SUPERCONVERGENCE PROPERTIES OF OPTIMAL CONTROL PROBLEMS

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Abstract. An optimal control problem for a 2-d elliptic equation is investigated with pointwise control constraints. This paper is concerned with discretization of the control by piecewise constant functions. The state and the adjoint state are discretized by linear finite elements. Approximations of the optimal solution of the continuous optimal control problem will be constructed by a projection of the discrete adjoint state. It is proved that these approximations have convergence order $h^2$.

Keywords: Linear-quadratic optimal control problems, error estimates, elliptic equations, numerical approximation, control constraints, superconvergence.

AMS subject classification: 49K20, 49M25, 65N30

1. Introduction. The paper is concerned with the discretization of the 2-d elliptic optimal control problem

$$J(u) = F(y, u) = \frac{1}{2} \| y - y_d \|_{L^2(\Omega)}^2 + \frac{\nu}{2} \| u \|_{L^2(\Omega)}^2$$

subject to the state equations

$$Ay + a_0 y = u \quad \text{ in } \Omega$$
$$y = 0 \quad \text{ on } \Gamma$$

and subject to the control constraints

$$a \leq u(t, x) \leq b \quad \text{ for a.a. } x \in \Omega,$$  \hspace{1cm} (1.3)

where $\Omega$ is a bounded domain and $\Gamma$ is the boundary of $\Omega$; $A$ denotes a second order elliptic operator of the form

$$Ay(x) = - \sum_{i,j=1}^{2} D_i (a_{ij}(x) D_j y(x))$$

where $D_i$ denotes the partial derivative with respect to $x_i$, and $a$ and $b$ are real numbers. Moreover, $\nu > 0$ is a fixed positive number. We denote the set of admissible controls by $U_{ad}$:

$$U_{ad} = \{ u \in L^2(\Omega) : a \leq u \leq b \text{ a.e. in } \Omega \}.$$

We discuss here the full discretization of the control and the state equations by a finite element method. The asymptotic behaviour of the discretized problem is studied, and superconvergence results are established.

The approximation of the discretization for semilinear elliptic optimal control problems is discussed in Arada, Casas, and Tröltzsch [1]. The optimal control problem (1.1)–(1.3) is a linear-quadratic

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counterpart of such a semilinear problem. Our aim is to construct controls \( \tilde{u} \) which have an approximation order of \( h^2 \). This higher convergence order is the difference to [1].

The discretization of optimal control problems by piecewise constant functions is well investigated, we refer to Falk [7], Geveci [8]. Piecewise constant and piecewise linear discretization in space are discussed for a parabolic problem in Malanowski [12]. Theory and numerical results for elliptic boundary control problems are contained in Casas and Tröltzsch [5] and Casas, Mateos, and Tröltzsch [4].

Piecewise linear control discretizations for elliptic optimal control problems are studied by Casas and Tröltzsch, see [5]. In an abstract optimization problem, piecewise linear approximations are investigated in Rösch [14]. In all papers, the convergence order is \( h \) or \( h^{3/2} \).

A quadratic convergence result is proved by Hinze [10]. In that approach only the state equation is discretized. The control is obtained by a projection of the adjoint state to the set of admissible controls.

In this paper, we combine the advantages of the different approaches. After solving a fully discretized optimal control problem, a control \( \tilde{u} \) is calculated by the projection of the adjoint state \( p_h \) in a post-processing step. Although the approximation of the discretized solution is only of order \( h \), we will show that this post-processing step improves the convergence order to \( h^2 \).

The paper is organized as follows: In section 2 the discretizations are introduced and the main results are stated. Section 3 contains auxiliary results. The proofs of the superconvergence results are placed in section 4. The paper ends with numerical experiments shown in section 5.

2. Discretization and superconvergence results. Throughout this paper, \( \Omega \) denotes a convex bounded open subset in \( \mathbb{R}^2 \) of class \( C^{1,1} \). The coefficients \( a_{ij} \) of the operator \( A \) belong to \( C^{0,1}(\Omega) \) and satisfy the ellipticity condition

\[
m_0 |\xi|^2 \leq \sum_{i,j=1}^2 a_{ij}(x)\xi_i \xi_j \quad \forall (\xi, x) \in \mathbb{R}^2 \times \Omega, \quad m_0 > 0.
\]

Moreover, we require \( a_{ij}(x) = a_{ji}(x) \) and \( y_d \in L^p(\Omega) \) for some \( p > 2 \). For the function \( a_0 \in L^\infty(\Omega) \), we assume \( a_0 \geq 0 \). Next, we recall some results from Bonnans and Casas [2].

**Lemma 2.1.** [2] For every \( p > 2 \) and every function \( g \in L^p(\Omega) \), the solution \( y \) of

\[
Ay + a_0y = g \quad \text{in } \Omega, \quad y|_\Gamma = 0,
\]

belongs to \( H_0^1(\Omega) \cap W^{2,p}(\Omega) \). Moreover, there exists a positive constant \( c \), independent of \( a_0 \) such that

\[
\|y\|_{W^{2,p}(\Omega)} \leq c\|g\|_{L^p(\Omega)}.
\]

Next, we introduce the adjoint equation

\[
Ap + a_0p = y - y_d \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma
\]  

Due to Lemma 2.1, the state equation and the adjoint equation admit unique solutions in \( H_0^1(\Omega) \cap W^{2,p}(\Omega) \), if \( y_d \in L^p(\Omega) \) for \( p > 2 \). This space is embedded in \( C^{0,1}(\Omega) \).
We call the solution $y$ of (1.2) for a control $u$ associated state to $u$ and write $y(u)$. In the same way, we call the solution $p$ of (2.1) corresponding to $y(u)$ associated adjoint state to $u$ and write $p(u)$.

Introducing the projection

$$
\Pi_{[a,b]}(f(x)) = \max(a, \min(b, f(x))),
$$

we can formulate the necessary and sufficient first-order optimality condition for (1.1)–(1.3).

