Edge-Oriented FEM Stabilization Techniques for Incompressible Flow

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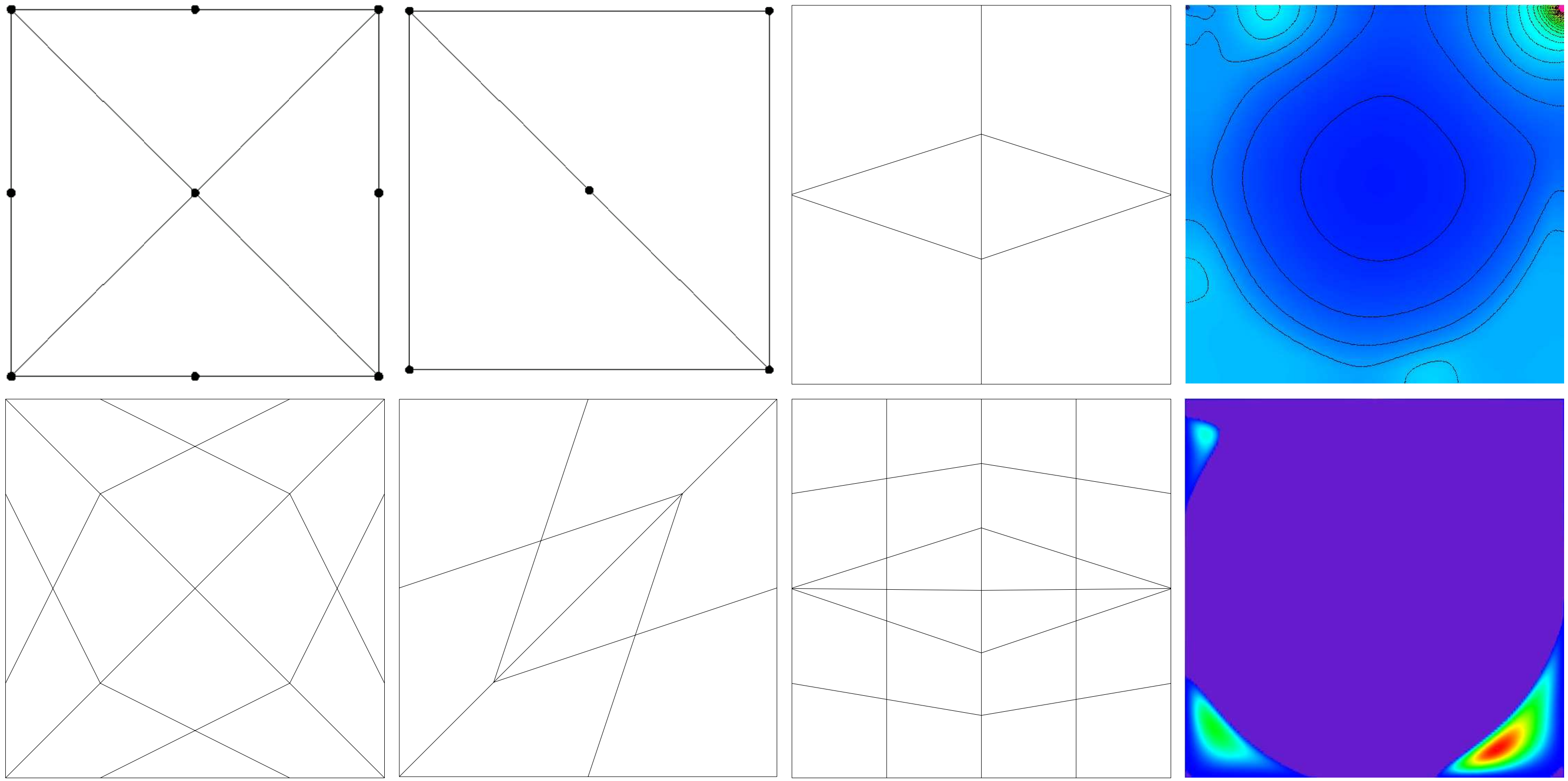
- Overview on EO-FEM stabilization
- Challenges
- Numerical investigations

‘Nonparametric’ Nonconforming $\tilde{Q}_1/P_0$

→ LBB condition for **highly deformed** meshes
→ Fast scalar and PSC **multigrid** solvers
→ Highly efficient for **fully nonstationary** flows



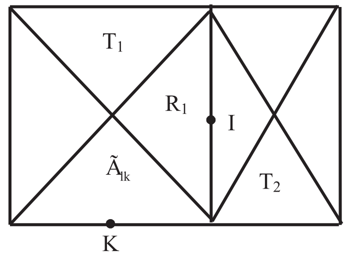
→ **Korn’s Inequality:** $\|D(\mathbf{v}_h)\|_0 \geq \beta\|\mathbf{v}_h\|_{1,h}$?
→ Stabilization of the **convective term** ‘$\mathbf{u}_h \cdot \nabla \mathbf{u}_h$’ ?

"Classical" Stabilization Methods

- Streamline diffusion (SD)

$$S = \sum_{\tau \in \tau_h} \delta_\tau \int_\tau (\mathbf{u}_h \cdot \nabla \mathbf{v}_h)(\mathbf{u}_h \cdot \nabla \mathbf{w}_h) dx \qquad (1)$$

- Samarski's upwind (UPW)



$$S = \sum_l \sum_{k \in \Lambda_l} \oint_{\Gamma_{lk}} \mathbf{u}_h \cdot \mathbf{n}_{lk} d\gamma [1 - \lambda_{lk}(\mathbf{u}_h)(\mathbf{v}_h(m_k) - \mathbf{v}_h(m_l))] w_h(m_l) \qquad (2)$$

Based on the local Reynolds number $Re_\tau = \frac{\|\mathbf{u}\|_\tau \cdot h_h}{\nu}$, we can define

$$\delta_\tau = \delta^* \cdot \frac{h_\tau}{\|\mathbf{u}\|_\Omega} \cdot \frac{2Re_\tau}{1 + Re_\tau}, \quad \lambda_{lk}(u_h) = \begin{cases} \dfrac{\frac{1}{2} + \delta^* Re_\tau}{1 + \delta^* Re_\tau} & \textbf{\textit{if }} Re_\tau \geq 0 \\ \dfrac{1}{2(1 - \delta^* Re_\tau)} & \textbf{\textit{otherwise}} \end{cases} \qquad (3)$$

