FEM Multigrid Techniques for Viscoelastic Flow

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http://www.mathematik.tu-dortmund.de/lsiii
www.featflow.de
Multiscale CFD problems

Inertia turbulence

- Re >> 1
- Numerical instabilities + problems

![Turbulence flow inside a pipe. From ProPipe](image)

- Turbulence Models
- Stabilization Techniques

Characteristics:

- Irregular temporal behaviour and spatially disordered
- Broad range of spatial/temporal scales
Multiscale CFD problems

Elastic turbulence

- $Re \ll 1$, $We \gg 1$ (less inertia, more elasticity)
- Numerical instabilities + problems (HWNP)

- Flow models: Oldroyd, Maxwell,…
- Stabilization: EEME, EEVS, DEVSS/DG, SD, SUPG,…
Nonlinear flow models

\[
\begin{aligned}
\partial_t u + (\nabla u) u &= \nabla \cdot \sigma - \nabla p + (1 - \gamma \theta) j \\
\nabla \cdot u &= 0 \\
\partial_t \theta + (\nabla u) \theta &= \nabla \cdot k \nabla \theta + \sigma : D \\
\sigma^p + \Lambda \delta_t \sigma^p &= 2 \eta_p D,
\end{aligned}
\]

\[
\begin{aligned}
\sigma &= \sigma^s + \sigma^p, \quad \eta_0 = \eta_s + \eta_p, \\
\sigma^s &= 2 \eta_s (D, \Theta) D, \\
D &= \frac{1}{2} \left( \nabla u + \nabla u^T \right) \\
\delta_t \sigma^p &= \partial_t \sigma + (\nabla u) \sigma + \frac{1-a}{2} \left( \sigma \nabla u + \nabla u^T \sigma \right) - \frac{1+a}{2} \left( \nabla u \sigma + \sigma \nabla u^T \right)
\end{aligned}
\]
Required: I) special models

\[
T + \Lambda \delta_t^a T = 2\eta_0 \left( D + \Lambda_r \delta_t^a D \right)
\]

Oldroyd A
Oldroyd B
Maxwell A
Maxwell B
Jeffreys

\[
T + \Lambda \delta_t^a T + B(T) = 2\eta D
\]

Phan-Thien Tanner
Phan-Thien
Giesekus
Required: II) special numerics

- Special FEM Techniques
- Multigrid Solvers
- Stabilization for high Re and We numbers
- Implicit Approaches
- Space-Time Adaptivity
- Grid Deformation Methods
- Newton Methods
Problems remain

Different highly developed models

Oldroyd A/B, Maxwell A/B, Jeffreys, PTT, Giesekus

… nevertheless, despite „good“ models and „good“ numerics, the HWNP („High Weissenberg Number Problem“) still exists for critical We, resp., De numbers…

Kinetic Energy for two different velocity inflow

Zoom shows oscillation...!!
Our numerical approach

Fully implicit monolithic FEM
Multigrid solver!
Numerical techniques

• The FEM techniques have to handle the following challenging points
  ☼ Stable FE spaces for velocity and pressure fields, and velocity and extra-stress fields → \( Q_2/P_1/Q_2 \) or \( Q_1(nc)/P_0/Q_1(nc) \) (new: \( Q_2(nc)/P_1/Q_2(nc) \))
  ☼ Special treatment of the convective terms \( u \cdot \nabla u \), \( u \cdot \nabla \Theta \), \( u \cdot \nabla \sigma \)
    → edge-oriented/interior penalty FEM, TVD/FCT
  ☼ high Weissenberg number problem (HWNP) → LCR

• The (nonlinear) solvers have to deal with different source of nonlinearity
  ☼ nonlinear viscosities → Newton method via divided differences
  ☼ the strong coupling of equations → monolithic multigrid approach
  ☼ complex geometries and meshes
Newton solver

Solve for the residual of the nonlinear system algebraic equations

\[ R(x) = 0, \quad x = (u, \Theta, \sigma, p) \]

Newton method with damping results in iterations of the form

\[ x^{n+1} = x^n + \omega^n \left( \frac{\partial R(x^n)}{\partial x} \right)^{-1} R(x^n) \]

\( \therefore \) Continuous Newton: on variational level (before discretization)
\( \rightarrow \) The continuous Frechet operator can be calculated

\( \therefore \) Inexact Newton: on matrix level (after discretization)
\( \rightarrow \) The Jacobian matrix is approximated using finite differences as

\[ \left[ \frac{\partial R(x^n)}{\partial x} \right]_{ij} \approx \frac{R_i(x^n + \varepsilon e_j) - R_i(x^n - \varepsilon e_j)}{2\varepsilon} \]
Multigrid solver

