Edge-Oriented FEM Stabilization

Accuracy, Robustness and Efficient Solvers

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- Overview on EO-FEM stabilization
- Numerical analysis + efficient solvers
- *Optimal convergence for quadratic FEM*

Edge-Oriented (EO) Stabilization Methods

EO is a new FEM technique which introduces **additional ”jump terms”** into the weak formulation in a consistent way (Brenner, Burman, Hansbo, John, Larson, T.):

- to control the **nonconformity** arising from the pressure term in Darcy’s law

$$j(\mathbf{u}, \mathbf{v}) = \sum_{\text{edge E}} \gamma \nu \frac{1}{|E|} \int_E [\mathbf{n} \cdot \mathbf{u}][\mathbf{n} \cdot \mathbf{v}] d\sigma \tag{1}$$

- to guarantee discrete **Korn’s inequality** for **nonconforming FEM**

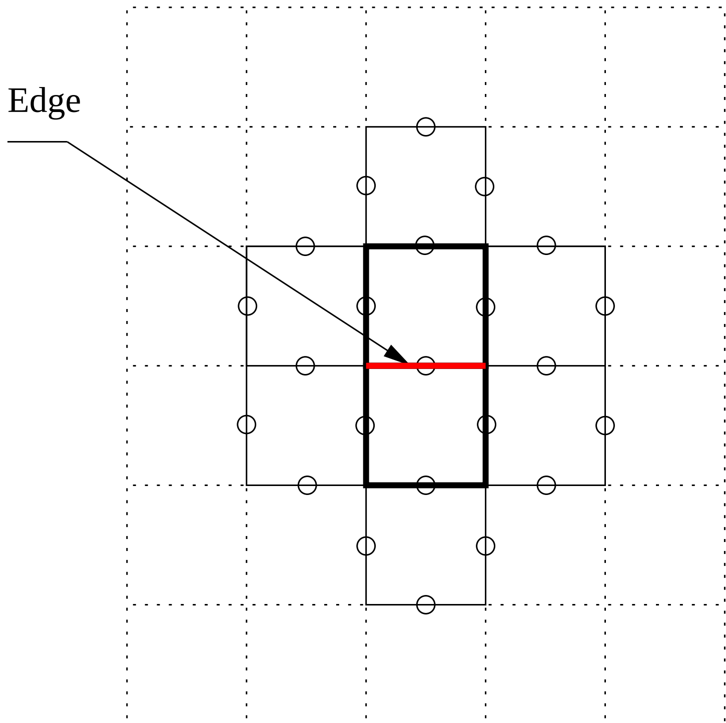
$$j_1(\mathbf{u}, \mathbf{v}) = \sum_{\text{edge E}} \gamma \nu \frac{1}{|E|} \int_E [\mathbf{u}][\mathbf{v}] d\sigma \tag{2}$$

- to stabilize **convection** dominated problem (for **all FEM**)

$$j_2(\mathbf{u}, \mathbf{v}) = \sum_{\text{edge E}} \gamma |E|^\alpha \int_E [\nabla \mathbf{u}][\nabla \mathbf{v}] d\sigma \tag{3}$$

Challenges for EO-FEM Stabilization

- **Robustness** on complex (anisotropic) meshes
- Time-dependent case + **higher order finite element spaces**
- Efficient FEM data structures for ”new” **sparsity**
- **Multigrid** for (4-th order?) EO-FEM stabilisation



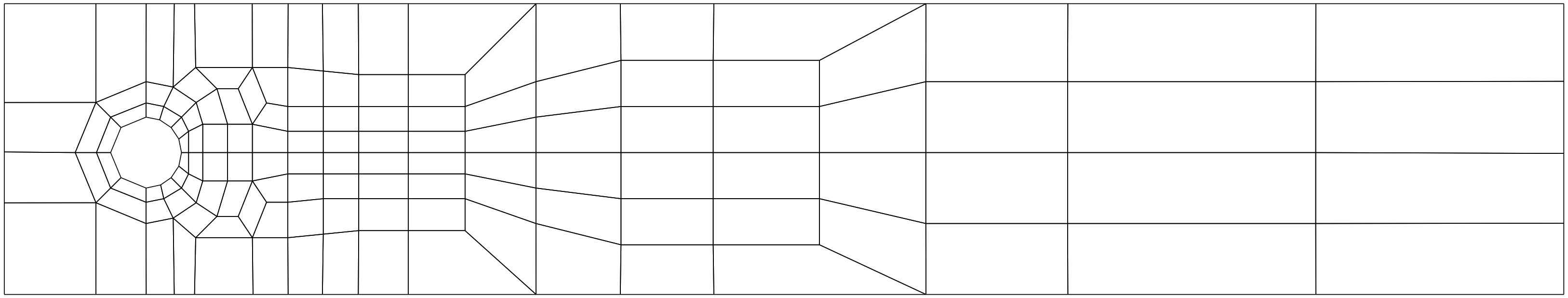
New: Unified Approach

$$\sum_{\text{edge E}} \max(\gamma\nu|E|, \gamma^*|E|^2) \int_E [\nabla \mathbf{u}][\nabla \mathbf{v}] d\sigma$$

