Fluid-Structure Interaction Problems:

FEM Multigrid Techniques and Benchmarking

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Multiphase FSI Problems: Elastic Solids

- **Liquid – Rigid Solid**
  - Particulate Flow
  - Robofish

- **Liquid – Elastic Solid**
  - Biomechanics
  - Medical applications
  - Aeroelasticity
Multiphase FSI Problems: Rigid Solids

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- **Liquid – Elastic Solid**
  - Biomechanics
  - Medical applications
  - Aeroelasticity
Required: Special Numerics for FSI

- Special FEM Techniques
- Stabilization for high Re, Pe, We,… Numbers
- Implicit Approaches
- Grid Deformation Methods
- Multigrid Solvers
- Space-Time Adaptivity
- Fictitious Boundary Methods

*Computational mesh (can be) independent of ‘internal objects’*
Challenges for Numerics

- Special FEM discretization techniques to handle the following challenging points
  - Stable FE spaces for velocity and pressure fields, and velocity and extra-stress fields
    \[ \rightarrow Q2/P1/Q2, \ Q1(nc)/P0/Q1(nc) \] (new: \[ Q2(nc)/P1/Q2(nc) \])
  - Special treatment of the „convective“ terms
    \[ \rightarrow \text{edge-oriented/interior penalty EO-FEM, TVD/FCT} \]
  - Special treatment of the „reactive" terms in viscoelastic problems
    \[ \rightarrow \text{LCR + EO-FEM} \]

- Special (nonlinear) solvers to deal with different sources of nonlinearity
  - nonlinear operators \[ \rightarrow \text{Newton method via divided differences} \]
  - stiff coupling of equations \[ \rightarrow \text{monolithic/operator splitting multigrid} \]
  - complex geometries and meshes
Nonlinear Solvers

Solve for the residual of the nonlinear system algebraic equations

\[ R(x) = 0, \quad x = (u, \Theta, \sigma, p) \]

Use Newton method with damping results in iterations of the form

\[ x^{n+1} = x^n + \omega^n \left[ \frac{\partial R(x^n)}{\partial x} \right]^{-1} R(x^n) \]

- Continuous Newton: on variational level (before discretization)
  - The continuous Frechet operator can be **analytically calculated**

- Inexact Newton: on matrix level (after discretization)
  - The Jacobian matrix is **approximated** using finite differences as

\[
\left[ \frac{\partial R(x^n)}{\partial x} \right]_{ij} \approx \frac{R_i(x^n + \epsilon e_j) - R_i(x^n - \epsilon e_j)}{2\epsilon}
\]
Multigrid Solvers

- Standard geometric multigrid approach with full FEM grid transfer
- Smoother: Local/Global MPSC
  - Local MPSC via Vanka-like smoother

  → Monolithic multigrid solver

  \[
  \begin{bmatrix}
  u^{l+1} \\
  \sigma^{l+1} \\
  \Theta^{l+1} \\
  p^{l+1}
  \end{bmatrix} = \begin{bmatrix}
  u^l \\
  \sigma^l \\
  \Theta^l \\
  p^l
  \end{bmatrix} + \omega^l \left( \sum_{T \in T_a} [K + J]_T^{-1} \right) \begin{bmatrix}
  \text{Res}_u \\
  \text{Res}_\sigma \\
  \text{Res}_\Theta \\
  \text{Res}_p
  \end{bmatrix}
  \]

- Global MPSC
  - solve for an intermediate \( u \) (generalized momentum equation)
  - solve for \( p \) (pressure Poisson equation)
  - update of \( u \) and \( p \)
  - solve for \( \Theta \) (tracer equation)
  - solve for \( \sigma \) (constitutive equation)

  → Decoupled multigrid solver
1) Aspects of (Elastic) FSI Problems

- incompressible Newtonian fluid (with nonlinear extensions)
  \[ \sigma^f = -pI + 2\nu D \]

- hyperelastic material, incompressible
  \[ \sigma^f = -pI + 2F\frac{\partial \Psi}{\partial F}F^T, \quad \det F = 1 \]
  \[ \Psi(F) = \alpha(I_c - 3)\text{Neo - Hook} \]
  \[ \Psi(F) = \alpha_1(I_c - 3) + \alpha_2(I_c - 3) + \alpha_3(|\text{Fe}| - 1)^2\text{Mooney - Rivlin + anisotropic} \]
  where \( C = FF^T \) and \( I_c = \text{tr}C, \quad I_c = \frac{1}{2}(\text{tr}C^2 - (\text{tr}C)^2) \)

