Edge-Oriented FEM Stabilization Techniques for Incompressible Flow

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- Overview on EOFEM stabilization
- Challenges and unified approach
- Numerical results

Edge-Oriented Stabilization Methods (I)

The following jump terms were introduced (diffusion coefficient ν, parameter γ):

- to achieve the same accuracy for nonconforming SD-FEM as for conforming SD-FEM by John et al. (1997)
- to control the nonconformity arising from the pressure term in **Darcy's law** by Burman and Hansbo (2003)

$$j(\mathbf{u},\mathbf{v}) = \sum_{\text{edge E}} \gamma\nu \frac{1}{|E|} \int_E [\mathbf{n}\cdot\mathbf{u}][\mathbf{n}\cdot\mathbf{v}]d\sigma \qquad (1)$$

- to guarantee **discrete Korn's inequality** by Hansbo and Larson (2002), and by Brenner (2004)

$$j_1(\mathbf{u},\mathbf{v}) = \sum_{\text{edge E}} \gamma\nu \frac{1}{|E|} \int_E [\mathbf{u}][\mathbf{v}]d\sigma \qquad (2)$$

Edge-Oriented Stabilization Methods (II)

- to stabilize **convection dominated** problem by Burman and Hansbo (2003)

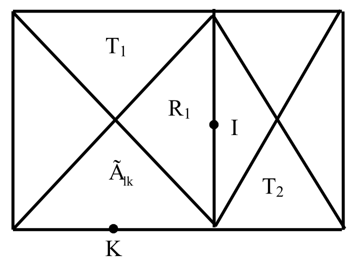
$$
\begin{aligned}
j_{2,\alpha}(\mathbf{u},\mathbf{v}) &= \sum_{\text{edge E}} \gamma |E|^2 \int_E [\nabla \mathbf{u}][\nabla \mathbf{v}] d\sigma \\
j_{3,\alpha}(\mathbf{u},\mathbf{v}) &= \sum_{\text{edge E}} \gamma |E|^2 \int_E [\mathbf{n}\cdot\nabla \mathbf{u}][\mathbf{n}\cdot\nabla \mathbf{v}] d\sigma \\
j_{4,\alpha}(\mathbf{u},\mathbf{v}) &= \sum_{\text{edge E}} \gamma |E|^2 \int_E [\mathbf{t}\cdot\nabla \mathbf{u}][\mathbf{t}\cdot\nabla \mathbf{v}] d\sigma \\
j_{5,\alpha}(\mathbf{u},\mathbf{v}) &= \sum_{\text{edge E}} \gamma |E|^2 \int_E [(\mathbf{t}\cdot\nabla \mathbf{u})\cdot\mathbf{n}][(\mathbf{t}\cdot\nabla \mathbf{v})\cdot\mathbf{n}] d\sigma \\
j_{6,\alpha}(\mathbf{u},\mathbf{v}) &= \sum_{\text{edge E}} \gamma |E|^2 \int_E [(\mathbf{n}\cdot\nabla \mathbf{v})\cdot\mathbf{t}][(\mathbf{n}\cdot\nabla \mathbf{v})\cdot\mathbf{t}] d\sigma
\end{aligned}
\tag{3}
$$

"Classical" Stabilization Methods

- Streamline diffusion (SD)

$$S = \sum_{\tau \in \tau_h} \delta_\tau \int_\tau (\mathbf{u}_h \cdot \nabla \mathbf{v}_h)(\mathbf{u}_h \cdot \nabla \mathbf{w}_h) dx \tag{4}$$

- Samarski's upwind (UPW)



$$S = \sum_l \sum_{k \in \Lambda_l} \oint_{\Gamma_{lk}} \mathbf{u}_h \cdot \mathbf{n}_{lk} d\gamma [1 - \lambda_{lk}(\mathbf{u}_h)(\mathbf{v}_h(m_k) - \mathbf{v}_h(m_l))] w_h(m_l) \tag{5}$$

Based on the local Reynolds number $Re_\tau = \frac{\|\mathbf{u}\|_\tau \cdot h_h}{\nu}$, we can define

$$\delta_\tau = \delta^* \cdot \frac{h_\tau}{\|\mathbf{u}\|_\Omega} \cdot \frac{2Re_\tau}{1 + Re_\tau}, \quad \lambda_{lk}(u_h) = \begin{cases} \dfrac{\frac{1}{2} + \delta^* Re_\tau}{1 + \delta^* Re_\tau} & \textit{\textbf{if}}\ Re_\tau \geq 0 \\ \dfrac{1}{2(1 - \delta^* Re_\tau)} & \textit{\textbf{otherwise}} \end{cases} \tag{6}$$