**Lemma 2.2.** A necessary and sufficient condition for the optimality of a control $\bar{u}$ with corresponding state $\bar{y} = y(\bar{u})$ and adjoint state $\bar{p} = p(\bar{u})$, respectively, is that the equation

$$
\bar{u}(x) = \Pi_{[a,b]}(-\frac{1}{\bar{p}})
$$

holds.

Since the optimal control problem is strictly convex, we obtain the existence of a unique optimal solution. The optimality condition can be formulated as variational inequality (3.11). A standard pointwise a.e. discussion of this variational inequality leads to the above formulated projection formula, see [12].

We are now able to introduce the discretized problem. We define a finite-element based approximation of the optimal control (1.1)–(1.3). To this aim, we consider a family of triangulations $(T_h)_{h>0}$ of $\Omega$. With each element $T \in T_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set $T$ and $\sigma(T)$ is the diameter of the largest ball contained in $T$. The mesh size of the grid is defined by $h = \max_{T \in T_h} \rho(T)$. We suppose that the following regularity assumptions are satisfied.

(A1) There exist two positive constants $\rho$ and $\sigma$ such that

$$
\frac{\rho(T)}{\sigma(T)} \leq \sigma, \quad \frac{h}{\rho(T)} \leq \rho
$$

hold for all $T \in T_h$ and all $h > 0$.

(A2) Let us define $\hat{\Omega}_h = \bigcup_{T \in T_h} T$, and let $\Omega_h$ and $\Gamma_h$ denote its interior and its boundary, respectively.

We assume that $\hat{\Omega}_h$ is convex and that the vertices of $T_h$ placed on the boundary of $\Gamma_h$ are points of $\Gamma$. From [13], estimate (5.2.19), it is known that

$$
|\Omega \setminus \Omega_h| \leq C h^2,
$$

where $|$ denotes the measure of the set. Next, to every boundary triangle $T$ of $T_h$ we associate another triangle $\hat{T}$ with curved boundary as follows: The edge between the two boundary nodes of $T$ is substituted by the corresponding curved part of $\Gamma$. We denote by $\hat{T}_h$ the union of these curved boundary triangles with the interior triangles to $\Omega$ of $T_h$, such that $\Omega = \bigcup_{\hat{T} \in \hat{T}_h} \hat{T}$. Moreover, we set

$$
U_h = \{ u \in L^\infty(\Omega) : u|_{\hat{T}} \text{ is constant on all } \hat{T} \in \hat{T}_h \}, \quad U_h^{ad} = U_h \cap U_{ad},
$$

$$
V_h = \{ y_h \in C(\Omega) : y_h \in P_1 \text{ for all } T \in T_h, \text{ and } y_h = 0 \text{ on } \Omega \setminus \Omega_h \},
$$

where $P_1$ is the space of polynomials of degree less or equal than 1. For each $u_h \in U_h$, we denote by $y_h(u_h)$ the unique element of $V_h$ that satisfies

$$
a(y_h(u_h), v_h) = \int_\Omega u_h v_h \, dx \quad \forall v_h \in V_h,
$$

(2.3)
where \( a : V_h \times V_h \to \mathbb{R} \) is the bilinear form defined by
\[
a(y_h, v_h) = \int_{\Omega} \left( a_0(x) y_h(x) v_h(x) + \sum_{i,j=1}^{2} a_{ij}(x) D_i y_h(x) D_j v_h(x) \right) \, dx.
\]

In other words, \( y_h(u_h) \) is the approximated state associated with \( u_h \). Because of \( y_h = v_h = 0 \) on \( \Omega \setminus \Omega_h \) the integrals over \( \Omega \) can be replaced by integrals over \( \Omega_h \). The finite dimensional approximation of the optimal control problem is defined by
\[
\inf J(u_h) = \frac{1}{2} \| y_h(u_h) - y_d \|_{L^2(\Omega)}^2 + \frac{\nu}{2} \| u_h \|_{L^2(\Omega)}^2 \quad u_h \in U^ad_h.
\]

The adjoint equation is discretized in the same way
\[
a(p_h(u_h), v_h) = \int_{\Omega} (y_h(u_h) - y_d) v_h \, dx \quad \forall v_h \in V_h.
\]

The approximation order of the solutions of (2.4) in \( L^2 \)-sense is investigated in [1].

We will go a different way. For our superconvergence result we need an additional assumption for \( \bar{u} \). We know already that the associated adjoint state \( \bar{p} \) belongs to a space \( W^{2,p}(\Omega) \) for a certain \( p > 2 \). The optimal control \( \bar{u} \) is obtained by the projection formula (2.2). Therefore, we can classify the triangles \( T_i \) in two sets \( K_1 \) and \( K_2 \):
\[
K_1 = \{ T_i : \bar{u} \text{ is only Lipschitz continuous on } T_i \}, \quad K_2 = \{ T_i : \bar{u} \in W^{2,p}(T_i) \}
\]

This classification is correct: \( W^{2,p}(\Omega) \) is embedded in \( C^{0,1}(\Omega) \). Moreover, the projection operator is continuous from \( C^{0,1}(\Omega) \) to \( C^{0,1}(\Omega) \). Clearly, the number of triangles in \( K_1 \) grows for decreasing \( h \). Nevertheless, the following additional assumption is fulfilled in many practical cases:

(A3) \( |K_1| \leq c \cdot h \).

Let \( \bar{u} \) be the optimal solution of (1.1)–(1.3). Next, we denote by \( S_i \) the centroid of the triangle \( T_i \).

We define a piecewise constant function by the values of \( \bar{u}(S_i) \)
\[
w_h(x) = \bar{u}(S_i) \quad \text{if } x \in T_i.
\]

It is easy to verify that \( w_h \in U^ad_h \).

Now we are able to formulate our first superconvergence result.