Edge-Oriented (EO) Stabilization Methods

EO is a new FEM technique which introduces **additional ”jump terms”** into the weak formulation in a consistent way (Brenner, Burman, Hansbo, John, Larson, T.):

- to control the **nonconformity** arising from the pressure term in Darcy's law

$$j(\mathbf{u},\mathbf{v}) = \sum_{\text{edge E}} \gamma\nu \frac{1}{|E|} \int_E [\mathbf{n}\cdot\mathbf{u}][\mathbf{n}\cdot\mathbf{v}]d\sigma \tag{4}$$

- to guarantee discrete **Korn's inequality** for **nonconforming FEM**

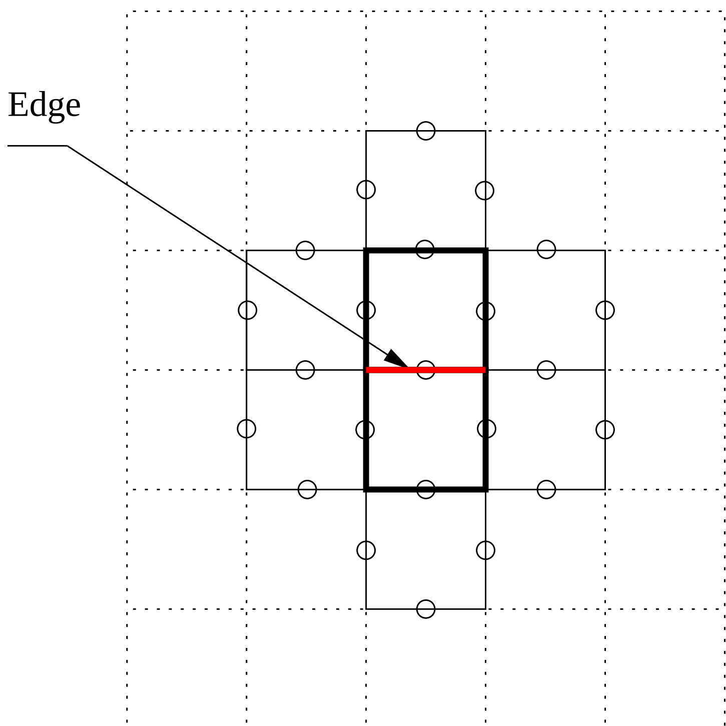
$$j_1(\mathbf{u},\mathbf{v}) = \sum_{\text{edge E}} \gamma\nu \frac{1}{|E|} \int_E [\mathbf{u}][\mathbf{v}]d\sigma \tag{5}$$

- to stabilize **convection** dominated problem (for **all FEM**)

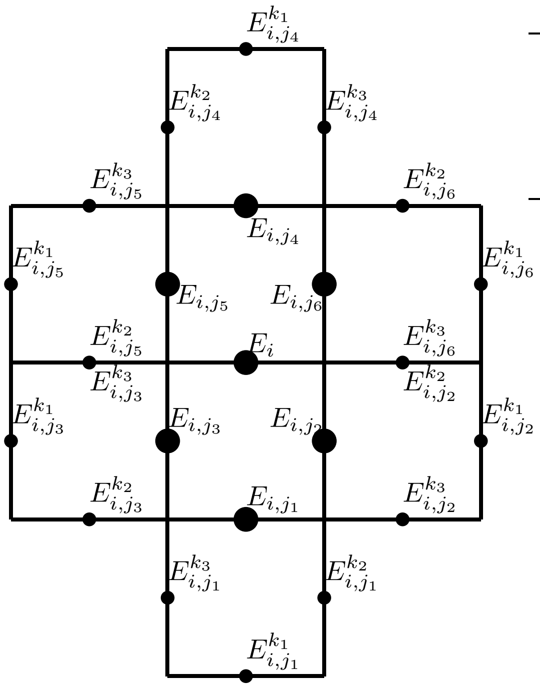
$$j_2(\mathbf{u},\mathbf{v}) = \sum_{\text{edge E}} \gamma|E|^\alpha \int_E \text{''}[\nabla\mathbf{u}]\text{''}\ \text{''}[\nabla\mathbf{v}]\text{''}d\sigma \tag{6}$$

Challenges for EO-FEM Stabilization

- **Robustness** on unstructured meshes
- Time-dependent case + **higher order finite element spaces**
- Efficient FEM data structures for ”new” **sparsity**
- **Multigrid** for (4-th order?) EO-FEM stabilisation