- or St. Venant-Kirchhoff material, compressible
  \[ \sigma^s = \frac{1}{J}F(\lambda^s(\text{tr}E)I + \mu^sE)F^T \]
  where \( E = \frac{1}{2}(F^TF - I) \)
Monolithic ALE-FEM Approach

\[ R(x) = 0 \quad x = (u_h, v_h, p_h) \in U_h \times V_h \times P_h \]

\[
Mu_h - \frac{k}{2}(M^s v_h + L^f u_h) = \text{rhs}(u^n_h, v^n_h)
\]

\[
(M^f + \beta M^s)v_h + \frac{k}{2}N_1(v_h, u_h) + \frac{1}{2}N_2(v_h, u_h) + \frac{k}{2}(S^s(u_h) + S^f(v_h)) - kBp_h = \text{rhs}(u^n_h, v^n_h, p^n_h)
\]

\[ C(u_h) + B^{fT}v_h = 1 \]

\[
\frac{\partial R}{\partial x}(x) = \begin{bmatrix}
  M - \frac{k}{2}L^f \\
  \frac{1}{2} \frac{\partial (N_1 + S^s + S^f)}{\partial u_h} + k \frac{\partial B}{\partial u_h} p_h \\
  \frac{1}{2} \frac{\partial N_2}{\partial v_h} + \frac{k}{2} \frac{\partial (N_1 + S^2_f)}{\partial v_h} \\
  B^{sT} + \frac{\partial B^{fT}}{\partial u_h} v_h \\
  B^{fT} \\
  0
\end{bmatrix}
\]
Monolithic ALE-FEM Approach

\[ R(x) = 0 \quad x = (u_h, v_h, p_h) \in U_h \times V_h \times P_h \]

\[ Mu_h - \frac{k}{2}(M^s v_h + L^f u_h) = \text{rhs}(u_h^n, v_h^n) \]

\[ (M^f + \beta M^s)v_h + \frac{k}{2}N_1(v_h, u_h) + \frac{1}{2}N_2(v_h, u_h) + \frac{k}{2}(S^s(u_h) + S^f(v_h)) - kBp_h = \text{rhs}(u_h^n, v_h^n, p_h^n) \]

\[ C(u_h) + B^{fT}v_h = 1 \]

\[ \begin{bmatrix}
    S_{uu} & S_{uv} & 0 \\
    S_{vu} & S_{vv} & kB \\
    c_u B_s^T & c_v B_f^T & 0
\end{bmatrix}
\begin{bmatrix}
    u \\
    v \\
    p
\end{bmatrix} =
\begin{bmatrix}
    f_u \\
    f_v \\
    f_p
\end{bmatrix} \]

Typical discrete saddle-point problem
Multigrid Solver for Q2/Q2/P1

- standard geometric multigrid approach
- smoother by local MPSC-Ansatz (Vanka-like smoother)

\[
\begin{bmatrix}
  u^{l+1} \\
  v^{l+1} \\
  p^{l+1}
\end{bmatrix} =
\begin{bmatrix}
  u^l \\
  v^l \\
  p^l
\end{bmatrix} - \omega \sum_{\text{Patch}\Omega_i} \begin{bmatrix}
  S_{u\Omega_i} & S_{v\Omega_i} & 0 \\
  S_{v\Omega_i} & S_{v\Omega_i} & kB_{\Omega_i} \\
  c_u B_{s\Omega_i}^T & c_v B_{f\Omega_i}^T & 0
\end{bmatrix}^{-1} \begin{bmatrix}
  \text{def}_u^l \\
  \text{def}_v^l \\
  \text{def}_p^l
\end{bmatrix}
\]

- full inverse of the local problems by LAPACK (39 x39 systems)
- alternatives: simplified local problems (3x3 systems) or ILU(k)
- combination with GMRES/BiCGStab methods possible
- full (canonical) FEM prolongation, restriction by \( R = P^T \)

Very accurate, flexible and highly efficient FSI solver

(⇒ FSI Benchmarks)
2) Aspects of Particulate Flow

Fluid flow is modelled by the Navier-Stokes equations:

$$\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) - \nabla \cdot \sigma = f, \quad \nabla \cdot u = 0, \quad \sigma(X, t) = -pI + \mu [\nabla u + (\nabla u)^T]$$