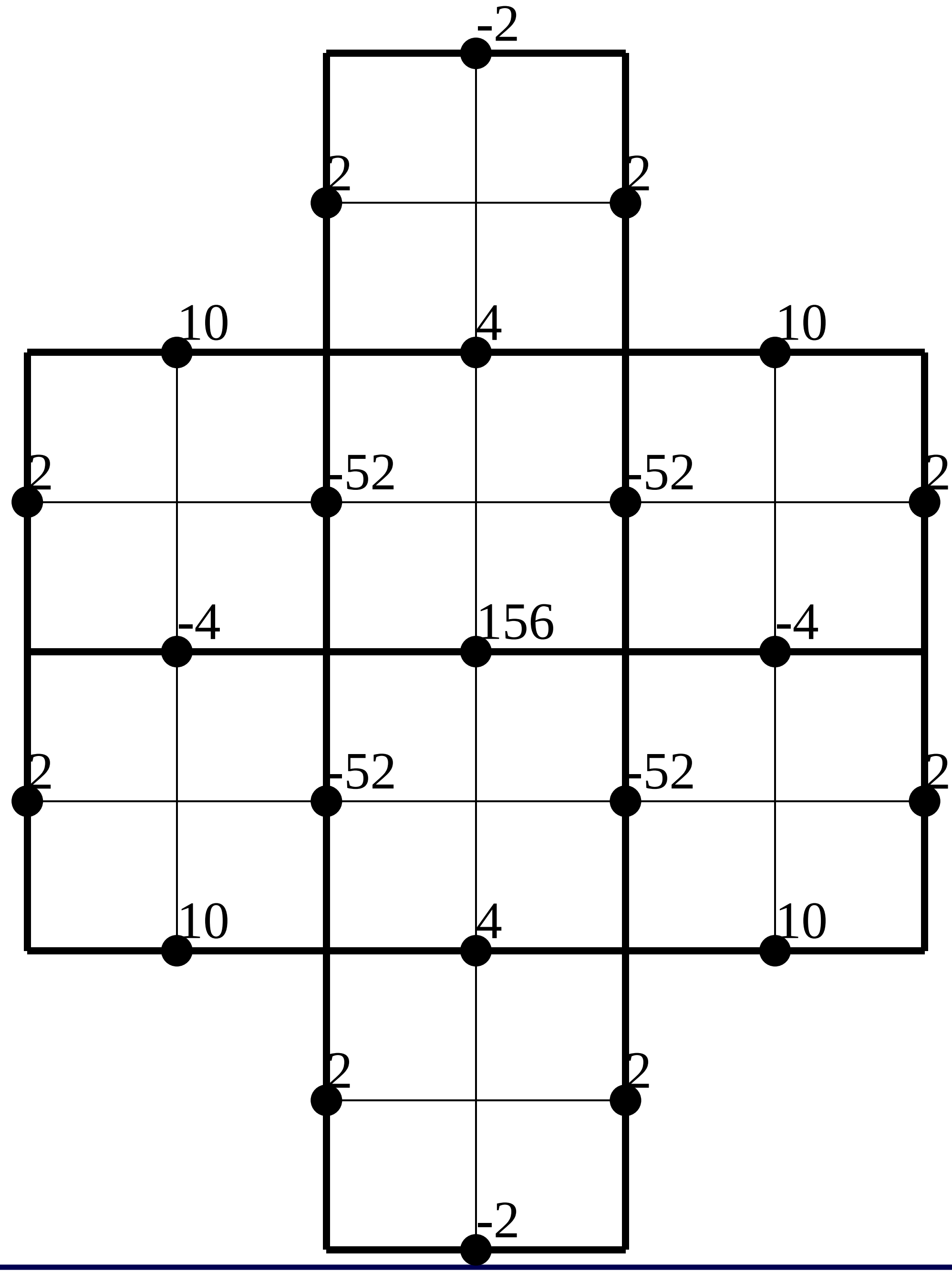
Aspects for (EO)FEM Stabilization

1. **Aspect of necessity:** When do (nonconforming) FEM methods fail?
   - Lack of coercivity for nonconforming low order approximations for symmetric deformation tensor formulation (small Re numbers)
   - Whenever convective operators are dominant (medium and high Re numbers)

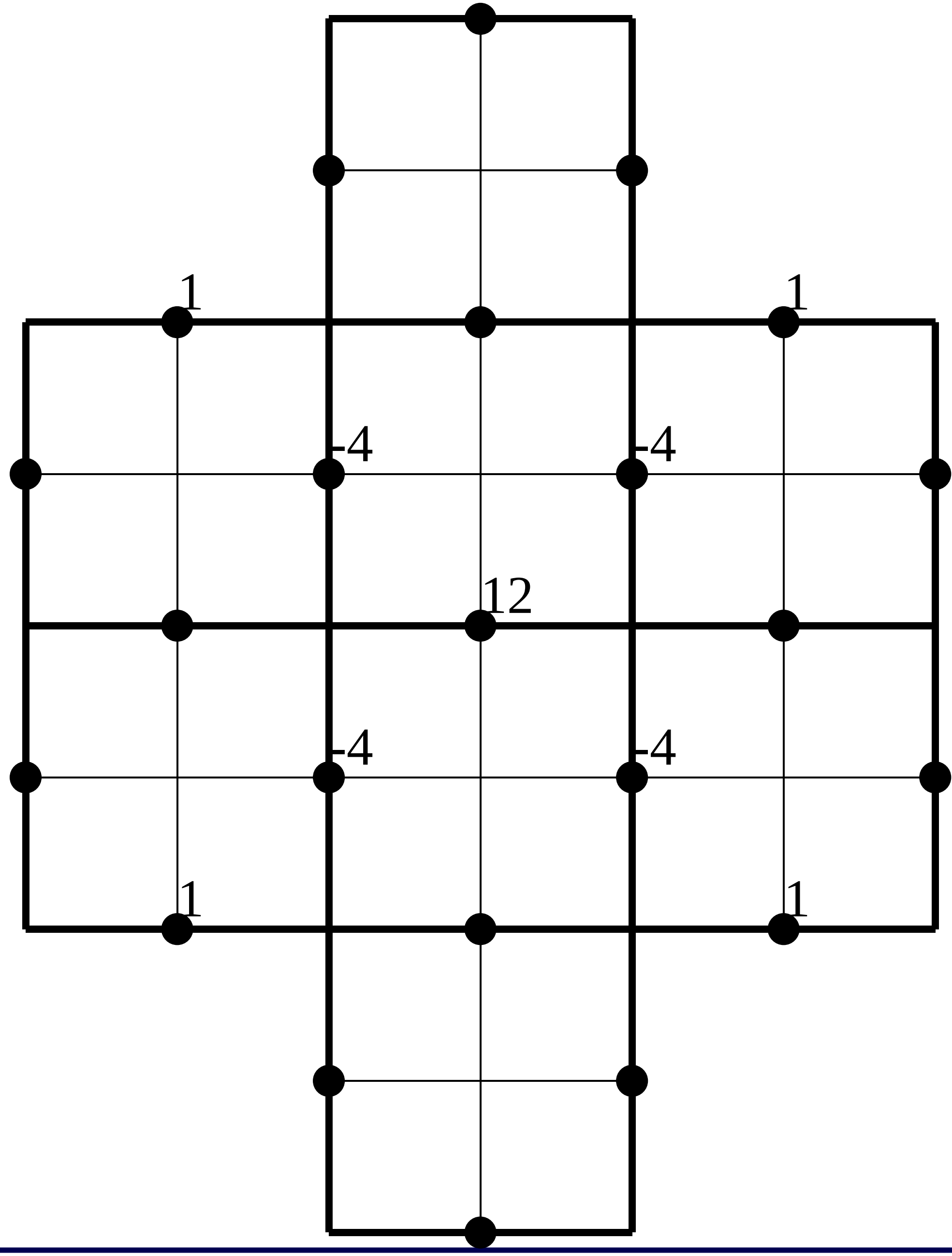
2. **Aspect of robustness:** The ability to handle a wide range of situations?
   - Wide range of Re numbers
   - Higher order finite element spaces
   - User-defined parameters and complex (anisotropic) meshes
   - Time-dependent case
   - Nonlinear material models