**Theorem 2.3.** Assume that the assumptions (A1)–(A3) hold. Let \( u_h \) be the solutions of (2.4). Then the estimate
\[
\| u_h - w_h \|_{L^2(\Omega)} \leq ch^2
\]
holds true. The proof of Theorem 2.3 is contained in section 4.

Moreover, we can construct controls in a post-processing step. We start by the solution \( u_h \) of (2.4). The control \( \tilde{u} \) is calculated by a projection of the discrete adjoint state \( p_h(u_h) \) to the admissible set
\[
\tilde{u}(x) = \Pi_{[a,b]}(-\frac{1}{\nu} (p_h(u_h))(x)).
\]

**Theorem 2.4.** Assume that the assumptions (A1)–(A3) hold. Let \( \bar{u} \) be the control constructed above. Then the estimate
\[
\| \bar{u} - \tilde{u} \|_{L^2(\Omega)} \leq ch^2
\]
holds true. The proof of Theorem 2.4 is also derived in section 4.
3. Auxiliary results. First, we recall some well known results for FEM-approximations [6]. We start with the so-called Aubin-Nitsche Lemma.

**Lemma 3.1.** Let \((A1)\) and \((A2)\) be fulfilled and \(u \in L^2(\Omega)\). Then we have

\[
\|y(u) - y_h(u)\|_{L^2(\Omega)} \leq ch^2 \|u\|_{L^2(\Omega)} \tag{3.1}
\]

\[
\|p(u) - p_h(u)\|_{L^2(\Omega)} \leq ch^2 (\|u\|_{L^2(\Omega)} + \|yd\|_{L^2(\Omega)}). \tag{3.2}
\]

Next, we prove an estimate for the numerical integration.

**Lemma 3.2.** Let \(f\) be a function belonging to \(H^2(T_i)\) for all \(i\) in a certain index set \(I\). Then the estimates

\[
\left| \int_{T_i} (f(x) - f(S_i)) \, dx \right| \leq ch^2 \sqrt{|T_i|} |f|_{H^2(T_i)}
\]

and

\[
\sum_{i \in I} \left| \int_{T_i} (f(x) - f(S_i)) \, dx \right| \leq ch^2 \left( \sum_{i \in I} |f|^2_{H^2(T_i)} \right)^{1/2}
\]

are valid.

**Proof.** The proof is almost standard. First, we remark that \(|.|_{H^2(T_i)}\) denotes the \(H^2\)-seminorm. Next, we transform the integral to an integral over the reference element by \(E \hat{x} = x\) and apply the Bramble-Hilbert-Lemma:

\[
\left| \int_{T_i} (f(x) - f(S_i)) \, dx \right| = \left| \frac{|T_i|}{|T|} \int_{\hat{T}} (f(E \hat{x}) - f(S_i)) \, d\hat{x} \right|
\]

\[
\leq c|T_i| \left( \int_{\hat{T}} \sum_{|\alpha| = 2} |D^\alpha f(E \hat{x})|^2 \, d\hat{x} \right)^{1/2}
\]

\[
\leq ch^2 |T_i| \left( \frac{|T|}{|T_i|} \int_{T_i} \sum_{|\alpha| = 2} |D^\alpha f(x)|^2 \, dx \right)^{1/2}
\]

\[
\leq ch^2 \sqrt{|T_i|} |f|_{H^2(T_i)}
\]

This implies

\[
\sum_{i \in I} \left| \int_{T_i} (f(x) - f(S_i)) \, dx \right| \leq ch^2 \left( \sum_{i \in I} |f|^2_{H^2(T_i)} \right)^{1/2}
\]

by the Cauchy-Schwarz inequality. \(\square\)

**Lemma 3.3.** Let \(w_h\) be the functions defined by \((2.6)\). In addition, we assume that the assumptions \((A1)-(A3)\) are fulfilled. Then the estimate

\[
\|y_h(\bar{u}) - y_h(w_h)\|_{L^2(\Omega)} \leq ch^2 \|\bar{p}\|_{W^{2,p}(\Omega)} \tag{3.3}
\]

holds true.
Proof. We start with the transformation
\[ \|y_h(\bar{u}) - y_h(w_h)\|_{L^2(\Omega)}^2 = (y_h(\bar{u}) - y_h(w_h), y_h(\bar{u}) - y_h(w_h))_{L^2(\Omega)} \]
\[ = (p_h(\bar{u}) - p_h(w_h), \bar{u} - w_h)_{L^2(\Omega)} \]
\[ = \int_{K_1} (p_h(\bar{u}) - p_h(w_h))(\bar{u} - w_h) \, dx \]
\[ + \int_{K_2} (p_h(\bar{u}) - p_h(w_h))(\bar{u} - w_h) \, dx \]
(3.4)

It remains to estimate these two integrals. The $K_1$-part can be estimated by the following arguments: The function $\bar{p}$ belongs to $W^{2,p}(\Omega)$ with $p > 2$. Hence, we have
\[ \|\bar{u}\|_{C^0(\overline{\Omega})} \leq \frac{1}{\nu} \|\bar{p}\|_{C^0(\overline{\Omega})} \leq c \|\bar{p}\|_{W^{2,p}(\Omega)}. \]