- ”fixed” constants γ, γ^*, no ”h_K” on unstructured meshes
- for discrete Korn’s inequality
- for medium/high Reynolds numbers, even for pure transport
- for Q_1, $\tilde{Q}_1$, Q_2 FEM spaces

Different Philosophy: Not looking at ”local Re/Peclet number”, but checking ”smoothness” of (discrete) solution only!

Numerical Analysis



Mesh information			$\tilde{Q}_1/Q_0$
Level	Elements	Vertices	Unknowns
1	156	130	702
2	572	520	2686
3	2184	2080	10608
4	8528	8320	42016
5	33696	33280	167232
6	**133952**	**133120**	**667264**

Outer Fixed-Point iteration + Oseen solver with Vanka smoother, resp., local MPSC

Stationary Flow around Cylinder: Stokes

deformation formulation							
Stab.	EO		SD+$\gamma j_1(\mathbf{u},\mathbf{v})$		UPW+$\gamma j_1(\mathbf{u},\mathbf{v})$		central
Level	γ		γ				
	0.001	0.01	0.0	0.1	0.0	0.1	
Drag ($C_D = 3142.4$)							
4	3132.5	3133.6	3132.4	3133.1	3132.4	3133.1	3132.4
5	3139.9	3139.9	3139.9	3140.1	3139.9	3140.1	3139.9
6	3141.8	3141.6	3141.8	3141.8	3141.8	3141.8	3141.8
NL/AVMG							
4	4/3	4/2	**4/29**	4/2	**4/29**	4/2	**4/29**
5	4/3	4/2	**5/98**	4/2	**4/99**	4/2	**5/98**
6	4/3	4/2	**5/154**	4/2	**5/154**	4/2	**5/154**

- Regarding accuracy: SD = UPW = C = EO, **but: Multigrid!!!**
- EO stabilization is a must for nonconforming FEM with multigrid

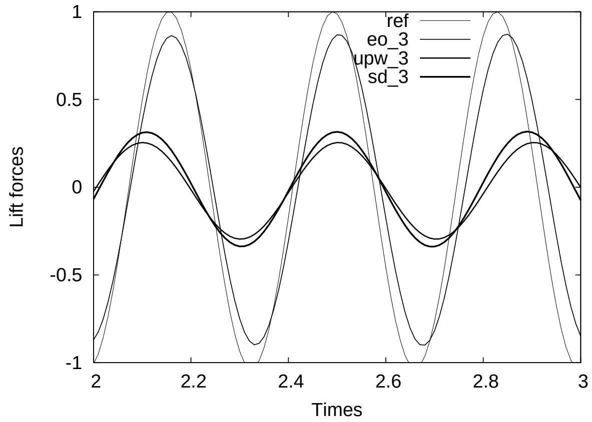
Stationary Flow around Cylinder: $Re = 20$

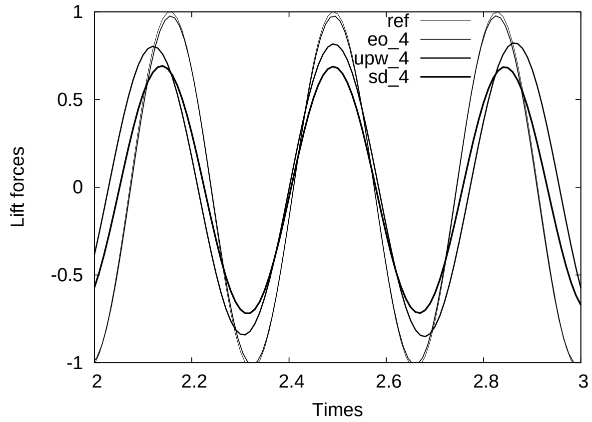
deformation formulation								
Stab.	EO			SD+$\gamma j_1(\mathbf{u},\mathbf{v})$		UPW+$\gamma j_1(\mathbf{u},\mathbf{v})$		central
Level	γ			γ				
	0.0001	0.001	0.01	0.0	0.1	0.0	0.1	
Drag ($C_D = 5.5795$)								
4	5.5846	5.5838	5.5811	5.6261	5.6264	5.5847	5.5850	5.5865
5	5.5810	5.5807	5.5790	5.5974	5.5975	5.5810	5.5811	5.5814
6	5.5799	5.5798	5.5793	5.5856	5.5856	5.5799	5.5799	5.5800
NL/AVMG								
4	12/3	12/2	12/12	**12/2**	12/2	**12/5**	12/2	19/2
5	12/3	12/2	12/8	**12/5**	12/2	**12/11**	12/2	21/2
6	12/4	12/2	12/8	**12/9**	12/2	**12/12**	12/2	26/4