3. **Aspect of efficiency:**
   - Multigrid for (4-th order?) FEM stabilisation techniques
   - New sparsity of the matrix and its contradiction to FEM data structures

Modified Sparsity for EOFEM



Modified Sparsity for EOFEM



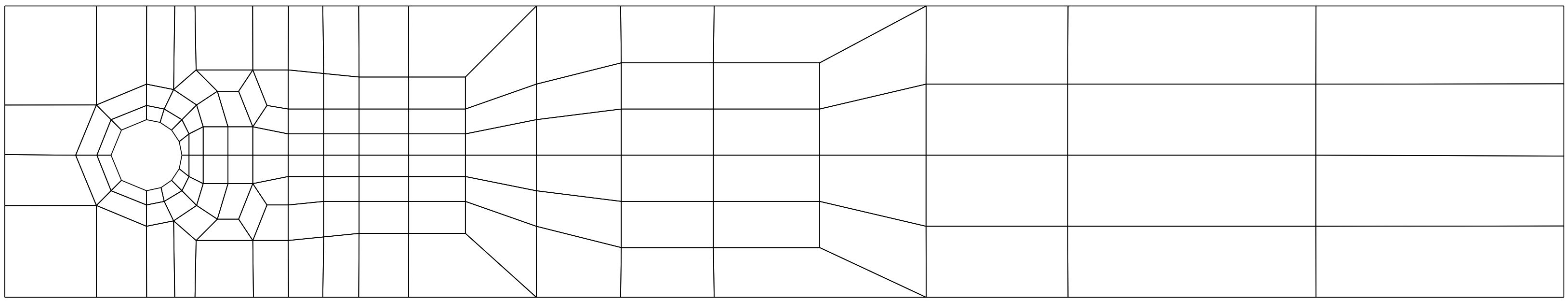
Aim: Unified Approach

- only ”one” jump term for different problems
- ”fixed” constant γ, ”h” on unstructured meshes
- for discrete Korn’s inequality
- for medium and high Reynolds numbers, even for pure transport
- for problems with variable viscosity
- for Q_1, $\tilde{Q}_1$, Q_2 spaces

$$\sum_{\text{edge E}} \max(\gamma\nu|E|, \gamma^*|E|^2) \int_E [\nabla \mathbf{u}][\nabla \mathbf{v}]d\sigma \quad \text{with}\, \gamma, \gamma^* \in [0.0001, 0.1] \tag{7}$$

Different Philosophy: Not looking at local Re/Peclet number or adding residual terms, but checking ”smoothness” of (discrete) solution only!

Benchmark Details



Mesh information			$\tilde{Q}_1/Q_0$
Level	Elements	Vertices	Unknowns
1	156	130	702
2	572	520	2686
3	2184	2080	10608
4	8528	8320	42016
5	33696	33280	167232
6	133952	133120	667264

Stationary Flow around Cylinder: Stokes

deformation formulation							
Stab.	EO		SD+$\gamma j_1(\mathbf{u},\mathbf{v})$		UPW+$\gamma j_1(\mathbf{u},\mathbf{v})$		central
Level	γ		$(\delta^* = 0.1)\gamma$				
	0.001	0.01	0.0	0.1	0.0	0.1	
Drag ($C_D = 3142.4$)							
4	3132.5	3133.6	3132.4	3133.1	3132.4	3133.1	3132.4
5	3139.9	3139.9	3139.9	3140.1	3139.9	3140.1	3139.9
6	3141.8	3141.6	3141.8	3141.8	3141.8	3141.8	3141.8
NL/AVMG							
4	4/3	4/2	4/29	4/2	4/29	4/2	4/29
5	4/3	4/2	5/98	4/2	4/99	4/2	5/98
6	4/3	4/2	5/154	4/2	5/154	4/2	5/154

- Regarding accuracy: SD = UPW = C = EO, **but:** Multigrid!!!
- Edge-oriented stabilization is a must for deformation formulation

Stationary Flow around Cylinder: Stokes

gradient formulation							
Stab.	EO		SD		UPW		Central
Level	γ		δ^*				
	0.001	0.01	0.1	0.5	0.1	1.0	
Drag ($C_D = 3142.4$)							
4	3127.4	3127.4	3127.4	3127.4	3127.4	3127.4	3127.4
5	3138.6	3138.6	3138.6	3138.6	3138.6	3138.6	3138.6
6	3141.5	3141.5	3141.5	3141.5	3141.5	3141.5	3141.5
NL/AVMG							
4	3/2	3/2	3/2	3/2	3/2	3/2	3/2
5	4/2	4/2	4/2	4/2	4/2	4/2	4/2
6	4/2	4/2	4/2	4/2	4/2	4/2	4/2

- No need for any edge stabilization, but no negative side effect

Stationary Flow around Cylinder: $Re = 20$

gradient formulation								
Stab.	EO			SD		UPW		Central
Level	γ			δ^*				
	0.0001	0.001	0.01	0.1	0.5	0.1	1.0	
Drag ($C_D = 5.5795$)								
4	5.5855	5.5864	5.5901	5.6417	5.7977	5.6005	5.7460	5.6040
5	5.5813	5.5815	5.5823	5.6020	5.6655	5.5841	5.6197	5.5862
6	5.5800	5.5800	5.5803	5.5868	5.6092	5.5806	5.5882	5.5812
NL/AVMG								
4	12/3	12/3	12/11	12/3	11/2	11/3	10/3	17/2
5	12/2	12/2	12/9	12/2	12/2	12/2	11/3	12/2
6	12/2	12/2	12/8	12/2	12/2	12/2	12/2	12/2

- SD and UPW are more sensitive w.r.t. the ”free” δ^*
- For EO, only the multigrid solver is slightly sensitive to over-stabilization