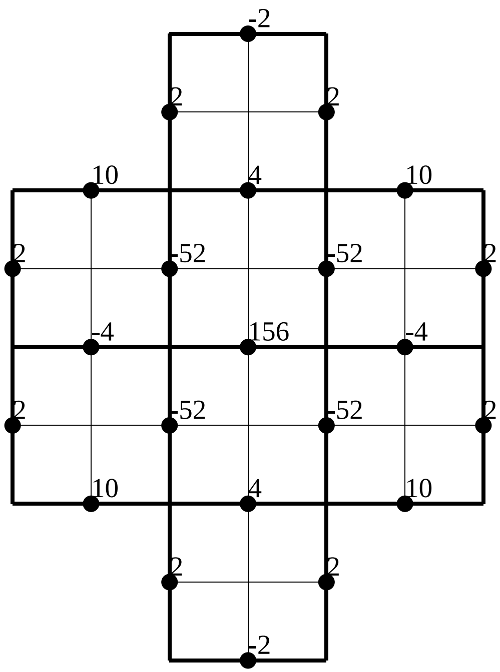


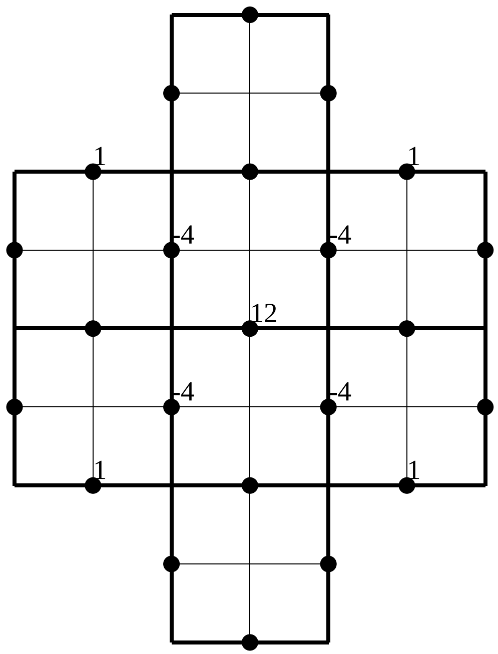
Edge-oriented Data Structures



		2D mesh			3D mesh	
		FEM	EO		FEM	EO
	NEL	Matrix entries		NEL	Matrix entries	
1	4	60	128	27	918	3474
2	16	232	628	216	7236	32436
3	64	912	2732	1728	57456	277632
4	246	3616	11356	13824	457920	2292768
5	1024	14400	46268	110592	3656448	18627264
6	4096	57472	186748	884736	29223936	150155136

Edge-oriented Data Structures





	Quadrature technique	
	exact Gauss	1x1 Gauss
1	128	76
2	628	328
3	2732	1360
4	11356	5536
5	46268	22336
6	186748	89728

Edge-oriented Data Structures

	edge-oriented storage technique				standard FEM storage technique			
	without jump stab.		with jump stab.		without jump stab.		with jump stab.	
	MG	Time	MG	Time	MG	Time	MG	Time
2	23	2.68	13	2.10	23	2.42	13	10.00
3	66	41.05	9	7.83	66	32.60	9	32.16
4	191	542.36	8	28.52	191	442.18	8	115.16
5	535	6209.84	9	133.62	535	5426.03	9	524.12
6	1225	63614.97	8	525.36	1225	49502.54	8	1905.98

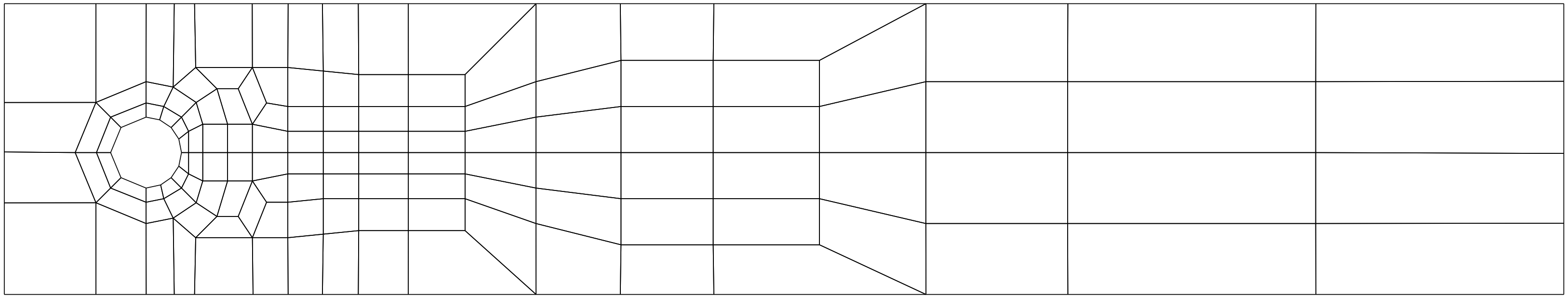
New: Unified Approach

$$j(\mathbf{u},\mathbf{v}) = \sum_{\text{edge E}} \max(\gamma\nu|E|, \gamma^*|E|^2) \int_E [\nabla\mathbf{u}][\nabla\mathbf{v}]d\sigma$$

- ”fixed” constants γ, γ^*, no ”h_K” on unstructured meshes
- for discrete Korn’s inequality
- for medium/high Reynolds numbers, even for **pure transport**
- for Q_1, $\tilde{Q}_1$, Q_2 FEM spaces
- Theory (?)

Different Philosophy: Not looking at ”local Re/Peclet number”, but checking ”smoothness” of (discrete) solution only!

Numerical Analysis



Mesh information			$\tilde{Q}_1/Q_0$
Level	Elements	Vertices	Unknowns
1	156	130	702
2	572	520	2686
3	2184	2080	10608
4	8528	8320	42016
5	33696	33280	167232
6	**133952**	**133120**	**667264**

Outer Fixed-Point iteration + Oseen solver with Vanka smoother, resp., local MPSC

Stationary Flow around Cylinder: Stokes

gradient formulation							
Stab.	EO		SD		UPW		Central
Level	γ		δ^*				
	0.001	0.01	0.1	0.5	0.1	1.0	
Drag ($C_D = 3142.4$)							
4	3127.4	3127.4	3127.4	3127.4	3127.4	3127.4	3127.4
5	3138.6	3138.6	3138.6	3138.6	3138.6	3138.6	3138.6
6	3141.5	3141.5	3141.5	3141.5	3141.5	3141.5	3141.5
NL/AVMG							
4	3/2	3/2	3/2	3/2	3/2	3/2	3/2
5	4/2	4/2	4/2	4/2	4/2	4/2	4/2
6	4/2	4/2	4/2	4/2	4/2	4/2	4/2

- No need for any edge stabilization, but no negative side effect

Stationary Flow around Cylinder: Stokes

deformation formulation							
Stab.	EO		SD+$\gamma j_1(\mathbf{u},\mathbf{v})$		UPW+$\gamma j_1(\mathbf{u},\mathbf{v})$		central
Level	γ		γ				
	0.001	0.01	0.0	0.1	0.0	0.1	
Drag ($C_D = 3142.4$)							
4	3132.5	3133.6	3132.4	3133.1	3132.4	3133.1	3132.4
5	3139.9	3139.9	3139.9	3140.1	3139.9	3140.1	3139.9
6	3141.8	3141.6	3141.8	3141.8	3141.8	3141.8	3141.8
NL/AVMG							
4	4/3	4/2	**4/29**	4/2	**4/29**	4/2	**4/29**
5	4/3	4/2	**5/98**	4/2	**4/99**	4/2	**5/98**
6	4/3	4/2	**5/154**	4/2	**5/154**	4/2	**5/154**