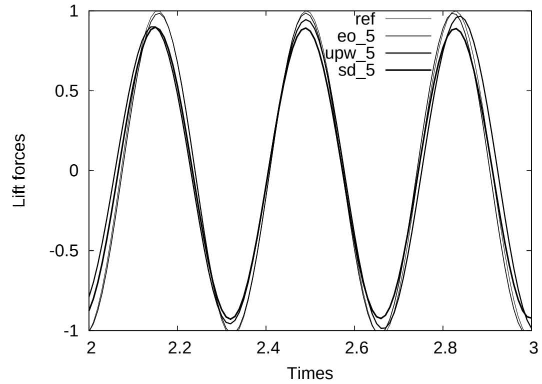
- SD and UPW require additional stabilization (EO) for multigrid only
- For EO, there is no need for any additional stabilization + more accurate

Nonstat. Flow around Cylinder: $Re = 100$

Lift coefficient for periodically oscillating flow







Excellent results for amplitude and frequency!!!

Similar multigrid behaviour as for SD and UPW!

Standing Vortex: $Re = \infty$

Incompressible Navier-Stokes equations for inviscid flow ($Re = \infty$) in a unit square

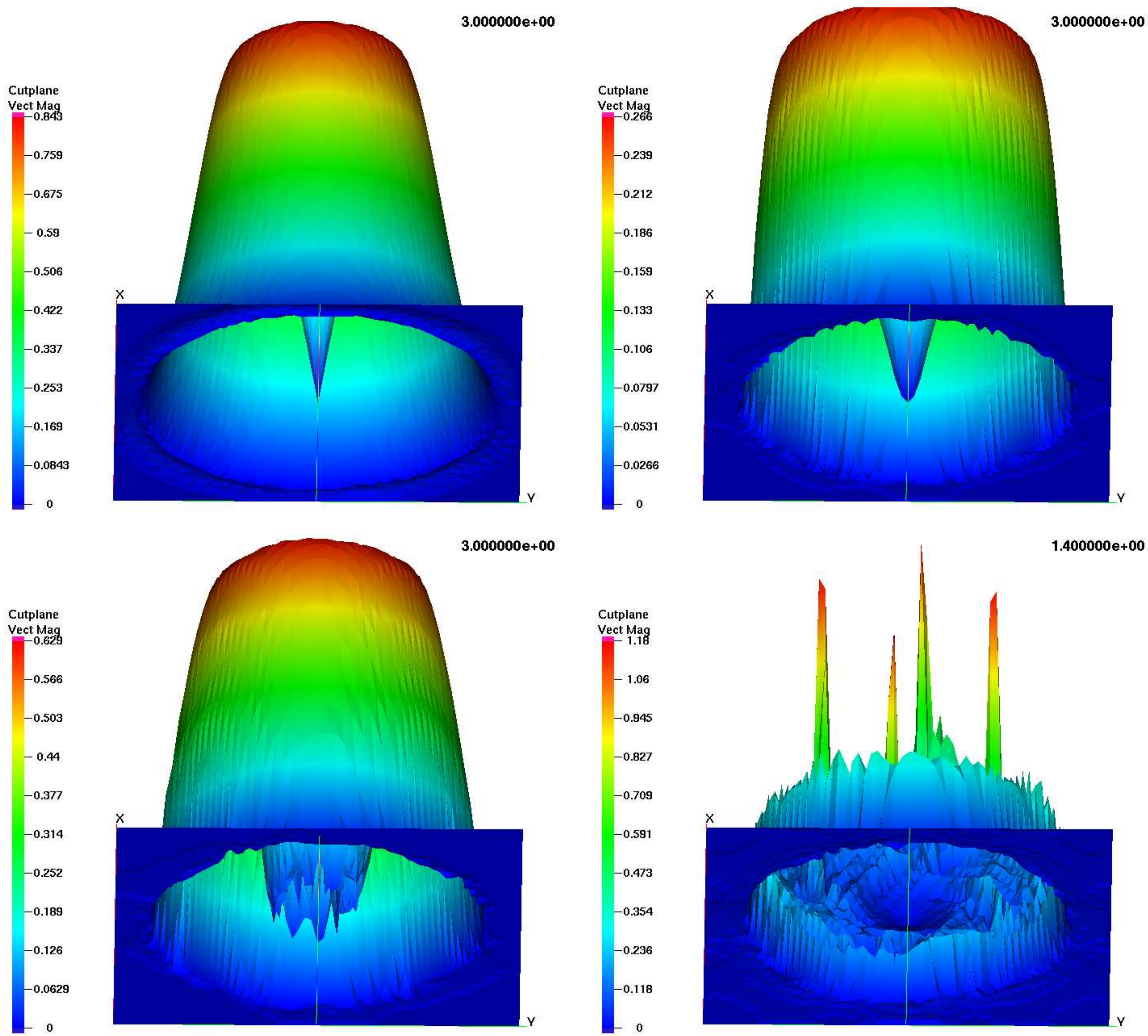
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega = (0,1) \times (0,1). \tag{4}$$