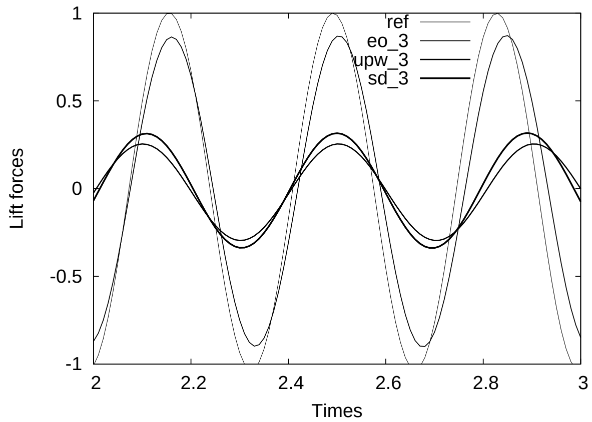
Stationary Flow around Cylinder: $Re = 20$

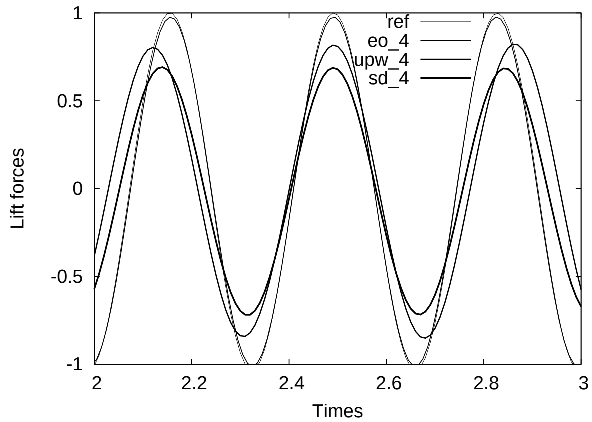
deformation formulation								
Stab.	EO			SD+$\gamma j_1(\mathbf{u},\mathbf{v})$		UPW+$\gamma j_1(\mathbf{u},\mathbf{v})$		central
Level	γ			$(\delta^* = 0.1)\gamma$				
	0.0001	0.001	0.01	0.0	0.1	0.0	0.1	
Drag ($C_D = 5.5795$)								
4	5.5846	5.5838	5.5811	5.6261	5.6264	5.5847	5.5850	5.5865
5	5.5810	5.5807	5.5790	5.5974	5.5975	5.5810	5.5811	5.5814
6	5.5799	5.5798	5.5793	5.5856	5.5856	5.5799	5.5799	5.5800
NL/AVMG								
4	12/3	12/2	12/12	12/2	12/2	12/5	12/2	19/2
5	12/3	12/2	12/8	12/5	12/2	12/11	12/2	21/2
6	12/4	12/2	12/8	12/9	12/2	12/12	12/2	26/4

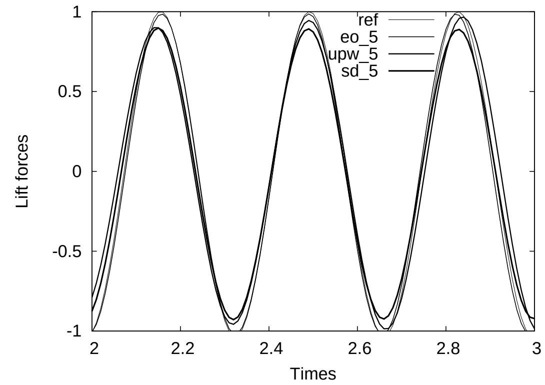
- SD and UPW require additional stabilization for multigrid only
- For EO, there is no need for any additional stabilization

Nonstat. Flow around Cylinder: $Re = 100$

- Lift coefficient for periodically oscillating flow







stab.	EO	SD	UPW	EO	SD	UPW
Level	Maximum amplitude			Strouhal number		
3	0.8750	0.3171	0.2543	0.29126	0.25862	0.23904
4	0.9753	0.6878	0.8214	0.29810	0.26906	0.28436
5	0.9858	0.8864	0.9664	0.30075	0.28708	0.29557
ref	~ 1.0060			~ 0.3020		

- **UPW:** Good results for the amplitude (level 5)
- **EO:** Excellent results for amplitude and frequency!!!

Standing Vortex: $Re = \infty$

Consider the incompressible Navier-Stokes equations for inviscid flow ($Re = \infty$) in a unit square

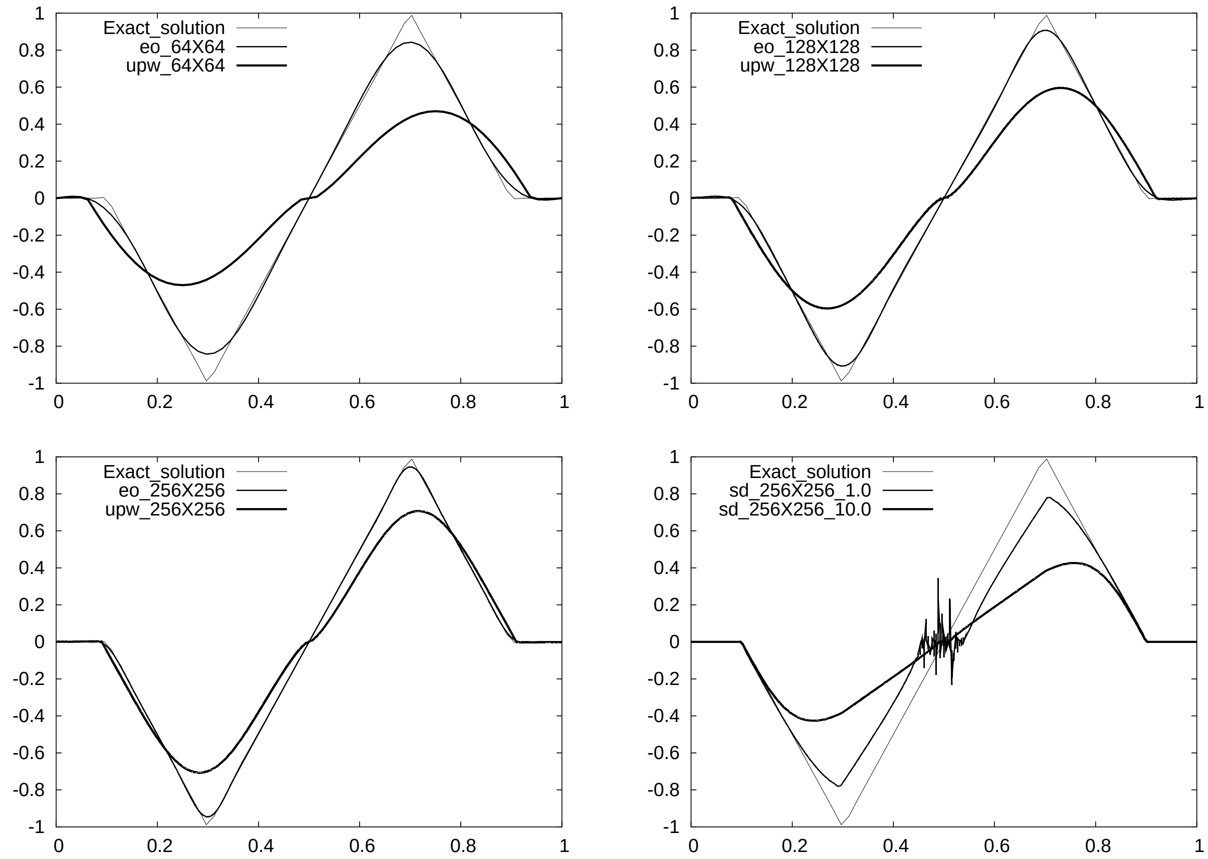
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega = (0,1) \times (0,1). \tag{8}$$