Because of $\bar{u}(S_i) = w_h(S_i)$ and the fact that $w_h$ is constant on $T_i$, this implies $|\bar{u}(x) - w_h(x)| \leq c\|\bar{p}\|_{W^{2,p}(\Omega)} \cdot |x - S_i| \leq ch\|\bar{p}\|_{W^{2,p}(\Omega)}$. Consequently, we obtain
\[ \left| \int_{K_1} (p_h(\bar{u}) - p_h(w_h))(\bar{u} - w_h) \, dx \right| \leq \sum_{T_i \in K_1} \int_{T_i} |(p_h(\bar{u}) - p_h(w_h))(\bar{u} - w_h)| \, dx \]
\[ \leq \sum_{T_i \in K_1} ch\|\bar{p}\|_{W^{2,p}(\Omega)} \|p_h(\bar{u}) - p_h(w_h)\|_{C(T_i)} \int_{T_i} dx \]
\[ \leq ch\|\bar{p}\|_{W^{2,p}(\Omega)} \|p_h(\bar{u}) - p_h(w_h)\|_{C(T_i)} \int_{K_1} dx \]
\[ \leq ch^2 \|\bar{p}\|_{W^{2,p}(\Omega)} \|p_h(\bar{u}) - p_h(w_h)\|_{C(T_i)} \]  
(3.5)

by means of assumption (A3). For a triangle $T_i$ of the $K_2$-part we have for an arbitrary function $v_h \in V_h$
\[ \int_{T_i} w_h v_h \, dx = \int_{T_i} \bar{u}(S_i) v_h \, dx = \int_{T_i} \bar{u}(S_i) v_h(S_i) \, dx. \]

This is a formula for the numerical integration of $\bar{u} v_h$. Consequently, we obtain by Lemma 3.2
\[ \left| \int_{K_2} (\bar{u} - w_h) v_h \, dx \right| \leq \sum_{T_i \in K_2} \left| \int_{T_i} (\bar{u} - \bar{u}(S_i)) v_h \, dx \right| \]
\[ \leq ch^2 \left( \sum_{T_i \in K_2} |\bar{u} v_h|^2_{H^2(T_i)} \right)^{1/2}. \]  
(3.6)

Next, we divide each triangle $T_i$ of $K_2$ in an "active" ($A_i$) and an "inactive" part ($I_i$) with $A_i \cup I_i = T_i$. The optimal control $\bar{u}$ is constant on the active component $A_i$ ($\bar{u} = a$ or $\bar{u} = b$). Therefore, the seminorm is 0 on these parts. On the inactive parts $I_i$, we have
\[ \bar{u} = -\frac{1}{\nu} \bar{p}. \]

Therefore, we can estimate
\[ |\bar{u} v_h|_{H^2(T_i)} = |\bar{u} v_h|_{H^2(I_i)} = \frac{1}{\nu} |\bar{p} v_h|_{H^2(I_i)} \leq c |\bar{p} v_h|_{H^2(T_i)}. \]
Hence, we can continue by

\[ \left| \int_{K_2} (\bar{u} - w_h)v_h \, dx \right| \leq ch^2 \left( \sum_{T_i \in K_2} |\bar{u}v_h|_{H^1(T_i)}^2 \right)^{1/2} \]

\[ \leq ch^2 \left( \sum_{T_i \in K_2} |\bar{\bar{u}}v_h|_{H^1(T_i)}^2 \right)^{1/2} \]

\[ \leq ch^2 \left( \sum_{T_i \in K_2} \sum_{|\alpha|,|\beta|=1} \|D^{\alpha+\beta}\bar{\bar{u}}v_h\|_{L^2(T_i)}^2 + \|D^\alpha D^\beta v_h\|_{L^2(T_i)}^2 \right)^{1/2} \]

\[ \leq ch^2 \|\bar{\bar{p}}\|_{W^{2,p}(\Omega)} \|v_h\|_{H^1_0(\Omega)}. \tag{3.7} \]

by means of Hölder’s inequality in the last step. Next, we set \( v_h = p_h(\bar{u}) - p_h(w_h) \) and obtain

\[ \left| \int_{K_2} (\bar{u} - w_h)(p_h(\bar{u}) - p_h(w_h)) \, dx \right| \leq ch^2 \|\bar{\bar{p}}\|_{W^{2,p}(\Omega)} \|p_h(\bar{u}) - p_h(w_h)\|_{H^1_0(\Omega)}. \tag{3.8} \]

Inserting (3.5) and (3.8) in (3.4), we get

\[ \|y_h(\bar{u}) - y_h(w_h)\|_{L^2(\Omega)} \leq ch^2 \|\bar{\bar{p}}\|_{W^{2,p}(\Omega)} (\|p_h(\bar{u}) - p_h(w_h)\|_{C(\Omega)} + \|p_h(\bar{u}) - p_h(w_h)\|_{H^1_0(\Omega)}). \]

We benefit now from the fact that \( p_h(\bar{u}) \) and \( p_h(w_h) \) are the solutions of the discretized adjoint equation (2.5). Hence, we have

\[ \|p_h(\bar{u}) - p_h(w_h)\|_{C(\Omega)} + \|p_h(\bar{u}) - p_h(w_h)\|_{H^1_0(\Omega)} \leq c \|y_h(\bar{u}) - y_h(w_h)\|_{L^2(\Omega)} \]

with a positive constant \( c \) independent of \( h \). The C-estimate can be obtained as follows: Take the adjoint equation (2.1) and the discretized adjoint equation (2.5), but with a right hand side \( f_h \in V_h \) instead of \( y - y_d \). Then we find for the corresponding solutions \( z \) and \( z_h \) of the continuous and discretized adjoint equation

\[ \|z_h\|_{C(\Omega)} \leq \|z_h - z\|_{C(\Omega)} + \|z\|_{C(\Omega)} \]

\[ \leq ch\|f_h\|_{L^2(\Omega)} + c\|z\|_{H^1(\Omega)} \]

\[ \leq ch\|f_h\|_{L^2(\Omega)} + c\|f_h\|_{L^2(\Omega)}. \]

Substituting \( f_h = y_h(\bar{u}) - y_h(w_h) \) and \( z_h = p_h(\bar{u}) - p_h(w_h) \) delivers the desired estimate. For the estimate of the first expression, we refer to Braess [3].