- Regarding accuracy: SD = UPW = C = EO, **but: Multigrid!!!**
- EO stabilization is a must for nonconforming FEM with multigrid

Stationary Flow around Cylinder: $Re = 20$

gradient formulation								
Stab.	EO			SD		UPW		Central
Level	γ			δ^*				
	0.0001	0.001	0.01	0.1	0.5	0.1	1.0	
Drag ($C_D = 5.5795$)								
4	5.5855	5.5864	5.5901	5.6417	5.7977	5.6005	5.7460	5.6040
5	5.5813	5.5815	5.5823	5.6020	5.6655	5.5841	5.6197	5.5862
6	5.5800	5.5800	5.5803	5.5868	5.6092	5.5806	5.5882	5.5812
NL/AVMG								
4	12/3	12/3	12/11	12/3	11/2	11/3	10/3	17/2
5	12/2	12/2	12/9	12/2	12/2	12/2	11/3	12/2
6	12/2	12/2	12/8	12/2	12/2	12/2	12/2	12/2

- SD and UPW are more sensitive w.r.t. the ”free” δ^*
- For EO, only the multigrid solver is slightly sensitive to over-stabilization

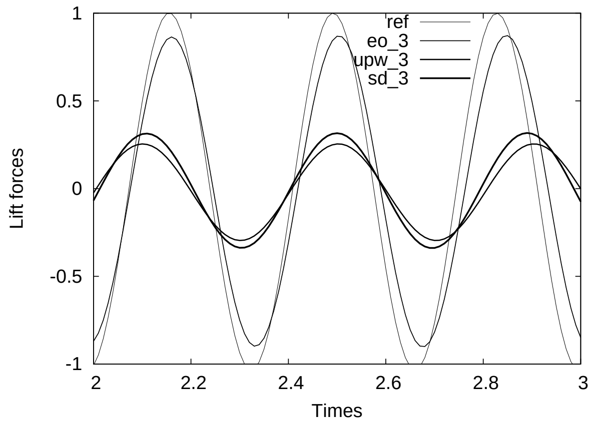
Stationary Flow around Cylinder: $Re = 20$

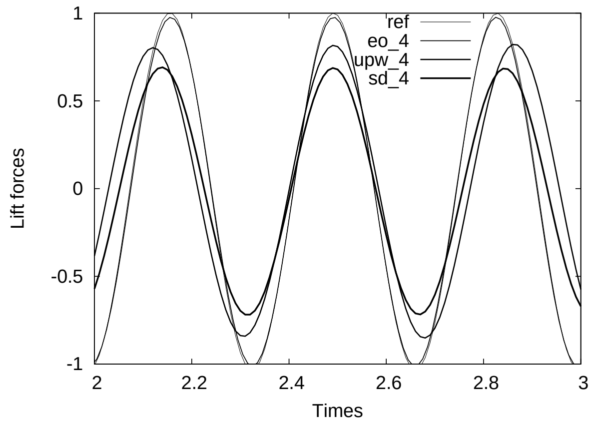
deformation formulation								
Stab.	EO			SD+$\gamma j_1(\mathbf{u},\mathbf{v})$		UPW+$\gamma j_1(\mathbf{u},\mathbf{v})$		central
Level	γ			γ				
	0.0001	0.001	0.01	0.0	0.1	0.0	0.1	
Drag ($C_D = 5.5795$)								
4	5.5846	5.5838	5.5811	5.6261	5.6264	5.5847	5.5850	5.5865
5	5.5810	5.5807	5.5790	5.5974	5.5975	5.5810	5.5811	5.5814
6	5.5799	5.5798	5.5793	5.5856	5.5856	5.5799	5.5799	5.5800
NL/AVMG								
4	12/3	12/2	12/12	**12/2**	12/2	**12/5**	12/2	19/2
5	12/3	12/2	12/8	**12/5**	12/2	**12/11**	12/2	21/2
6	12/4	12/2	12/8	**12/9**	12/2	**12/12**	12/2	26/4

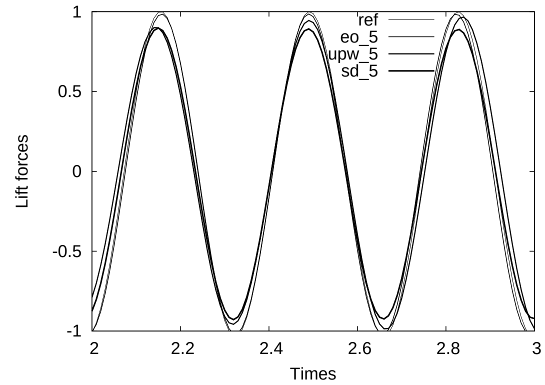
- SD and UPW require additional stabilization (EO) for multigrid only
- For EO, there is no need for any additional stabilization + more accurate

Nonstat. Flow around Cylinder: $Re = 100$

- Lift coefficient for periodically oscillating flow







stab.	EO	SD	UPW	EO	SD	UPW
Level	Maximum amplitude			Strouhal number		
3	0.8750	0.3171	0.2543	0.29126	0.25862	0.23904
4	0.9753	0.6878	0.8214	0.29810	0.26906	0.28436
5	0.9858	0.8864	0.9664	0.30075	0.28708	0.29557
ref	~ 1.0060			~ 0.3020		

- **UPW:** Good results for the amplitude (level 5)
- **EO:** Excellent results for amplitude and frequency!!!