$$\mathbf{u}_r = 0, \quad \mathbf{u}_\theta = \begin{cases} 5r, & r < 0.2, \\ 2 - 5r, & 0.2 \leq r \leq 0.4, \\ 0, & r > 0.4, \end{cases} \tag{9}$$

where $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$ denotes the distance from the center

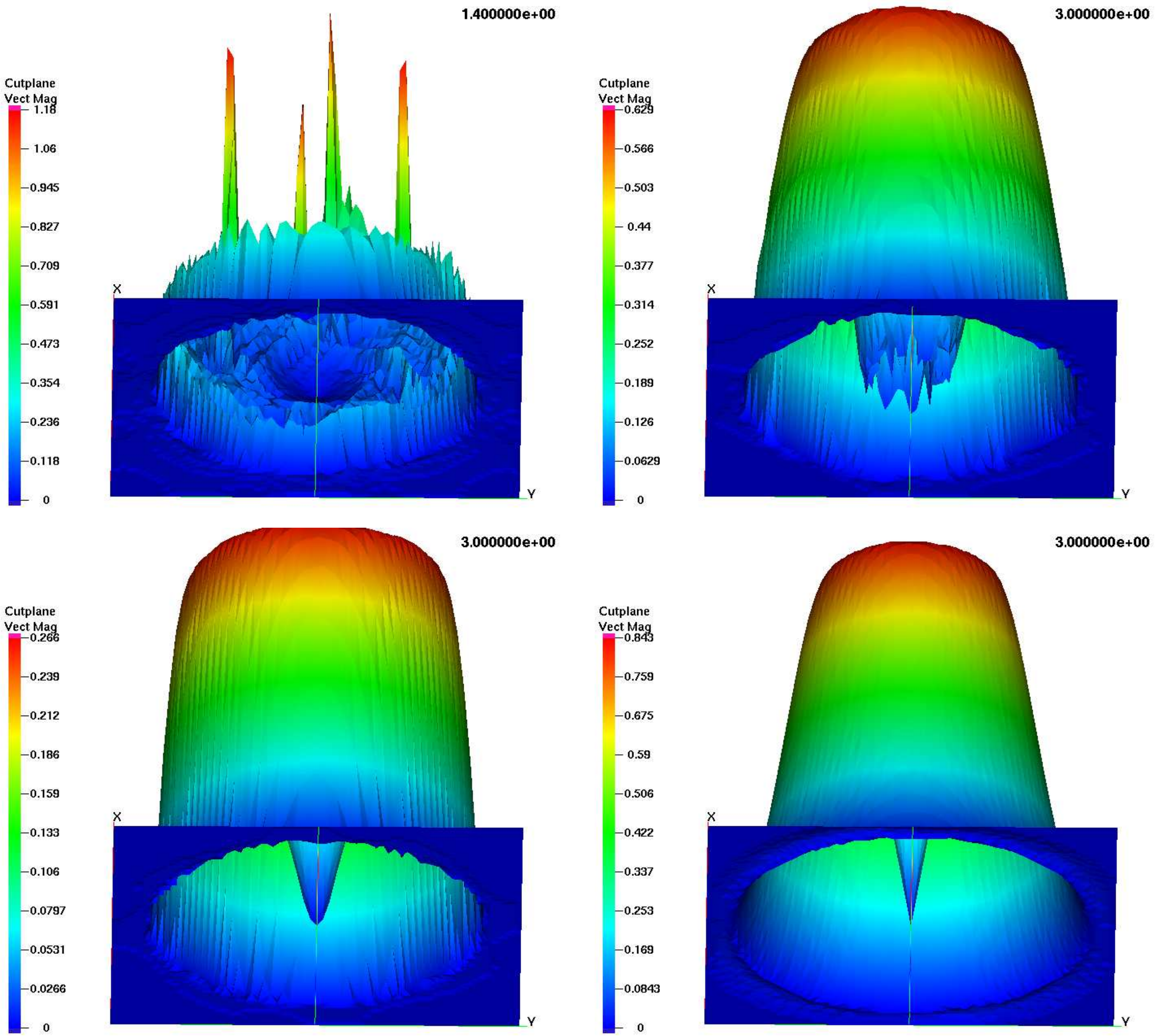
Which discretization schemes preserve the original vortex !?