Finally, we get

\[ \|y_h(\bar{u}) - y_h(w_h)\|_{L^2(\Omega)} \leq ch^2 \|\bar{\bar{p}}\|_{W^{2,p}(\Omega)} \]

which is exactly inequality (3.3). \( \square \)

**Corollary 3.4.** Assume that the assumptions of Lemma 3.3 are fulfilled. Then, we have

\[ \|p_h(\bar{u}) - p_h(w_h)\|_{L^2(\Omega)} \leq ch^2 \|\bar{\bar{p}}\|_{W^{2,p}(\Omega)}. \tag{3.9} \]

By means of Lemma 3.1, we obtain

\[ \|\bar{\bar{p}} - p_h(w_h)\|_{L^2(\Omega)} \leq ch^2 (\|\bar{\bar{p}}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}). \tag{3.10} \]
Lemma 3.5. The following variational inequalities are necessary and sufficient for the optimality of the unique solutions of (1.1)–(1.3) and (2.4), respectively.

\[
\begin{align*}
(p + \nu \bar{u}, u - \bar{u})_{L^2(\Omega)} &\geq 0 \quad \text{for all } u \in U_{\text{ad}}, \\
(p_h(u_h) + \nu u_h, \zeta_h - u_h)_{L^2(\Omega)} &\geq 0 \quad \text{for all } \zeta_h \in U_h^{\text{ad}}.
\end{align*}
\]  

(3.11) (3.12)

The variational inequality (3.11) is an equivalent formulation for the projection formula (2.2).

Next, we derive a variational inequality for the function \( w_h \). First, formula (3.11) is true for all \( u \in U_{\text{ad}} \). Therefore, we have pointwise a.e.

\[
(p(x) + \nu \bar{u}(x)) \cdot (u - \bar{u}(x)) \geq 0 \quad \forall u \in [a, b].
\]

We apply this formula for \( x = S_i \) and \( u = u_h(S_i) \). This is correct because of the continuity of \( \bar{u} \), \( \bar{p} \), and \( u_h \) in these points. We arrive at

\[
(p(S_i) + \nu \bar{u}(S_i)) \cdot (u_h(S_i) - \bar{u}(S_i)) \geq 0 \quad \text{for all } S_i.
\]

Due to (2.6), this is equivalent to

\[
(p(S_i) + \nu w_h(S_i)) \cdot (u_h(S_i) - w_h(S_i)) \geq 0 \quad \text{for all } S_i.
\]

We integrate this formula over \( T_i \) and add up over all \( i \)

\[
(p + \nu w_h, u_h - w_h)_{L^2(\Omega)} \geq 0
\]

(3.13)

with

\[
p(x) = \bar{p}(S_i) \quad \text{if } x \in T_i.
\]

Moreover, we can test inequality (3.12) with the function \( w_h \) and get

\[
(p_h(u_h) + \nu u_h, w_h - u_h)_{L^2(\Omega)} \geq 0.
\]

(3.14)

We add these two inequalities and obtain

\[
(p - p_h(u_h) + \nu (w_h - u_h), u_h - w_h)_{L^2(\Omega)} \geq 0.
\]

This is equivalent to

\[
\nu \|w_h - u_h\|^2_{L^2(\Omega)} \leq (p - p_h(u_h), u_h - w_h)_{L^2(\Omega)}.
\]

(3.15)

4. Superconvergence properties. Inequality (3.15) is the starting point for the proofs of the superconvergence results. Now, we are ready to prove Theorem 2.3.

Proof. For the right-hand side of (3.15), we find

\[
(p - p_h(u_h), u_h - w_h)_{L^2(\Omega)} = (p_h(w_h) - p_h(u_h), u_h - w_h)_{L^2(\Omega)} \\
+ (p - p_h(w_h), u_h - w_h)_{L^2(\Omega)} \\
+ (\bar{p} - \bar{p}, u_h - w_h)_{L^2(\Omega)}.
\]

(4.1)

Next we estimate these three terms. We start with

\[
(p_h(w_h) - p_h(u_h), u_h - w_h)_{L^2(\Omega)} = (y_h(w_h) - y_h(u_h), y_h(u_h) - y_h(w_h))_{L^2(\Omega)} \leq 0.
\]

(4.2)
The second term can be estimated by formula (3.10)
\[ (\bar{p} - p_h(w_h), u_h - w_h)_{L^2(\Omega)} \leq c h^2 (\|\bar{p}\|_{W^{2,p}\Omega} + \|y_d\|_{L^2(\Omega)}) \cdot \|w_h - u_h\|_{L^2(\Omega)}. \] (4.3)

The third term represents again a formula for the numerical integration. Using that \( u_h \) and \( w_h \) are constant on each triangle \( T_i \), we obtain by Lemma 3.2
\[ (\bar{p} - \hat{p}, u_h - w_h)_{L^2(\Omega)} = \sum_i \int_{T_i} (\hat{p}(x) - \bar{p}(x))(u_h(x) - w_h(x)) \, dx \]
\[ = \sum_i (u_h(S_i) - w_h(S_i)) \int_{T_i} (\hat{p}(S_i) - \bar{p}(x)) \, dx \]
\[ \leq \sum_i c h^2 |u_h(S_i) - w_h(S_i)| \sqrt{|T_i|} \cdot |\bar{p}|_{H^2(T_i)} \]
\[ \leq c h^2 \cdot \|w_h - u_h\|_{L^2(\Omega)} \cdot \|\bar{p}\|_{W^{2,p}(\Omega)}. \] (4.4)

Inserting (4.2)–(4.4) in (4.1), we get
\[ (\bar{p} - p_h(u_h), u_h - w_h)_{L^2(\Omega)} \leq c h^2 (\|\bar{p}\|_{W^{2,p}\Omega} + \|y_d\|_{L^2(\Omega)}) \cdot \|w_h - u_h\|_{L^2(\Omega)}. \]

Next, we combine this inequality with (3.15)
\[ \nu \|w_h - u_h\|_{L^2(\Omega)}^2 \leq c h^2 (\|\bar{p}\|_{W^{2,p}\Omega} + \|y_d\|_{L^2(\Omega)}) \cdot \|w_h - u_h\|_{L^2(\Omega)}. \]

This formula is equivalent to
\[ \|w_h - u_h\|_{L^2(\Omega)} \leq c h^2 (\|\bar{p}\|_{W^{2,p}\Omega} + \|y_d\|_{L^2(\Omega)}), \]

which was the assertion of Theorem 2.3.