Motion of particles is described by the Newton-Euler equations, i.e., the translational velocities and angular velocities of the p-th particle satisfy:

$$M_p \frac{dU_p}{dt} = F_p + F'_p + (\Delta M_p)g, \quad I_p \frac{d\omega_p}{dt} + \omega_p \times (I_p \omega_p) = T_p.$$

$F_p$ and $T_p$ are the hydrodynamical forces and the torque at mass center acting on the p-th particle and $F'_p$ are the collision forces

$$F_p = -\int_{\Gamma_p} \sigma \cdot n_p \, d\Gamma_p, \quad T_p = -\int_{\Gamma_p} (X - X_p) \times (\sigma \cdot n_p) \, d\Gamma_p.$$
No slip boundary conditions at interface $\Gamma_p$ between particles and fluid i.e., for any $X \in \Gamma_p$, the velocity $u(X)$ is defined by:

$$u(X) = U_p + \omega_p \times (X - X_p)$$

The position $X_p$ of the p-th particle and its angle $\theta_p$ are obtained by integration of the kinematic equations:

$$\frac{dX_p}{dt} = U_p, \quad \frac{d\theta_p}{dt} = \omega_p$$
How to Calculate the Forces?

Hydrodynamic forces and torque acting on the i-th particle

\[
F_i = -\int_{\partial P_i} \sigma \cdot n_i \, d\Gamma_i, \quad T_i = -\int_{\partial P_i} (X - X_i) \times (\sigma \cdot n_i) \, d\Gamma_i
\]

Idea: ‘Replace the surface integral by a volume integral’ and use indicator functions (\( n_p \approx \nabla \alpha_p \))

\[
F_p = -\int_{\Gamma_p} \sigma \cdot n_p \, d\Gamma_p = -\int_{\Omega_T} \sigma \cdot \nabla \alpha_p \, d\Omega_T
\]

Fictitious Boundary Method on Generalized Tensorproduct Meshes
**Grid Deformation Methods**

**Idea**: construct transformation \( \phi, \ x = \phi(\xi, t) \) with \( \det \nabla \phi = f \)

\[ \Rightarrow \text{local mesh area } \approx f \]

1. Compute monitor function \( f(x, t) > 0, \ f \in C^1 \) and

\[ \int_{\Omega} f^{-1}(x, t) dx = |\Omega|, \ \forall t \in [0,1] \]

2. Solve \((t \in [0,1])\)

\[ \Delta v(\xi, t) = - \frac{\partial}{\partial t} \left( \frac{1}{f(\xi, t)} \right), \ \frac{\partial v}{\partial n} \bigg|_{\partial \Omega} = 0 \]

3. Solve the ODE system

\[ \frac{\partial}{\partial t} \phi(\xi, t) = f(\phi(\xi, t), t) \nabla v(\phi(\xi, t), t) \]

new grid points: \( x_i = \phi(\xi_i, 1) \)

Grid deformation preserves the (local) logical structure of the grid
(Semi-explicit) Operator-Splitting Approach

The algorithm for $t^n \rightarrow t^{n+1}$ consists of the following 5 substeps

1. Fluid velocity and pressure: $NSE\left(u_f^{n+1}, p^{n+1}\right) = BC\left(\Omega_p^n, u_p^n\right)$

2. Calculate hydrodynamic forces: $F_p^{n+1}$

3. Calculate velocity of particles: $u_p^{n+1} = g\left(F_p^{n+1}\right)$

4. Update position of particles: $\Omega_p^{n+1} = f\left(u_p^{n+1}\right)$

5. Align new mesh

→ Required: efficient calculation of hydrodynamic forces
→ Required: efficient treatment of (many) particle interaction
→ Required: efficient (dynamic) grid alignment
→ Required: fast (nonstationary) Navier-Stokes solver FEASTFLOW
Dynamic Adaptation: 2D Sedimentation
3D Examples
3) Benchmarking of Multiphase CFD

- Initiative “Rising Bubble”
  → quantitative validation and comparison of multiphase codes

- Initiative “Elastic FSI”
  → quantitative validation and comparison of monolithic vs. decoupled approaches

- Initiative “Particulate Flow”
  → quantitative validation and comparison with experimental configurations