Standing Vortex: $Re = \infty$

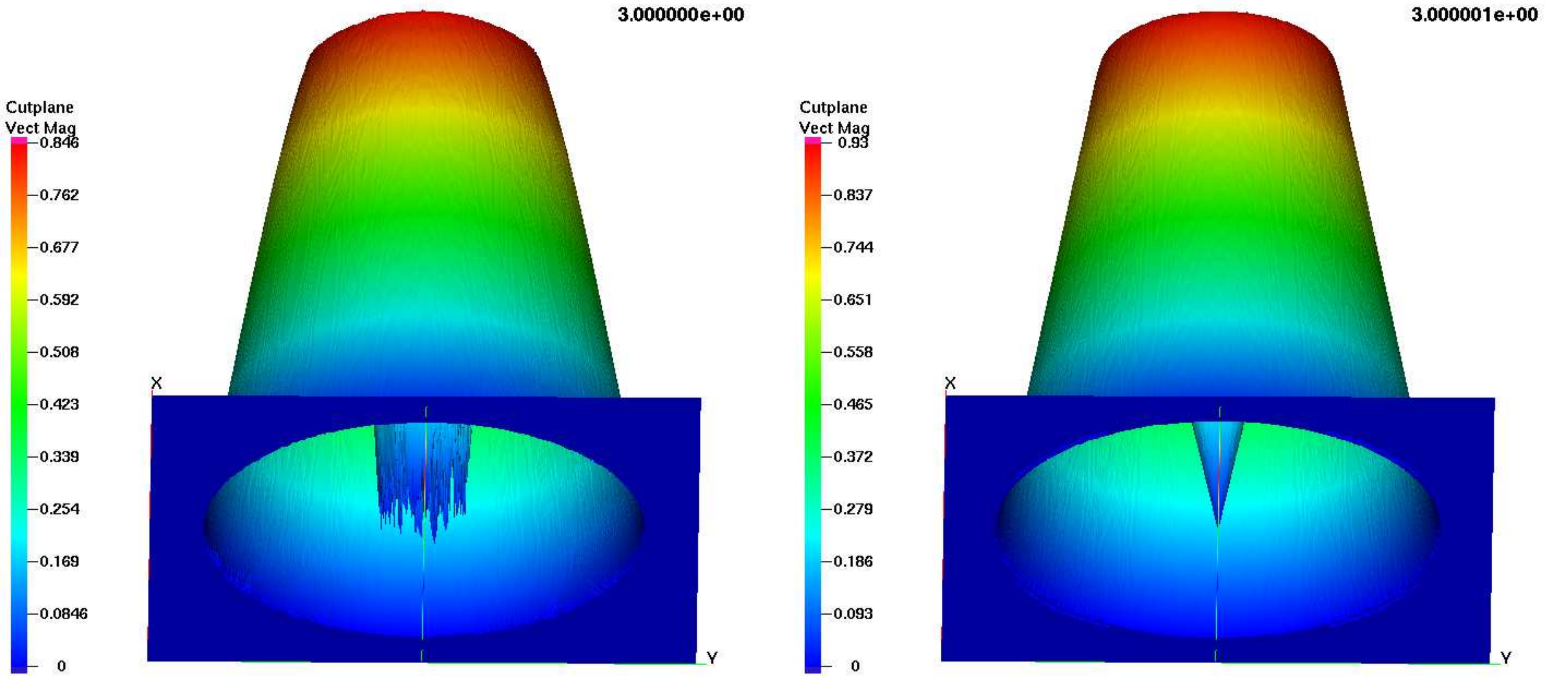


- UPW: significant smearing effects: only first order accuracy
- SD: entropy violating shocks regarding transonic rarefraction solution
- EO: preserves ”perfectly” the solution with high accuracy