☀ Standard geometric multigrid approach
☀ Full $Q_2$ and $P_1^{\text{disc}}$ grid transfer
☀ Smoother: Local/Global MPSC
   ☀ Local MPSC via Vanka-like smoother

\[
\begin{bmatrix}
  u^{l+1} \\
  \sigma^{l+1} \\
  \Theta^{l+1} \\
  p^{l+1}
\end{bmatrix} =
\begin{bmatrix}
  u^l \\
  \sigma^l \\
  \Theta^l \\
  p^l
\end{bmatrix} + \omega^l \sum_{T \in \mathcal{T}_h} [J]^l_T
\begin{bmatrix}
  \text{Res}_u \\
  \text{Res}_\sigma \\
  \text{Res}_\Theta \\
  \text{Res}_p
\end{bmatrix}_T
\]

Monolithic multigrid solver

☀ Global MPSC
   ☀ solve for an intermediate $\tilde{u}$ (generalized momentum equation)
   ☀ solve for $p$ (pressure Poisson equation)
   ☀ update of $u$ and $p$
   ☀ solve for $\Theta$ (tracer equation)
   ☀ solve for $\sigma$ (constitutive equation)

Decoupled multigrid solver
Problem reformulation

Old $\rightarrow$ $(u, p, \sigma^p)$

\[
\begin{align*}
\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) &= \nabla p - 2\eta_s \nabla \cdot D - \nabla \cdot \sigma^p, \\
\nabla \cdot u &= 0, \\
\Lambda \frac{\delta_a \sigma^p}{\delta t} + \sigma^p - 2\eta_p D &= 0,
\end{align*}
\]

Conformation tensor $\rightarrow$ $(u, p, \tau)$  This tensor is positive definite by design!!

Replace $\sigma^p$ in (1) with $\sigma^p = \frac{\eta_p}{\Lambda} (\tau - I) \rightarrow$ special discretization : TVD

\[
\begin{align*}
\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) &= \nabla p - 2\eta_s \nabla \cdot D - \frac{\eta_p}{\Lambda} \nabla \cdot \tau, \\
\nabla \cdot u &= 0, \\
\frac{\delta_a \tau}{\delta t} + \frac{1}{\Lambda} (\tau - I) &= 0,
\end{align*}
\]
Conformation tensor property

2 observations:
- positive definite $\rightarrow$ special discretizations like FCT/TVD
- exponential behaviour $\rightarrow$ approximation by polynomials???

$$\tau(t) = \int_{\infty}^{t} \frac{1}{\text{We}} \exp \left( -\frac{(t-s)}{\text{We}} \right) \mathbf{F}(s, t) \mathbf{F}(s, t)^T \, ds$$

Positive by design, so we can take its logarithm
Exponential behaviour

Driven cavity example:

as We number changes from We=0.5 to We=1.5, the stress value jumps very high

Old Formulation Vs Lcr

- We = 0.5 → We = 1.5

Cutline of Stress_11 component at y = 1.0
**Problem reformulation**

*Replace* $\tau$ *in (2) with* $\tau = \exp \psi$

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \nabla \mathbf{p} - 2\eta_s \nabla \cdot D - \frac{\eta_p}{\Lambda} \nabla \cdot (\exp \psi),
\]

\[
\nabla \cdot \mathbf{u} = 0,
\]

\[
\frac{\delta_a (\exp \psi)}{\delta t} + \frac{1}{\Lambda} (\exp \psi - 1) = 0,
\]

\[
\text{(3)}
\]

Gradient of exponential of $\psi \to ???$

Solvers $\to ???$
Experiences:

☼ Stresses grow exponentially
☼ Conformation stress is positive by design

Idea ("Kupferman Trick"):

☼ Decompose the velocity gradient inside the stretching part

\[ \nabla u = \Omega + B + N\sigma_c^{-1} \]

☼ Take the logarithm as a new variable (\(\psi = \log \sigma\)) using eigenvalue decomposition

\[ \psi = R \log(\lambda_T) R^T \]
LCR for Oldroyd-B model

\[ \tau(t) = \int_{t}^{t} \frac{1}{\text{We}} \exp \left( -\frac{(t-s)}{\text{We}} \right) F(s, t) F(s, t)^T \, ds \]

Oldroyd-B

\[ \Lambda \frac{\delta_a \sigma^p}{\delta t} + \sigma^p - 2\eta_p D = 0, \]

\[ \sigma^p = \frac{\eta_p}{\text{We}} (\tau - I) \]

\[ \frac{\delta_a \tau}{\delta t} + \frac{1}{\text{We}} (\tau - I) = 0, \]

\[ \nabla u = \Omega + B + N\sigma_c^{-1} \]

\[ \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \tau - (\Omega \tau - \tau \Omega) + 2B \tau = \frac{1}{\Lambda} (I - \tau) \]

\[ \tau = \exp \psi \quad \psi = R \log(\lambda_r) R^T \]