Standing Vortex: $Re = \infty$

Incompressible Navier-Stokes equations for inviscid flow ($Re = \infty$) in a unit square

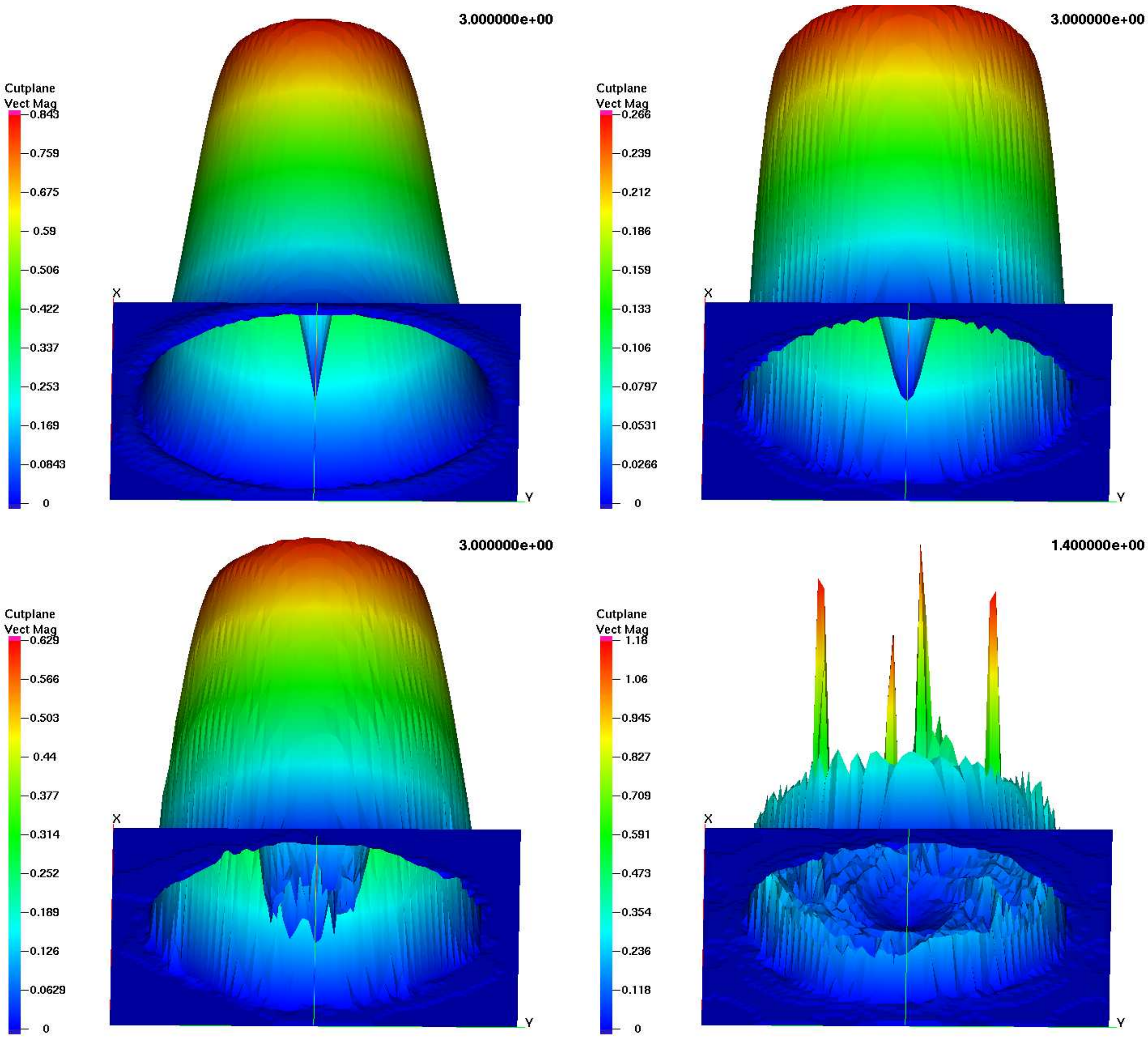
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega = (0,1) \times (0,1). \tag{7}$$

$$\mathbf{u}_r = 0, \quad \mathbf{u}_\theta = \begin{cases} 5r, & r < 0.2, \\ 2 - 5r, & 0.2 \leq r \leq 0.4, \\ 0, & r > 0.4, \end{cases} \tag{8}$$

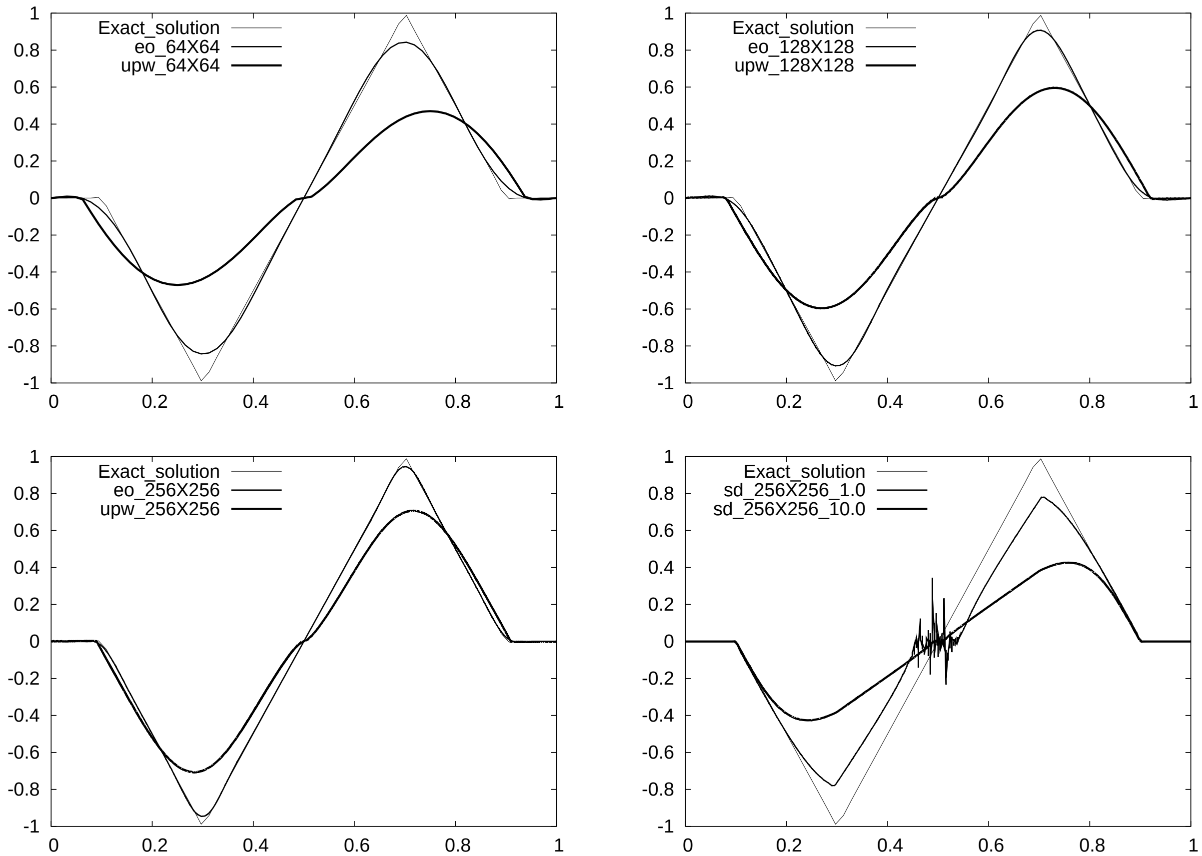
where $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$ denotes the distance from the center

