We present a Bayesian inversion method with functional representations of all quantities. The posterior density is given in terms of a polynomial basis, based on an adaptive stochastic Galerkin discretization. The sampling-free approach, using tensor trains, alleviates the curse of dimensionality by hierarchical subspace approximations of the respective low-rank manifolds. All computations are adjusted adaptively based on a posteriori error estimators or indicators. Convergence of the posterior can be shown with respect to the discretization parameters.

Explicit and parametric Bayesian inversion

- applies in parameter identification and upscaling
- parametrized model operator \( G(y) = \sum g_p(y) P_p(y) \), \( g_p = (G, P_p) \)
- finite measurements \( \delta \in \mathbb{R}^d \) of indirect quantity, observed by linear operator \( \mathcal{O} \)
- prior measure \( \pi_0 \) on parameters and noise measure \( \mathcal{N}(0, \Gamma) \) on measurement error \( \eta \)

Statistical inverse problem: Find \( y \in \Omega \) s.t. \( \delta = (\mathcal{O} \circ G)(y) + \eta \) \( \eta \sim \mathcal{N}(0, \Gamma) \)

Bayes’ theorem yields existence of posterior measure \( \pi \) in functional representation [1]:

\[
\sum_{p \in \mathbb{N}} a_p P_p(y) = \int_{\mathbb{R}^d} \left( \frac{1}{2} \mathcal{O}(G(y)) \mathcal{O}^T - \mathcal{O}(G(y)) \right) \overline{\mathcal{O}(G(y))} \, \pi(\cdot) \, d\pi(\cdot)
\]

Model reduction: Tensor formats

- high dimensional problem, curse of dimensionality: \( O(n^M) \)
- HT/TT allows for polynomial complexity: \( O(n^r M \epsilon) \) via

\[
U[x_1, \ldots, x_M] = \prod_{k=1}^M U_m[k_{m-1}; x_k; k_m]
\]

- features:
  - separation of variables and closedness of rank \( r \) manifold,
  - indirect access to tensor elements by hierarchical basis
- Creation by tensor recovery/reconstruction [2] or cross-approximation

Sampling free Bayesian inversion using Tensor Trains

- explicit forward solver yields surrogate model in TT format
  \[
  G^y(x) = \sum_{\alpha=1}^N \sum_{\mu} a_{\alpha}(\mu) P_{\alpha}(x)
  \]
- approximation of Bayesian potential in closed TT form by exact and anisotropic interpolation

Inverse scattering: Helmholtz problem

Consider two random media \( D_1(\omega) \), \( D_2(\omega) \),
\( D_1(\omega) \cup D_2(\omega) = \mathbb{R}^d \) separated by interface \( \Gamma(\omega) \).
Transmission and reflection problem for plane-wave incidence and known material parameters given by transformed Helmholtz equation

\[
- \nabla \cdot \left( \alpha(\Gamma(\omega), \cdot) \nabla \right) - \kappa^2(\Gamma(\omega), \cdot) = 0 \quad \text{in } \mathbb{R}^d
\]
boundary condition
radiation condition

Outlook and references

Adaptive functional representation combined with hierarchical model reduction:
- Statistical parameter identification for shape reconstruction in scattering applications
- Reconstruction of shapes of blood-cells from measured reflection intensities (with PTB)