LCR

\[ \partial_t \psi + (\nabla u) \psi - (\Omega \psi - \psi \Omega) + 2B = \frac{1}{\text{We}} (\exp(-\psi) - 1) \]
LCR for Oldroyd-B model

\[
\begin{align*}
\partial_t u + (\nabla u) u &= \nabla \cdot \sigma - \nabla p + (1 - \gamma \Omega) j \\
\nabla \cdot u &= 0 \\
\partial_t \psi + (\nabla u) \psi - (\Omega \cdot \psi - \psi \cdot \Omega) + 2 B &= \frac{1}{We} \left( \exp(-\psi) - I \right)
\end{align*}
\]

Standard discretization techniques \(\rightarrow\) EO-FEM, TVD
Standard nonlinear (Newton) and linear (MG) solvers

\(\rightarrow\) Increases the critical We number dramatically !!
Numerical tests

Driven cavity

☀ Velocity profile at the upper wall: \( u_x = 8(1 + \tanh(8(t - 0.5)))x^2(1 - x)^2 \)
☀ For \( t > 1 \), \( u_{in} = 16 \)
☀ Dirichlet Bc’s everywhere
☀ Stress field: Natural Bc’s

4 to 1 contraction

☀ Velocity profile at the inlet: \( u_x = \frac{3}{128}(16 - y^2) \)
☀ Out flow: Natural Bc’s
☀ Stress field: Natural Bc’s

Planar flow around cylinder

☀ Velocity profile at the inlet: \( u_x = 1.5(1 - y^2 / 4) \)
☀ Out flow: Natural Bc’s
☀ Stress field: Natural Bc’s
Driven cavity

As $We$ increases, numerical instabilities become visible.

EOFEM is able to throw this
As $We$ increases, numerical instabilities become visible. EOFEM is able to throw this
Driven cavity

As $\text{We}$ increases, numerical instabilities become visible. EOFEM is able to throw this
4:1 Contraction

Corner vortex, no lip

Experiment vs Numeric
4:1 Contraction

Lip vortex appears

Experiment vs Numeric
4:1 Contraction

Lip vortex grows

Experiment  vs  Numeric

Qualitatively close
### Planar flow around cylinder

<table>
<thead>
<tr>
<th>We</th>
<th>0.01</th>
<th>0.1</th>
<th>1.0</th>
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<tr>
<td>Linear Tol</td>
<td>0.1</td>
<td>0.01</td>
<td>0.1</td>
</tr>
<tr>
<td>R1</td>
<td>9/2 5/3</td>
<td>10/1 7/3</td>
<td>14/1 10/3</td>
</tr>
<tr>
<td>R2</td>
<td>9/3 5/5</td>
<td>10/2 7/4</td>
<td>16/2 10/5</td>
</tr>
<tr>
<td>R3</td>
<td>9/3 5/6</td>
<td>10/3 7/5</td>
<td>16/2 11/5</td>
</tr>
<tr>
<td>R4</td>
<td>9/3 5/6</td>
<td>10/3 9/5</td>
<td>13/3 11/5</td>
</tr>
</tbody>
</table>

*stable Newton and multigrid!*
Planar flow around cylinder

Natural outflow condition is sufficient!
Planar flow around cylinder

Drag coefficient planar flow around cylinder

DEVSS/DG & Q2/P1/Q1 (Hulsen et. al)
h-p FEM (Fan et. al)
DEVSS-G/DG (Caola et. al)
Spectral method (Owens)
DEVSS-G/SUPG (Liu)
FVM+LCR (Afonso et. al)
MFEM+LCR+EOFEM (Featflow)

Drag coefficient for different levels

R4
R3a1
R3a2

We can go further than ever before
Complex model

\[ \eta = \eta_0 \exp\left(a_1 + \frac{a_2}{a_3 + \theta}\right)(b_1 + b_2 \|D\|)^{-b^3} \]

**More verification: viscous heating effect to the flow behaviour**
We present a tool to simulate complex fluid flow with:

- Monolithic FEM
- Geometric Multigrid
- Edge oriented stabilization
- LCR for Oldroyd-B model
- Local adaptivity
- 2nd order accurate time integrator

Future work:
- Further investigation of Newton – multigrid solver
- Validation of NS EE SE
- Further complex viscoelastic fluid model
- Further coupling with structure part (FSI)