Which discretization schemes preserve the original vortex !?

Standing Vortex: $Re = \infty$



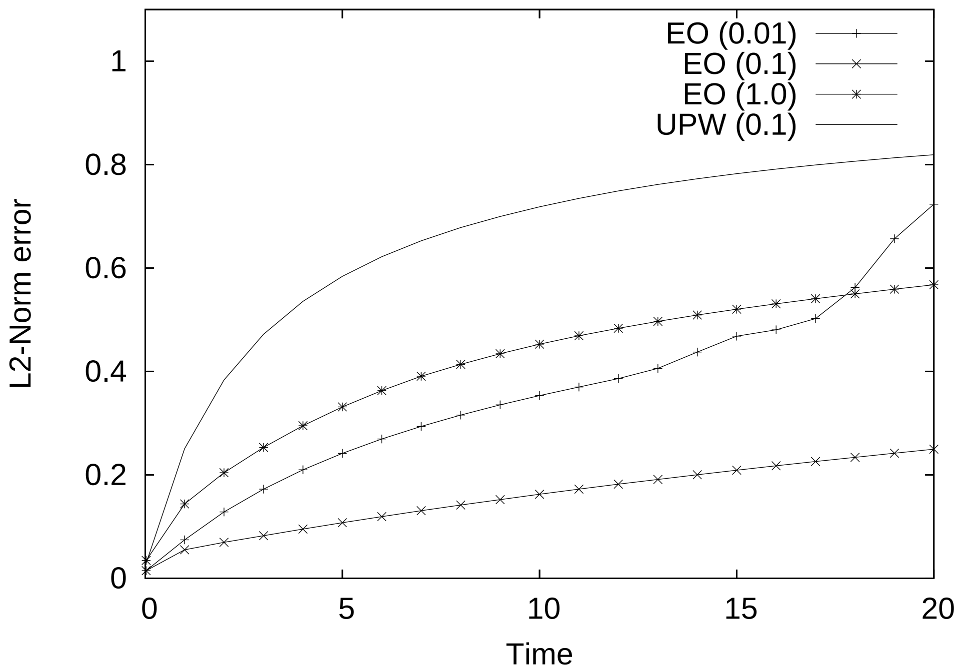
Standing Vortex: $Re = \infty$



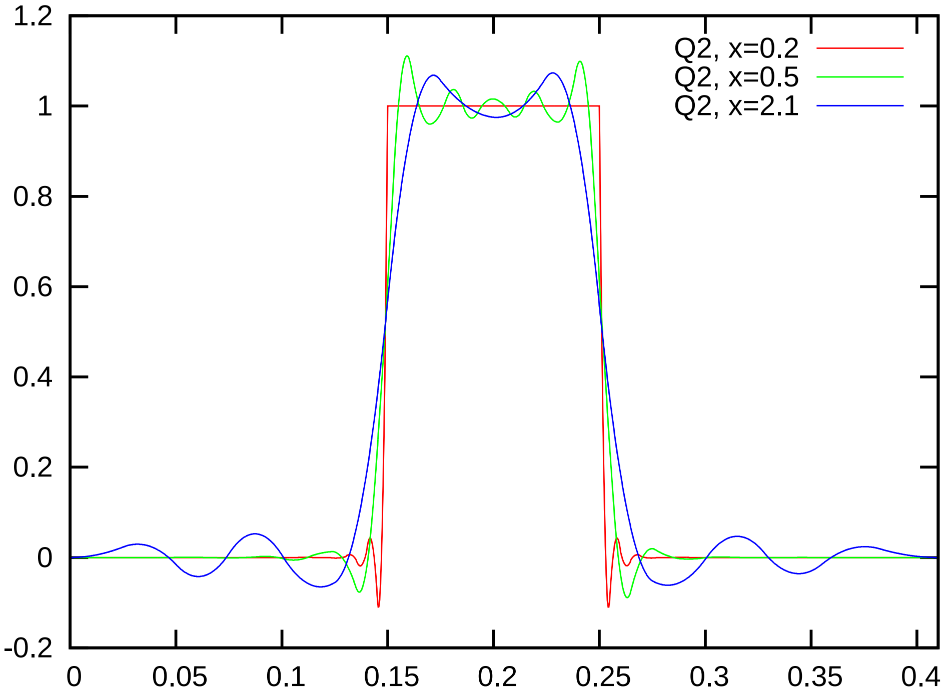
- UPW: significant smearing effects: only first order accuracy
- SD: problem with ‘entropy violating shock’
- EO: preserves ”perfectly” the solution with high accuracy