$$\mathbf{u}_r = 0, \quad \mathbf{u}_\theta = \begin{cases} 5r, & r < 0.2, \\ 2 - 5r, & 0.2 \leq r \leq 0.4, \\ 0, & r > 0.4, \end{cases} \tag{5}$$

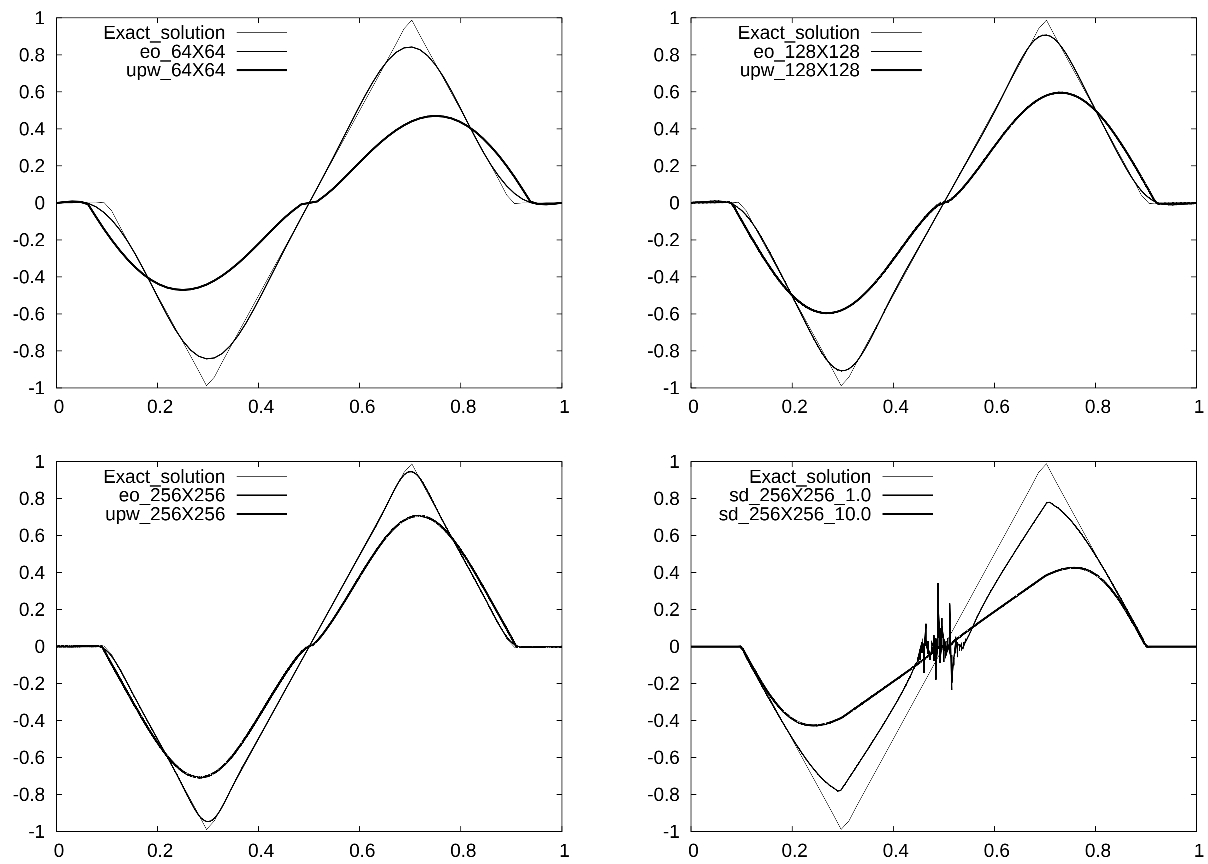
where $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$ denotes the distance from the center

Which discretization schemes preserve the original vortex !?