Theorem 2.3 means that the values of the numerical solution \( u_h \) in the centroids have already quadratic convergence rate. By the projection of the associated adjoint state \( p_h(u_h) \), we obtain an admissible control \( \bar{u} \) that has a quadratic convergence order with respect to the \( L^2 \)-norm. This was the assertion of Theorem 2.4.

\textbf{Proof.} We start with the result of Theorem 2.3
\[ \|w_h - u_h\|_{L^2(\Omega)} \leq c h^2 (\|\bar{p}\|_{W^{2,p}\Omega} + \|y_d\|_{L^2(\Omega)}). \]

This inequality implies
\[ \|p_h(w_h) - p_h(u_h)\|_{L^2(\Omega)} \leq c \|w_h - u_h\|_{L^2(\Omega)} \leq c h^2 (\|\bar{p}\|_{W^{2,p}\Omega} + \|y_d\|_{L^2(\Omega)}). \]

From Corollary 3.4, we know formula (3.10)
\[ \|\bar{p} - p_h(w_h)\|_{L^2(\Omega)} \leq c h^2 (\|\bar{p}\|_{W^{2,p}\Omega} + \|y_d\|_{L^2(\Omega)}). \]

Therefore, we obtain by the triangle inequality
\[ \|\bar{p} - p_h(u_h)\|_{L^2(\Omega)} \leq c h^2 (\|\bar{p}\|_{W^{2,p}\Omega} + \|y_d\|_{L^2(\Omega)}). \]

The projection operator \( \Pi_{[a,b]} \) is Lipschitz continuous with constant 1 from \( L^2(\Omega) \) to \( L^2(\Omega) \). Finally, we get
\[ \|\bar{u} - \bar{u}\|_{L^2(\Omega)} \leq c h^2 (\|\bar{p}\|_{W^{2,p}\Omega} + \|y_d\|_{L^2(\Omega)}). \]

The superconvergence result is proved.
COROLLARY 4.1. By the arguments of the proof of Theorem 2.4, we get another result. We find for the $L^\infty$-error
\[ \| \bar{u} - \tilde{u} \|_{L^\infty(\Omega)} \leq ch(\| \tilde{p} \|_{W^{2,p}(\Omega)} + \| y_d \|_{L^2(\Omega)}). \]

Proof. From formula (3.3)
\[ \| y_h(w_h) - y_h(\bar{u}) \|_{L^2(\Omega)} \leq ch^2 \| \tilde{p} \|_{W^{2,p}(\Omega)} \]
and the inequality
\[ \| y_h(w_h) - y_h(u_h) \|_{L^2(\Omega)} \leq c \| w_h - u_h \|_{L^2(\Omega)} \leq ch^2 (\| \tilde{p} \|_{W^{2,p}(\Omega)} + \| y_d \|_{L^2(\Omega)}), \]
we get by the triangle inequality
\[ \| y_h(\tilde{u}) - y_h(u_h) \|_{L^2(\Omega)} \leq ch^2 (\| \tilde{p} \|_{W^{2,p}(\Omega)} + \| y_d \|_{L^2(\Omega)}). \]
This inequality implies
\[ \| p_h(\tilde{u}) - p_h(u_h) \|_{L^\infty(\Omega)} \leq c \| y_h(\tilde{u}) - y_h(u_h) \|_{L^2(\Omega)} \leq ch^2 (\| \tilde{p} \|_{W^{2,p}(\Omega)} + \| y_d \|_{L^2(\Omega)}). \]
Denoting the solution of (2.5) with $\bar{y}$ instead of $y_h(u_h)$ by $\tilde{p}_h$, we continue with
\[ \| \tilde{p} - p_h(u_h) \|_{L^\infty(\Omega)} \leq \| \tilde{p} - \tilde{p}_h \|_{L^\infty(\Omega)} + \| \tilde{p}_h - p_h(\bar{u}) \|_{L^\infty(\Omega)} + \| p_h(\bar{u}) - p_h(u_h) \|_{L^\infty(\Omega)}. \]
(4.5)
The first term can be estimated by $ch \| \tilde{p} \|_{H^2(\Omega)}$ see [3]. For the second term, we use the argumentation of Lemma 3.1 with $z_h = p_h(\bar{u}) - p_h(u_h)$ and $f_h = y_h(\bar{u}) - \bar{y}$
\[ \| \tilde{p} - p_h(u_h) \|_{L^\infty(\Omega)} \leq ch \| \tilde{p} \|_{H^2(\Omega)} + c \| y_h(\bar{u}) - \bar{y} \|_{L^2(\Omega)} + ch^2 (\| \tilde{p} \|_{W^{2,p}(\Omega)} + \| y_d \|_{L^2(\Omega)}) \]
\[ \leq ch \| \tilde{p} \|_{H^2(\Omega)} + c \| y_h(u_h) \|_{L^2(\Omega)} + ch^2 (\| \tilde{p} \|_{W^{2,p}(\Omega)} + \| y_d \|_{L^2(\Omega)}) \]
\[ \leq ch (\| \tilde{p} \|_{W^{2,p}(\Omega)} + \| y_d \|_{L^2(\Omega)}). \]
The properties of the projection operator implies the assertion. \qed

COROLLARY 4.2. The first estimate can be improved, if all data are sufficiently smooth
\[ \| \tilde{p} - \tilde{p}_h \|_{L^\infty(\Omega)} \leq ch^2 | \ln h |^{3/2} \| \tilde{p} \|_{W^{2,\infty}(\Omega)}, \]
see Braess [3]. In this case, formula (4.5) implies
\[ \| \bar{u} - \tilde{u} \|_{L^\infty(\Omega)} \leq ch^2 | \ln h |^{3/2} (\| \tilde{p} \|_{W^{2,\infty}(\Omega)} + \| y_d \|_{L^2(\Omega)}). \]

5. Numerical tests. We have tested the convergence theory by two examples. In both cases, the Laplace operator $-\Delta$ was chosen for the elliptic operator $A$. The first example fits exactly to the presented theory.