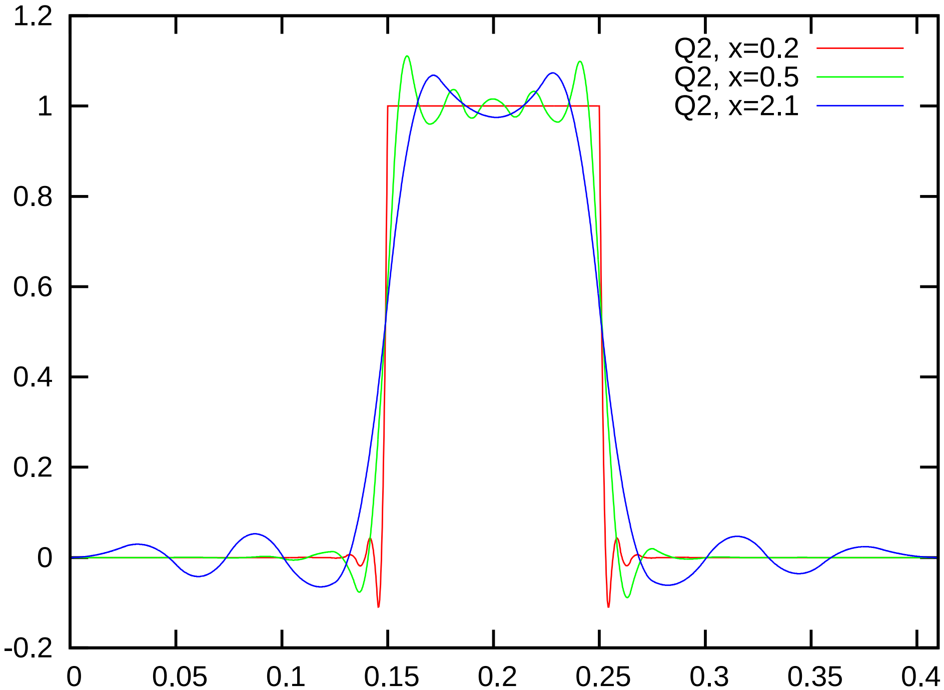
Standing Vortex

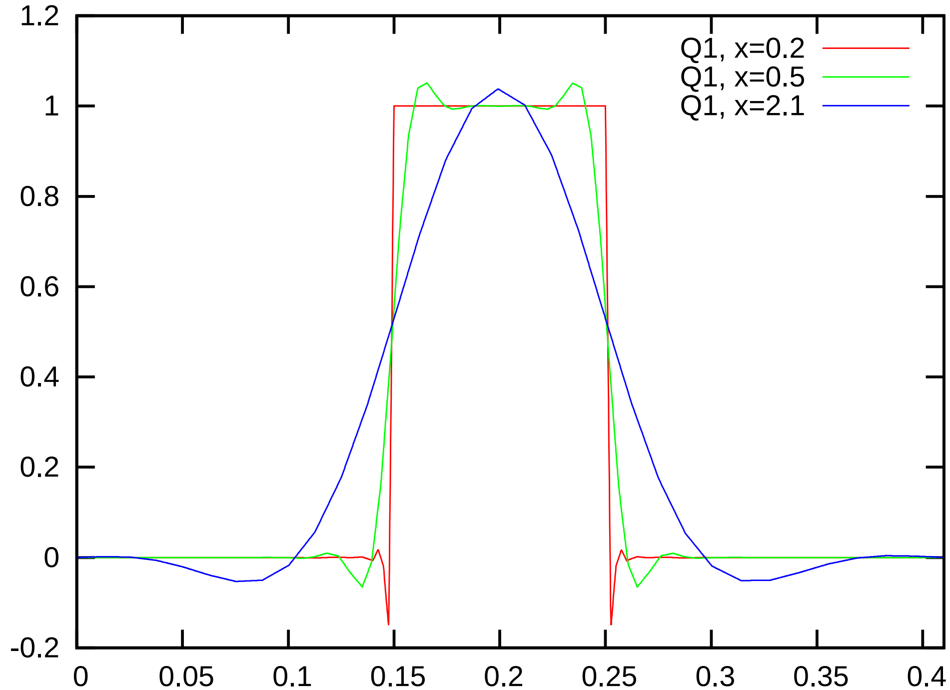


Standing Vortex

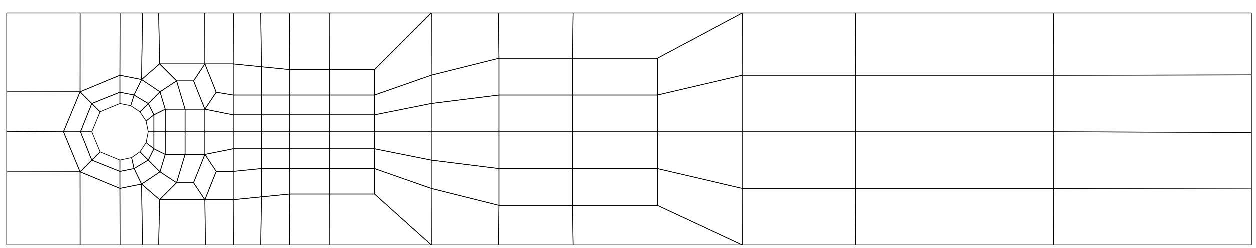


Scalar Transport Problem

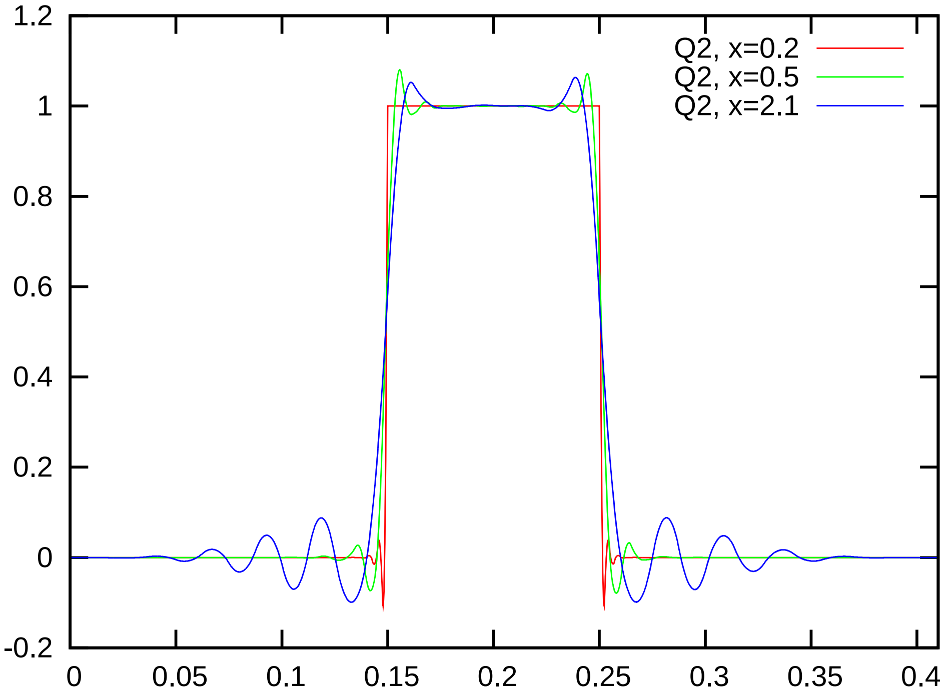


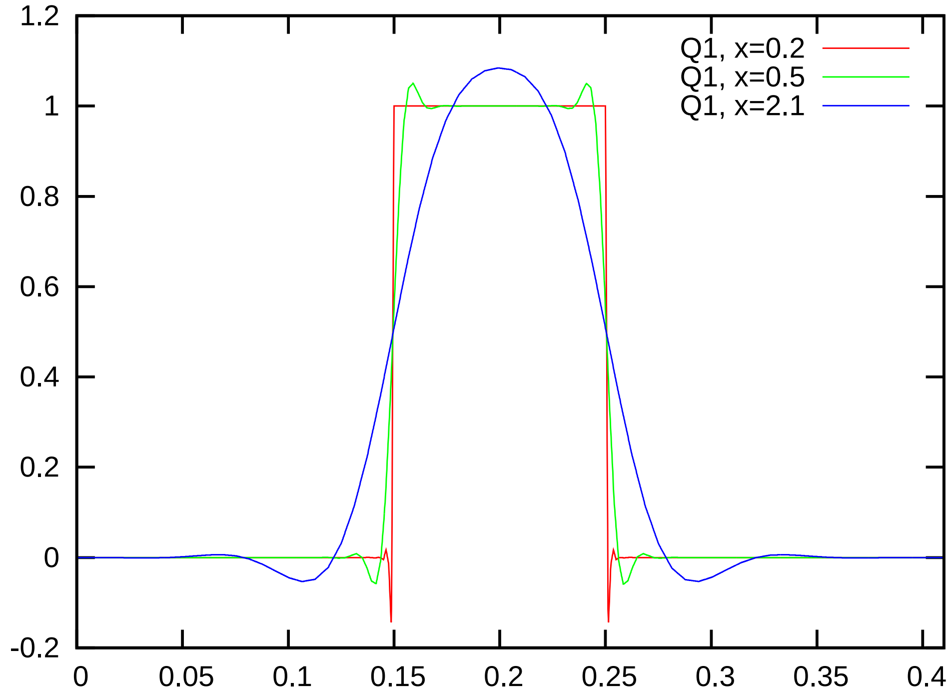


”Only inflow + inhomogeneous b.c.’s on cylinder”