Standing Vortex: $Re = \infty$

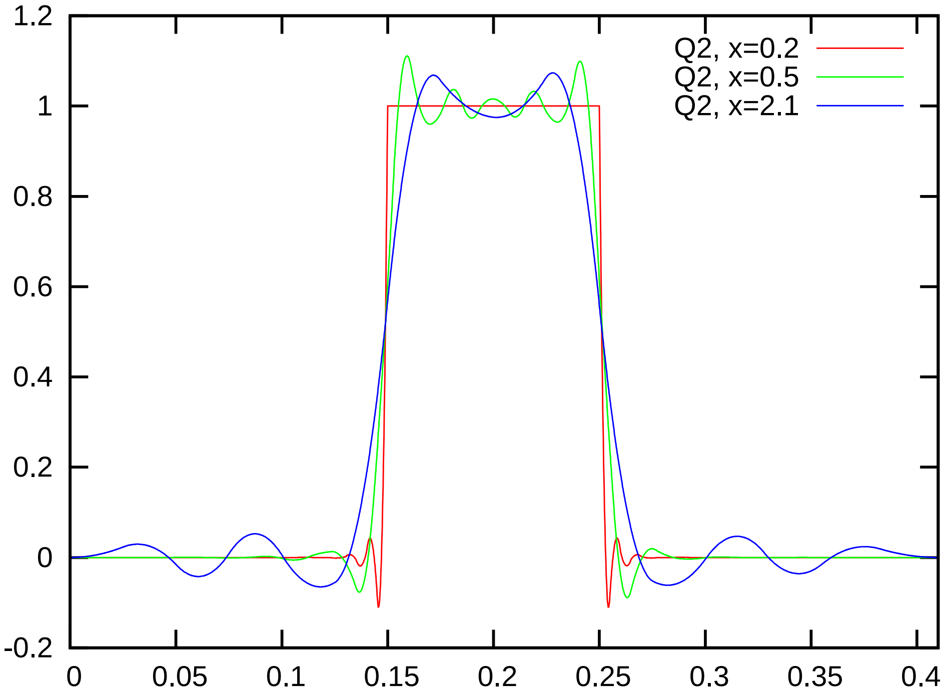


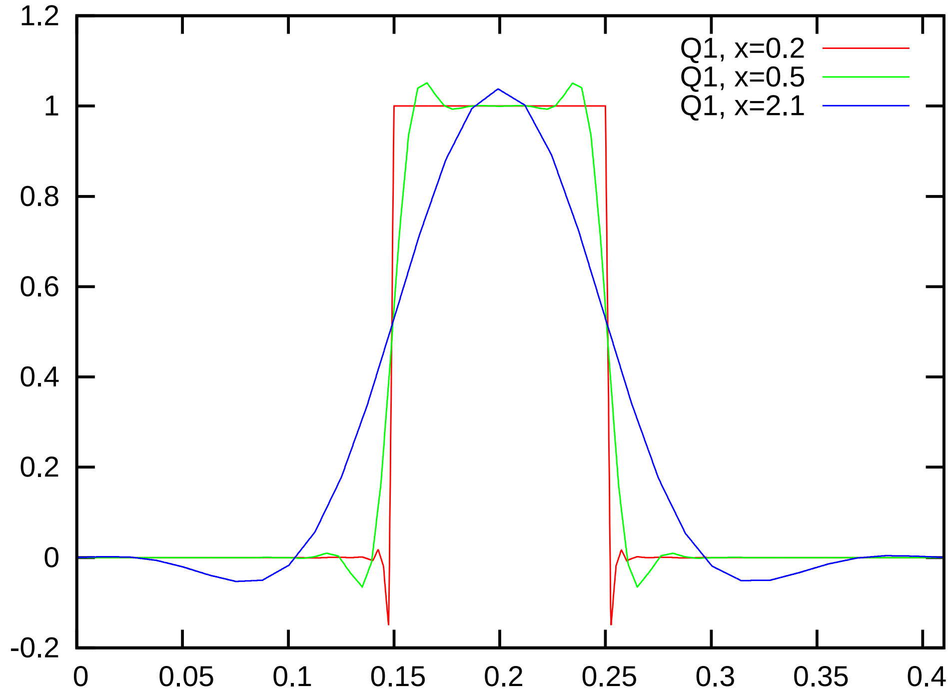
Standing Vortex: $Re = \infty$



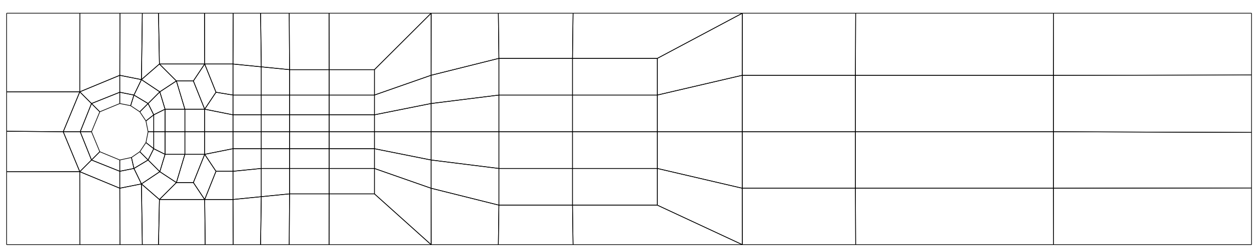
- UPW: significant smearing effects: only first order accuracy
- SD: entropy violating shocks regarding transonic rarefraction solution
- EO: preserves ”perfectly” the solution with high accuracy

