^{h,L} \), using the locally constant interpolation operator \( I_{2a}^{h,K} \), we can only show a slightly weaker result [16].

**Lemma 4.7.** The inequality holds:
\[
\| I_{2a}^h v_{2a} - v_{2a} \|_0 \leq c h \| v_{2a} \|_{2a} \quad \forall v_{2a} \in H_{2a}^a.
\]

We have not proved the second relation for \( I_{2a}^{h,K} \),
\[
\| \Pi_a v - I_{2a}^h \Pi_{2a} v \|_0 + h \| \Pi_a v - I_{2a}^h \Pi_{2a} v \|_h \leq c h^2 \| v \|_2 \quad \forall v \in V(\Omega) \cap H^2(\Omega). \tag{4.42}
\]

Nevertheless, this operator is one of those used in our test calculations. There is another approach for developing \( I_{2a}^{h,K} \). We consider the discretization of the generalized Stokes problem
\[
\alpha u - \varepsilon \Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \tag{4.43}
\]
with \( \alpha > 0, \varepsilon \geq 0 \). As \( \varepsilon \to 0 \), the effect of the Stokes operator decreases, and for \( \varepsilon = 0 \), we only have to solve a linear system for the mass matrix \( M_{h,h} \). As mentioned above, the streamfunction part of this matrix is spectrally equivalent to the Laplacian matrix which is discretized using conforming linear or bilinear finite elements. Hence, it seems logical to solve this system by conforming multigrid routines. This procedure, however, is exactly the same as the one proposed for \( I_{2a}^{h,K} \), if for the tangential part at the edges the operator \( I_{2a}^{h,L} \) is taken. In the subsequent section we carefully examine the numerical cost of solving the Stokes equations by the different schemes proposed.

| Table 1. Rates for grid I. |

<table>
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<th>( k )</th>
<th>( 3201 )</th>
<th>( 12545 )</th>
<th>( 49665 )</th>
<th>( 3201 )</th>
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<td></td>
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<tr>
<td>( I_{2h}^{h,K} )</td>
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<td></td>
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<td>GS1</td>
<td>0.404</td>
<td>0.834</td>
<td>1.220</td>
<td>0.714</td>
<td>3.440</td>
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<td>0.597</td>
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<td>0.162</td>
<td>0.165</td>
<td>0.815</td>
<td>0.808</td>
<td>0.849</td>
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</tr>
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<td>0.128</td>
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<td>0.111</td>
<td>0.544</td>
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Table 2. Rates for grid II.

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<tr>
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<td>0.356</td>
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<td>0.190</td>
<td>0.184</td>
<td>0.186</td>
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<td>GS2</td>
<td>0.136</td>
<td>0.143</td>
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<td>0.099</td>
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<tr>
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<td>0.241</td>
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Table 3. Rates for grid III.

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<table>
<thead>
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<tbody>
<tr>
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Table 4. Rates for ellipse.

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<td>1.880</td>
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<td>GS4</td>
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Table 5. Grid I with $I_{2h}^{h,K}$ and $\alpha = 10^n$, $n = 0, 3, 6, 9$.

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<tr>
<td>n</td>
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5. NUMERICAL RESULTS

Our aim is to examine the influence of different smoothing and transfer operators and different meshes on the efficiency and robustness of our multigrid algorithm. As a smoothing operator, we restrict ourselves to the Gauß–Seidel method which is about as expensive as the Jacobi-iteration, at least on scalar workstations like the SUN 4/260 used. The case of the ILU-method is studied in [19], in which we give an overview of possible renumbering strategies. For the grid transfer routines we expect that the projection method $I_{2h}^{h,P}$ will have the best convergence rates followed by $I_{2h}^{h,L}$ and $I_{2h}^{h,K}$. But these convergence rates are not a true measure since, for instance, Gaussian elimination has the best rates, namely zero. In order to take account of this discrepancy we introduce efficiency rates as a measure in real time.
We start with the standard problem of the Stokes Driven Cavity on a unit square,

\[ \Delta u - \nabla p = 0, \quad \nabla \cdot u = 0 \quad \text{in } \Omega \]

\[ u = 0 \quad \text{on } \partial \Omega \setminus \{ y = 1 \}, \quad u = (1,0)^T \quad \text{on } \partial \Omega \cap \{ y = 1 \} \]

with the coarse grids as typical representations of possible meshes (Fig. 10). The finer subdivisions are generated using the regular refinement process as described in Section 2. In the tables we present the number of unknowns (NEQ), the number m of pre- and postsmoothing steps, the convergence rate \( \kappa \) and the efficiency rate \( \gamma \),

\[ \kappa = \frac{\sqrt{|r^{(8)}|/|r^{(0)}|}}{\sqrt{|r^{(8)}|/|r^{(0)}|}}, \quad \gamma = -\frac{1000T_8}{8NEQ \log \kappa}. \]  

(5.1)

Here, \( r^{(8)} \) denotes the residual after 8 iterations, and \( T_8 \) the corresponding computational time. The efficiency rate \( \gamma \) measures the time in milliseconds needed to gain one digit per unknown. For the operator \( I^{h,J}_{2h} \) we used the adaptive step length control, while for \( I^{h,K}_{2h} \) a fixed value smaller than 1, and for \( I^{h,P}_{2h} \) the choice \( \alpha = 1 \), due to the character of a projection method, seems optimal. The numbers are generated using an F-cycle, since the V-cycle seems unstable sometimes, while the W-cycle shows no visible advantages. As finite element space, we use the space \( H_h = H^{(6)}_h \). The tables show that the projection method leads to the best convergence rates but at the expense of the greatest computational effort. The constant operator \( I^{h,K}_{2h} \) leads to surprisingly good results, but the robustness against grid irregularities seems to be lost, at least as compared to \( I^{h,L}_{2h} \), which produces the best results. Similar results can be obtained in the next domain, which simulates the flow around an ellipse. This is a practical example of domains (a coarse grid, see Fig. 11) as to concerning the Navier–Stokes equations. As a final example, we show the results for the class of problems (with \( \alpha \geq 0 \))

\[ \alpha u - \Delta u + \nabla p = f, \quad \nabla \cdot u = 0. \]  

(5.2)

\[ \begin{array}{ccc}
I & II & III \\
\end{array} \]

Figure 10. Used coarse grids.

\[ \begin{array}{c}
\end{array} \]

Figure 11. Coarse grid for ellipse configuration.
This configuration is a typical example for an unsteady calculation, where the mass matrix is weighted with $O(1/\Delta t)$. Here, the values $\alpha = 10^n$, $n = 0, 3, 6, 9$, are taken. Our theoretical considerations as to the influence as $\alpha \to \infty$ are justified. $I_{2h}^{h,K}$ is now the best but our favourite $I_{2h}^{h,L}$ was not much worse, only the projection method was surprisingly bad.

Summarizing, we have found an interpolation operator, namely $I_{2h}^{h,L}$, which satisfies all requirements: small numerical effort, good convergence rates, robust against grid and parameter variations, and theoretically analyzable. In connection with the Gauß-Seidel iteration as a smoother in our algorithm, we seem to have found a good candidate as a Black Box solver for linear systems in a fully nonstationary Navier–Stokes code [16–18].

REFERENCES