Both optimization problems were solved numerically by a primal-dual active set strategy, see for instance [11]. The discretization was already described in section 2: The control function $u$ is discretized by piecewise constant functions, whereas the state $y$ and the adjoint state $p$ were approximated by piecewise linear functions. We used uniform meshes. The additional numerical effort for the calculation of $\tilde{u}$ is very small. We have only to evaluate the pointwise projection of the function $-\frac{1}{p} p_h$ to the interval $[a, b]$.\textellipsis
In contrast to this, the numerical evaluation of the $L^2$-norm $\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}$ and the graphical representation are not so simple. Therefore, we shortly sketch these aspects. We want to point out that this additional effort is only needed to confirm the theoretical results. This effort is not necessary for the computation of the approximated optimal control.

For the computation of the $L^2$-norms $\|\bar{u} - u_k\|_{L^2(\Omega)}$ and $\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}$, respectively, we introduce sets $\tilde{K}_1$, $\tilde{K}_2$ analogue to the sets $K_1$ and $K_2$:

$$\tilde{K}_1 = \{ T_i : \tilde{u} \text{ is only Lipschitz continuous on } T_i \}, \quad \tilde{K}_2 = \{ T_i : \tilde{u} \in C^{\infty}(T_i) \}.$$  

Moreover, we set $M_1 = K_1 \cup \tilde{K}_1$, $M_2 = K_2 \cap \tilde{K}_2$. The numerical evaluation of $\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}$ differs on the sets $M_1$ and $M_2$. Therefore, we split the $L^2$-norm up into

$$\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}^2 = \|\bar{u} - \tilde{u}\|_{L^2(M_1)}^2 + \|\bar{u} - \tilde{u}\|_{L^2(M_2)}^2.$$  

In our examples, the part $\tilde{u}_{|K_2} \in C^{\infty}(K_2)$ is smooth. Thus, $\|\bar{u} - \tilde{u}\|_{L^2(M_2)}$ can be evaluated with sufficient accuracy applying an appropriate quadrature formula. In contrast to this, the difference $\tilde{u} - \bar{u}$ only belongs to $C^{0,1}(T_i)$ for all triangles $T_i \in M_1$. Hence, an arbitrary accurate quadrature formula would only admit an error of order $h$. Therefore, we introduce a subgrid of significant smaller mesh size in each triangle $T_i \in M_1$ and evaluate the norm $\|\bar{u} - \tilde{u}\|_{L^2(M_1)}$ on this subgrid to ensure sufficient accuracy. We want to point out that this subgrid is only used for the evaluation of the norm $\|\bar{u} - \tilde{u}\|_{L^2(M_1)}$ with a sufficient high accuracy.

**Example 1.** In this example, we investigate the Laplace equation with homogenous Dirichlet boundary conditions. Therfore, we choose $a_0 \equiv 0$ in (1.2). Thus, the state equation is given by

$$-\Delta y = u \quad \text{in } \Omega$$

$$y = 0 \quad \text{on } \Gamma. \quad (5.1)$$

Now, we define the optimal state by

$$\bar{y} = y_a - y_g$$

with an analytical part $y_a = \sin(\pi x_1) \sin(\pi x_2)$ and a less smooth function $y_g$. The function $y_g$ is defined as the solution of

$$-\Delta y_g = g \quad \text{in } \Omega$$

$$y_g = 0 \quad \text{on } \Gamma.$$  

The function $g$ is given by

$$g(x_1, x_2) = \begin{cases} u_f(x_1, x_2) - a & \text{if } u_f(x_1, x_2) < a \\ 0 & \text{if } u_f(x_1, x_2) \in [a, b] \\ u_f(x_1, x_2) - b & \text{if } u_f(x_1, x_2) > b \end{cases}$$

with $u_f(x_1, x_2) = 2\pi^2 \sin(\pi x_1) \sin(\pi x_2)$. Due to the state equation (5.1), we obtain for the exact optimal control $\bar{u}$

$$\bar{u}(x_1, x_2) = \begin{cases} a & \text{if } u_f(x_1, x_2) < a \\ u_f(x_1, x_2) & \text{if } u_f(x_1, x_2) \in [a, b] \\ b & \text{if } u_f(x_1, x_2) > b. \end{cases}$$

For the optimal adjoint state $\bar{\rho}$, we find

$$\bar{\rho}(x_1, x_2) = -2\pi^2 \nu \sin(\pi x_1) \sin(\pi x_2).$$
The desired state is given by
\[ y_d(x_1, x_2) = \bar{y} + \Delta \bar{p} = y_a - y_g + 4 \pi^4 \nu \sin(\pi x_1) \sin(\pi x_2). \]

It is easy to see that these functions fulfill the necessary and sufficient first-order optimality conditions. Moreover, the sets with
\[ -\frac{1}{\nu} \bar{p} = a \quad \text{or} \quad -\frac{1}{\nu} \bar{p} = b \]
are a finite number (here two) of curves \( \gamma_i \). Hence, the measure of the set \( K_1 \) is bounded by the total length of these curves
\[ |K_1| \leq 2h \sum |\gamma_i| \]
and Assumption (A3) is fulfilled. We chose \( \nu = 1 \) for the numerical calculations.