Scalar Transport Problem

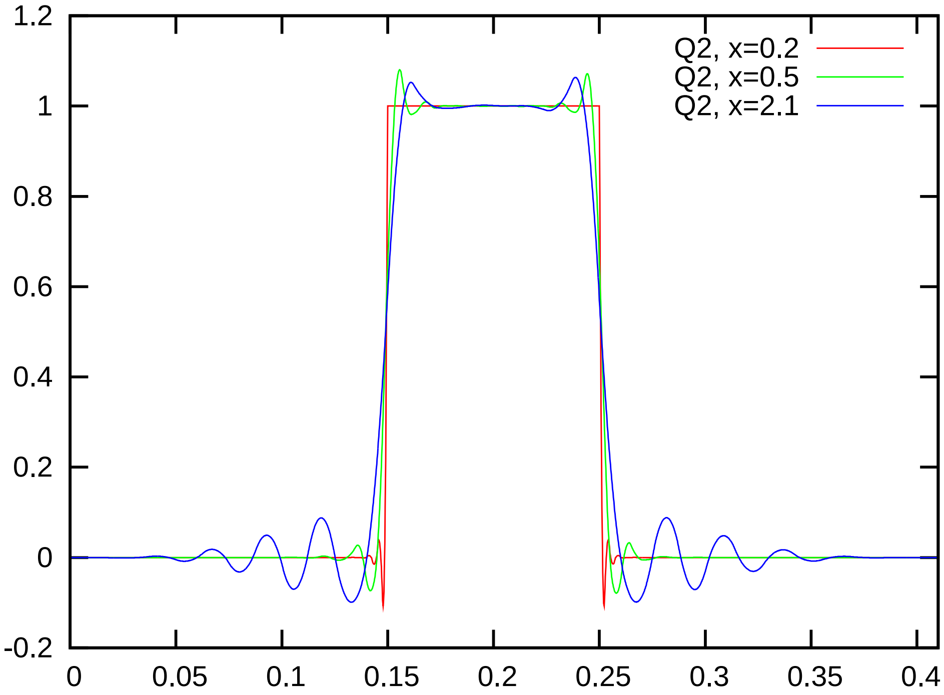


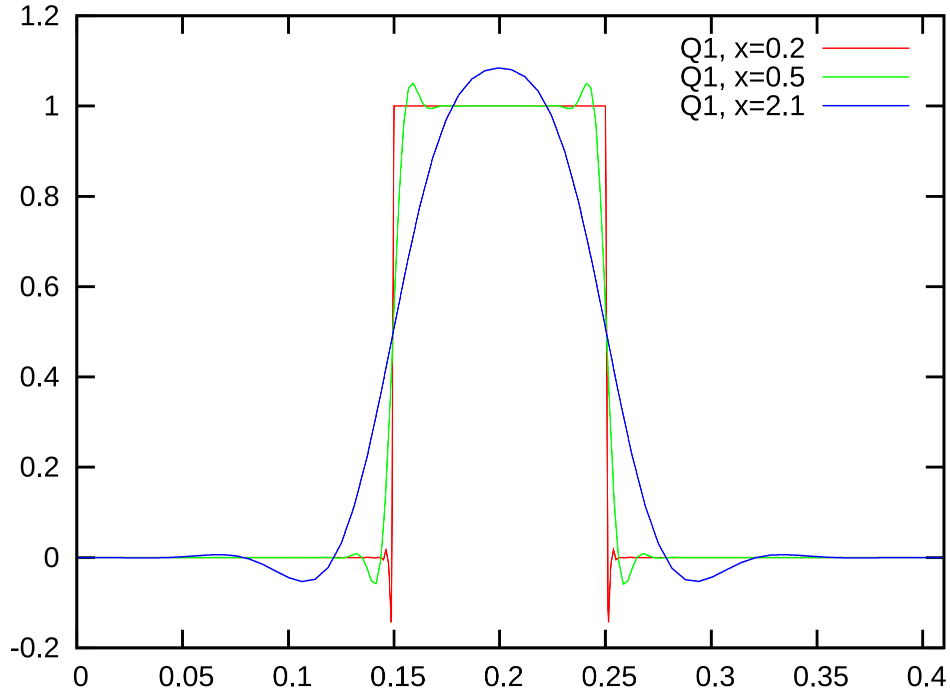


"Only inflow + inhomogeneous b.c.'s on cylinder"