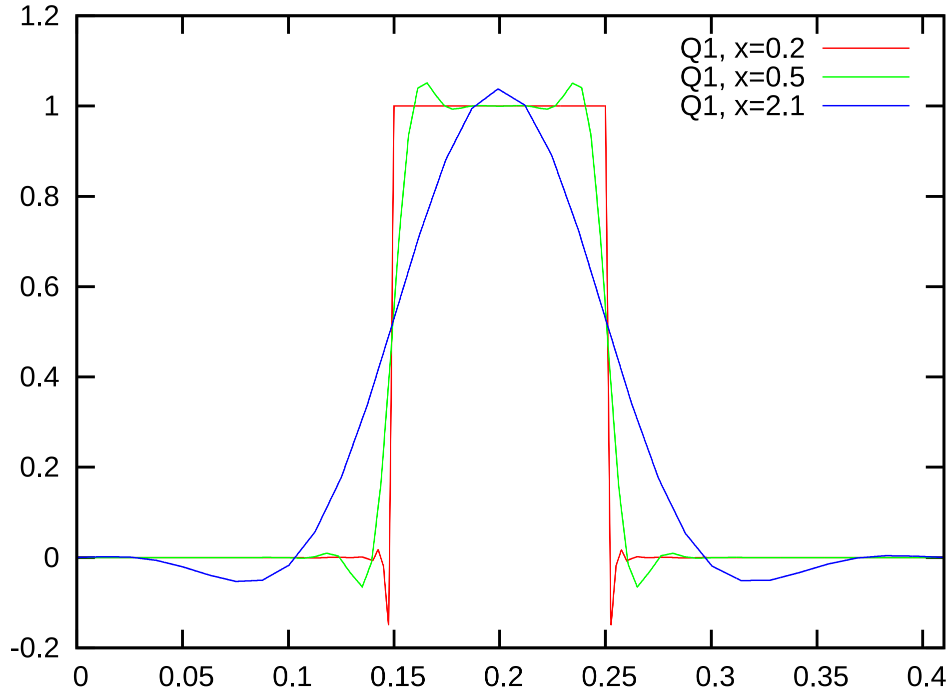
Standing Vortex: $Re = \infty$

	EO			UPW		
Level	T=1	T=2	T=3	T=1	T=2	T=3
6	0.0551	0.0695	0.0823	0.250	0.383	0.471
7	0.0252	0.0337	0.0401	0.154	0.249	0.324
8	0.0115	0.0153	0.0184	0.087	0.151	0.204
9	0.0052	0.0070	0.0084	0.049	0.087	0.120
10	0.0024	0.0032	0.0039	0.027	0.049	0.068

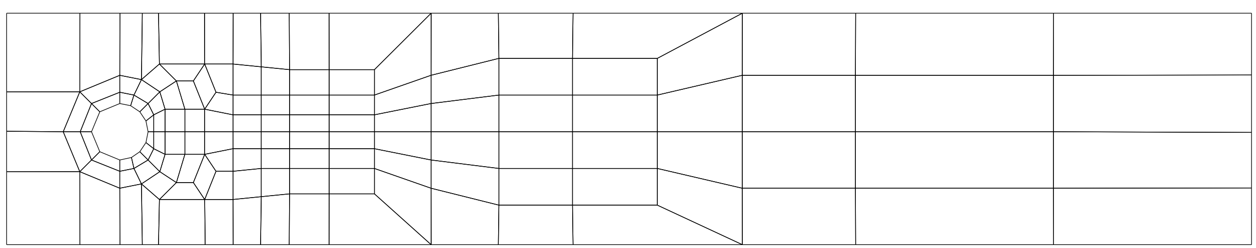


Scalar Transport Problem





”Only inflow + inhomogeneous b.c.’s on cylinder”