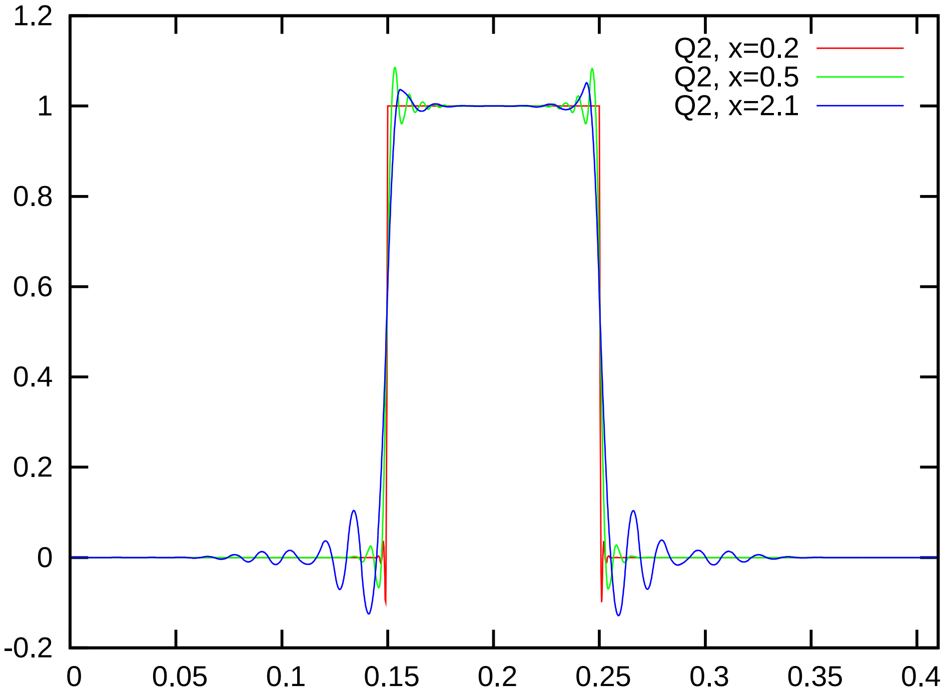


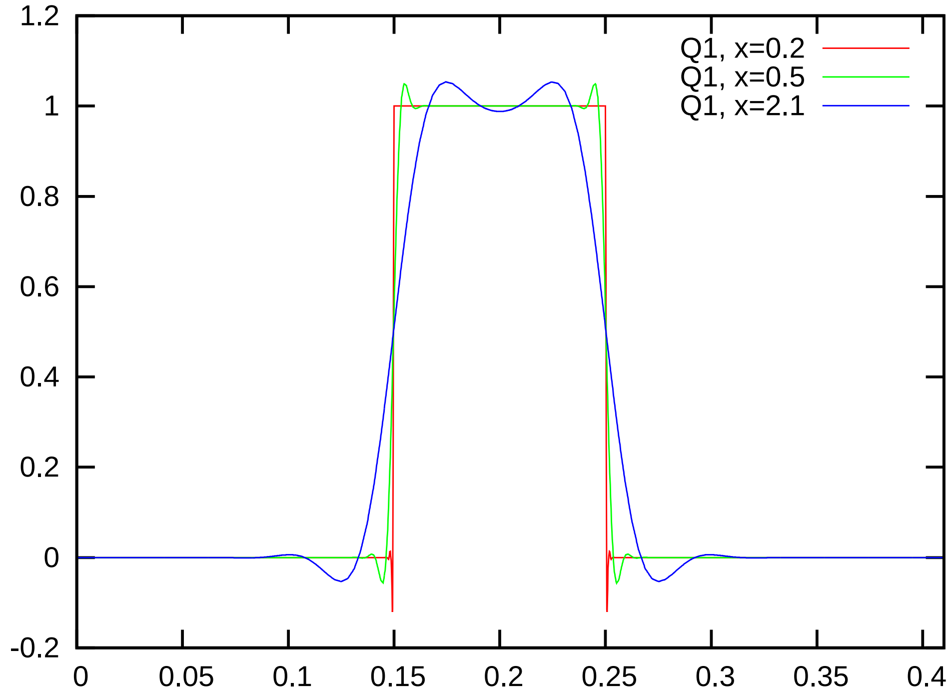
Scalar Transport Problem





Scalar Transport Problem





High accuracy, but problems with steep gradients!

Multigrid Solver

		Level					Level		
El.	ν	4	5	6	El.	ν	3	4	5
Q1	0	0.006	0.030	0.107	Q2	0	0.001	0.001	0.008
	0.0001	0.004	0.014	0.040		0.0001	0.001	0.001	0.002
	0.01	0.002	0.001	0.002		0.01	0.002	0.000	0.001
	1	0.002	0.004	0.004		1	0.009	0.001	0.001

Very efficient multigrid solver possible (Q_2!!!)

Very high accuracy for smooth data (Q_2!!!)

mesh width	ν=0.1		ν=0	
$h = 1/32$	4.95-5	1.03-3	6.12-5	1.17-3
$h = 1/64$	5.75-6	2.57-4	6.06-6	2.66-4
$h = 1/128$	6.88-7	6.42-5	6.97-7	6.48-5

Summary

- EO can handle problems in the limit of inviscid flow, resp., pure transport
- EO stabilizes the lack of coercivity for deformation tensor formulation
- EO applies to nonlinear viscosity (Power Law, Bingham, Schaeffer)
- Unified EOFEM stabilization for all relevant Re numbers is possible
- Higher order accuracy and fast multigrid for higher order FEM is possible
- But: special FEM data structure is necessary
- But: additional nonlinear(!) TVD/Shock-capturing is needed

The proposed EOFEM stabilisation is a ”new” candidate for a black box tool and can be applied with many FEM spaces to a large number of problems

Further improvement for grid adaptivity/alignment and transport problems with steep gradients (”shock-capturing”) is required