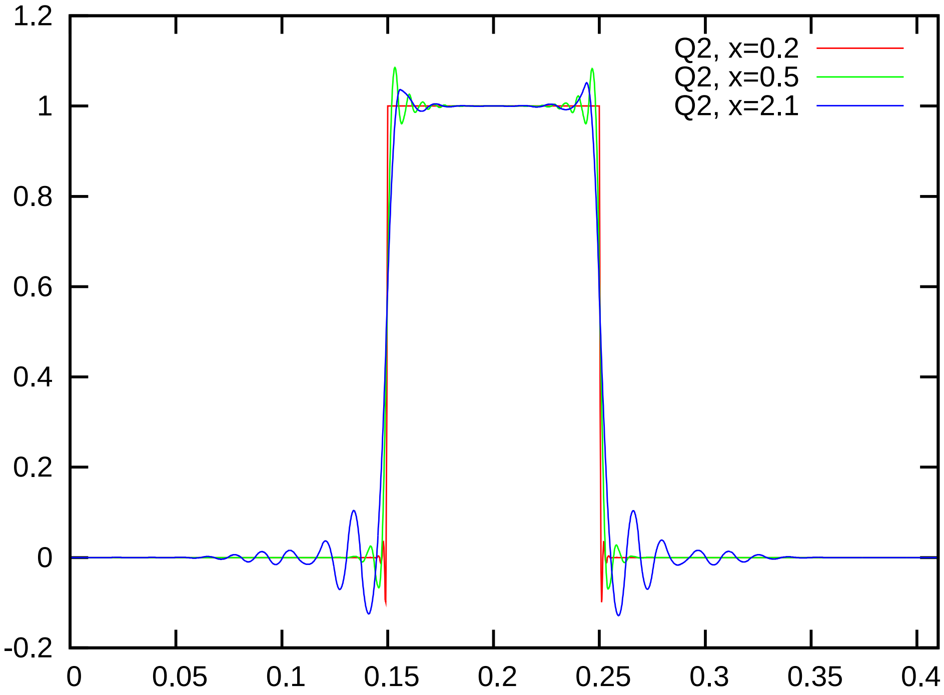


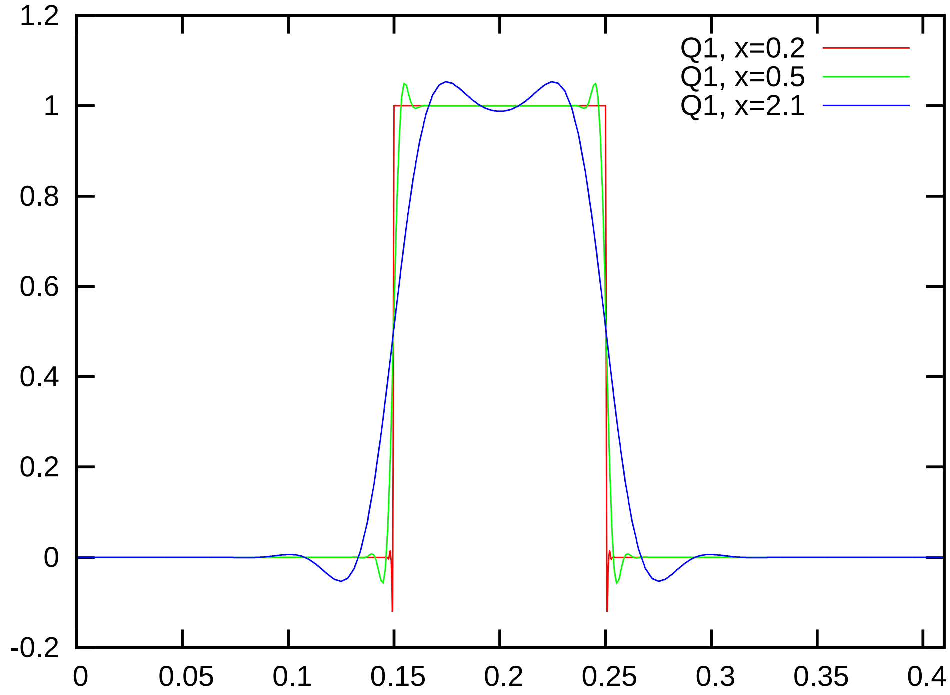
Scalar Transport Problem





Scalar Transport Problem





High accuracy with Q_2, but problems with steep gradients (also for Q_1)!

(Additional nonlinear(!) TVD/Shock-Capturing required!)

Multigrid Solver

		Level					Level		
El.	ν	4	5	6	El.	ν	3	4	5
Q1	0	0.006	0.030	0.107	Q2	0	0.001	0.001	0.008
	0.0001	0.004	0.014	0.040		0.0001	0.001	0.001	0.002
	0.01	0.002	0.001	0.002		0.01	0.002	0.000	0.001
	1	0.002	0.004	0.004		1	0.009	0.001	0.001

Very efficient multigrid solver possible (Q_2!!!)

Why is multigrid for Q_2 better than for Q_1 ???