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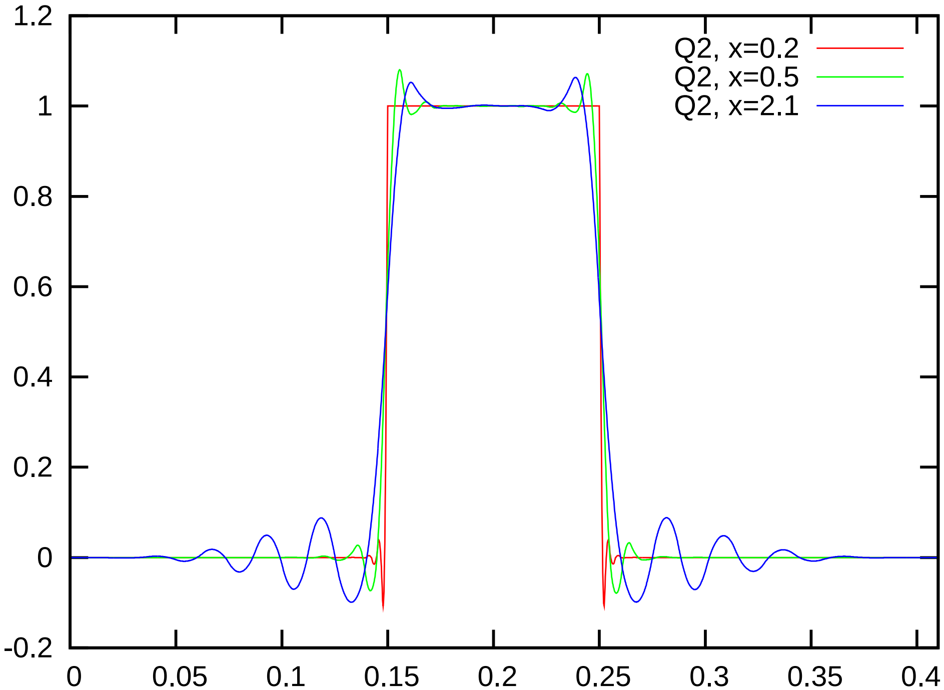
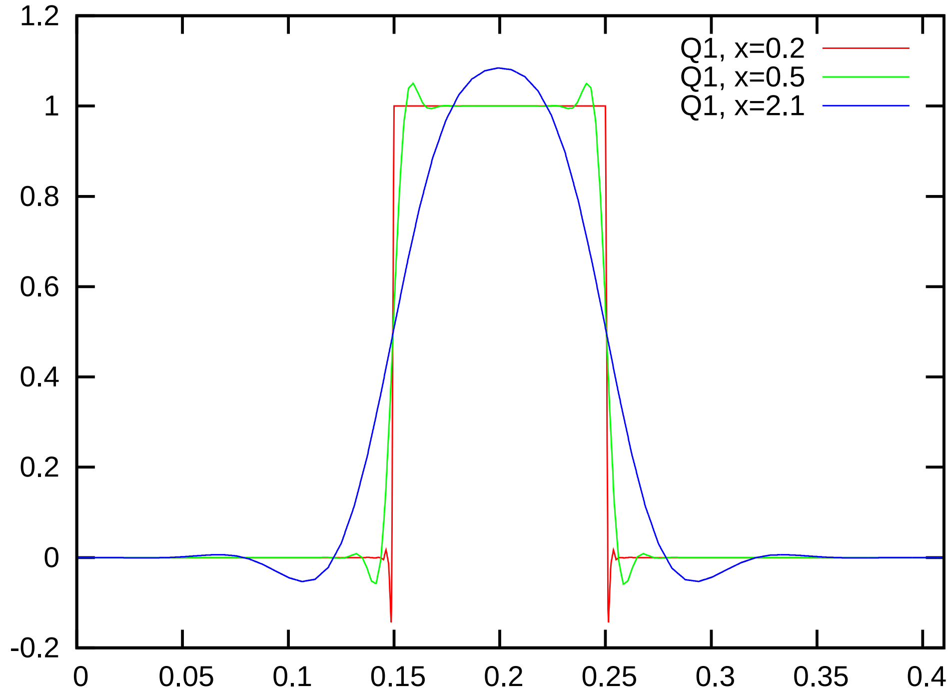
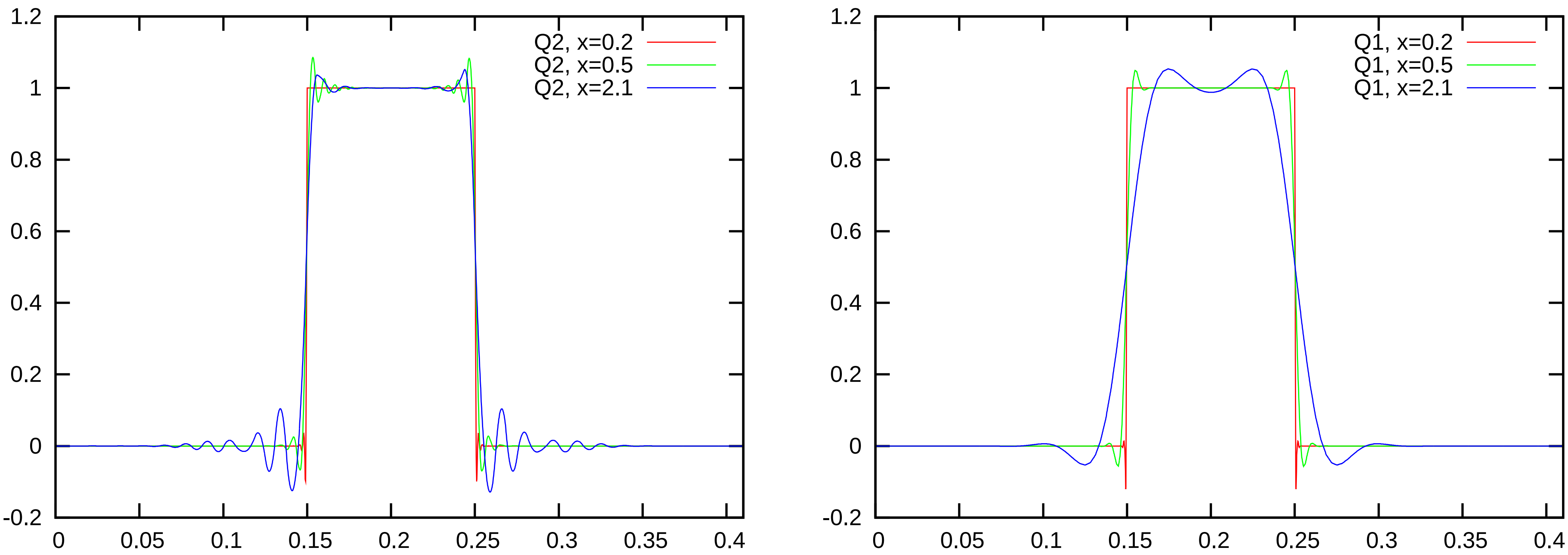




Scalar Transport Problem



High accuracy with Q_2, but problems with steep gradients (also for Q_1)!

(Additional nonlinear(!) TVD/Shock-Capturing required!)

Multigrid Solver

		Level					Level		
El.	ν	4	5	6	El.	ν	3	4	5
Q1	0	0.006	0.030	0.107	Q2	0	0.001	0.001	0.008
	0.0001	0.004	0.014	0.040		0.0001	0.001	0.001	0.002
	0.01	0.002	0.001	0.002		0.01	0.002	0.000	0.001
	1	0.002	0.004	0.004		1	0.009	0.001	0.001

Very efficient multigrid solver possible (Q_2!!!)

Very high accuracy for smooth data (Q_2!!!)

mesh width	ν=0.1		ν=0	
$h = 1/32$	4.95-5	1.03-3	6.12-5	1.17-3
$h = 1/64$	5.75-6	2.57-4	6.06-6	2.66-4
$h = 1/128$	6.88-7	6.42-5	6.97-7	6.48-5

Summary

The proposed EO-FEM stabilisation is a ”new” candidate for a black box tool and can be applied, with many FEM spaces, to a large number of problems

Further improvement for transport problems with steep gradients (”shock-capturing/TVD”) is required

Together with appropriate data structures, very efficient multigrid solvers are feasible