Standard Geometric Multigrid

- 2nd order elliptic boundary problem with H^2 regularity

$$\begin{aligned} -\Delta u = f & \qquad \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \qquad \text{on } \partial\Omega \end{aligned}$$

- approximation with conforming finite elements
- μ smoothing steps with damped Richardson method

$$|||u_h - u_h^{\mu+1}||| \quad \leq \quad \frac{c}{\mu} |||u_h - u_h^0|||$$

Doubling the number of smoothing steps should halfen the convergence rate

Averaged (last l steps) Rates: Q_1

m	ITE	ρ	l=2	l=4	l=8
4	25	7,707E-02			
6	22	5,279E-02			
8	21	4,027E-02	1,914		
12	19	2,723E-02	1,938		
16	17	2,044E-02	1,970	1,942	
24	16	1,340E-02	2,032	1,985	
32	15	1,030E-02	1,984	1,977	1,956
48	14	6,539E-03	2,049	2,041	2,006
64	13	5,026E-03	2,049	2,017	2,001
96	12	2,766E-03	2,364	2,201	2,143
128	12	2,204E-03	2,280	2,162	2,101
192	11	1,464E-03	1,889	2,113	2,092

$\Rightarrow$ Factor ≈ 2 as expected

Q_2 with Bilinear ("h/2") Interpolation

m	**ITE**	ρ	**l=2**	**l=4**	**l=8**
4	30	2,931E-01			
6	30	2,258E-01			
8	30	1,808E-01	1,621		
12	30	1,287E-01	1,754		
16	28	1,032E-01	1,752	1,685	
24	25	7,368E-02	1,747	1,751	
32	23	5,702E-02	1,810	1,781	1,726
48	20	3,916E-02	1,881	1,813	1,793
64	19	2,983E-02	1,911	1,860	1,823
96	17	2,021E-02	1,938	1,910	1,854
128	16	1,503E-02	1,985	1,948	1,901
192	15	1,032E-02	1,958	1,948	1,926

$\Rightarrow$ Factor ≈ 2 (but worse rates!)

Q_2 with Biquadratic Interpolation

m	ITE	ρ	l=2	l=4	l=8
4	30	1,426E-01			
6	26	8,112E-02			
8	23	6,391E-02	2,232		
12	20	4,032E-02	2,012		
16	18	2,554E-02	2,502	2,363	
24	14	1,011E-02	3,988	2,833	
32	13	4,971E-03	5,138	3,586	3,062
48	11	1,990E-03	5,079	4,501	3,442
64	10	1,167E-03	4,259	4,678	3,797
96	9	5,515E-04	3,609	4,281	4,181
128	9	3,370E-04	3,463	3,841	4,232
192	8	1,576E-04	3,499	3,553	4,003

$\Rightarrow$ Factor ≈ 4!!!

Multigrid for Quadratic FEM

Assumptions:

- H^3 regular problem
- **natural inclusion** (= quadratic interpolation) as grid transfer operator
- two-grid or W-cycle multigrid, μ smoothing steps
- Scale of norms, $s \in \mathbb{R}$:

$$|||u_h|||_s := \sqrt{(\mathcal{A}_h^s v_h, v_h)} \quad , \quad |||\cdot|||_1 \sim \|\cdot\|_1 \, , \, |||\cdot|||_0 = \|\cdot\|_0$$

Main result:

$$\boxed{|||u_h - u_h^{\mu+1}|||_{-1} \leq \frac{c}{\mu^2} |||u_h - u_h^0|||_{-1}}$$

Components of Multigrid Proof

With the error:

$$e_m := u_h - u_h^m$$

a) Smoothing property:

$$|||e_\mu|||_3 \leq \frac{c}{\mu^2}\, h^{-4}\, |||e_0|||_{-1}$$

b) Approximation property:

$$|||e_{\mu+1}|||_{-1} \leq c\, h^4\, |||e_\mu|||_3$$

FEM Spaces of Order s ($s > 2$)

Required key inequality:

??? $$\boxed{|||v_h|||_{1-s} \ \leq\ c\,\|v_h\|_{1-s}}$$ or $$\boxed{\|v_h\|_{s-1} \leq\ c\,|||v_h|||_{s-1}}$$???

In that case: If the problem is H^{s+1} regular, s polynomial degree:

$$\boxed{|||e_{\mu+1}|||_{1-s} \ \leq\ \frac{c}{\mu^s}\,|||e_0|||_{1-s}}$$

$\Rightarrow$ **Higher order FEM can be solved much faster!?**

Summary

The proposed EO-FEM stabilisation is a ”new” candidate for a black box tool and can be applied, with many FEM spaces, to a large number of problems

Further improvement for transport problems with steep gradients (”shock-capturing”) is required

Together with appropriate data structures, very efficient multigrid solvers are feasible

Multigrid for hp-FEM ???