Figures 5.1 and 5.2 show the numerical solutions \( \tilde{u} \) for \( h = 0.04 \) and \( h = 0.02 \). As described above, the function \( \tilde{u} \) is obtained by the pointwise projection \( \Pi_{[a,b]}(-\frac{1}{\nu} p_h) \) in a post-processing step. Therefore, \( \tilde{u} \) contains sharp breaks on the subset \( K_1 \). To visualize these breaks, we introduce new mesh points in all triangles \( T_i \in K_1 \). Notice that these new grid points are only used for the graphical presentation of the projection.

The following figures show the convergence behaviour of \( \| \bar{u} - u_h \|_{L^2(\Omega)} \) and \( \| \bar{u} - \tilde{u} \|_{L^2(\Omega)} \), respectively, for \( h = 0.04, 0.02, 0.01 \) and 0.005. In the figures, \( \bar{u} \) is denoted by \( u_{opt} \).
As one can see, the theoretical predictions are fulfilled and one obtains quadratic convergence for
\[ \|\bar{u} - \tilde{u}\|_{L^2(\Omega)}. \]
Furthermore, the absolute value of the error is significantly reduced by the projection, as the following table shows.

<table>
<thead>
<tr>
<th>( h/\sqrt{2} )</th>
<th>( |\bar{u} - u_h|_{L^2(\Omega)} )</th>
<th>( |\bar{u} - \tilde{u}|_{L^2(\Omega)} )</th>
<th>( |u_h - w_h|_{L^2(\Omega)} )</th>
<th>( |\bar{u} - w_h|_{L^2(\Omega)} )</th>
<th>( |\bar{u} - \bar{w}|_{L^\infty(\Omega)} )</th>
</tr>
</thead>
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<tr>
<td>0.04</td>
<td>0.34312</td>
<td>0.04856</td>
<td>0.05335</td>
<td>1.61552</td>
<td>0.13760</td>
</tr>
<tr>
<td>0.02</td>
<td>0.17155</td>
<td>0.01221</td>
<td>0.01342</td>
<td>0.81633</td>
<td>0.03513</td>
</tr>
<tr>
<td>0.01</td>
<td>0.08556</td>
<td>0.00306</td>
<td>0.00335</td>
<td>0.40975</td>
<td>0.00884</td>
</tr>
<tr>
<td>0.005</td>
<td>0.04281</td>
<td>0.00077</td>
<td>0.00084</td>
<td>0.20485</td>
<td>0.00221</td>
</tr>
</tbody>
</table>

Theoretical results with respect to the \( L^\infty(\Omega) \)-norm were addressed in the Corollaries 4.1 and 4.2. Again, we used finer subgrids for the numerical evaluation of the norms.

\[ \text{Fig. 5.5.} \ \|\bar{u} - u_h\|_{L^\infty(\Omega)} \]

\[ \text{Fig. 5.6.} \ \|\bar{u} - \bar{w}\|_{L^\infty(\Omega)} \]

**Example 2.** A Neumann boundary problem is studied in this example. In this case, the theoretical results does not exactly fit to the problem. The Lax-Milgram Theorem implies the existence of weak solutions \((\bar{y}, \bar{p} \in H^1(\Omega))\) for state equation and adjoint equation. The \( W^{2,p} - \)regularity of the solution of the adjoint equation is obtained by a result of Grisvard [9] Th. 4.4.1.2. For this special structure (Laplace-operator, homogeneous Neumann data, \( \Omega = \Omega_h \)), the proofs of our main results can be adapted. A general discussion of Neumann boundary conditions requires more detailed investigations. We discuss here the problem

\[ \begin{align*}
-\Delta y + cy = u & \quad \text{in } \Omega \\
\partial_n y = 0 & \quad \text{on } \Gamma,
\end{align*} \tag{5.2} \]

where \( \partial_n \) denotes the normal derivative with respect to the outward normal vector. Again, we construct the optimal state \( \bar{y} \) by \( \bar{y} = y_a - y_g \), with an analytical part \( y_a(x_1,x_2) = \cos(\pi x_1) \cos(\pi x_2) \). The function \( y_g \) is now determined by the following equation

\[ \begin{align*}
-\Delta y_g + cy_g &= g & \quad \text{in } \Omega \\
\partial_n y_g &= 0 & \quad \text{on } \Gamma,
\end{align*} \]

with the inhomogeneity

\[ g(x_1,x_2) = \begin{cases} 
  u_f(x_1,x_2) - a, & \text{if } u_f(x_1,x_2) < a \\
  0, & \text{if } u_f(x_1,x_2) \in [a,b] \\
  u_f(x_1,x_2) - b, & \text{if } u_f(x_1,x_2) > b
\end{cases} \]
and $u_f(x_1, x_2) = (2\pi^2 + c) \cos(\pi x_1) \cos(\pi x_2)$. The optimal control $\bar{u}$ is given by (5.2)

$$
\bar{u}(x_1, x_2) = \begin{cases} 
  a & \text{if } u_f(x_1, x_2) < a \\
  u_f(x_1, x_2) & \text{if } u_f(x_1, x_2) \in [a, b] \\
  b & \text{if } u_f(x_1, x_2) > b.
\end{cases}
$$

The optimal adjoint state is defined by

$$
\bar{p}(x_1, x_2) = -(2\pi^2 + c) \nu \sin(\pi x_1) \sin(\pi x_2).
$$

Moreover the desired state $y_d$ is chosen as

$$
y_d(x_1, x_2) = \bar{y} + \Delta \bar{p} - c \bar{p} = y_a - y_g + (4\pi^4 \nu + 4\pi^2 \nu c + \nu c^2) \sin(\pi x_1) \sin(\pi x_2).
$$

Again, it is easy to see that these functions fulfill the necessary and sufficient first-order optimality conditions. Assumption (A3) can be verified with the same arguments as in Example 1. In the numerical test, we chose $\nu = c = 1$. The projected function $\bar{u}$ for $h = 0.04$ and $h = 0.02$ is shown in figures 5.7 and 5.8. For the visualization of this projection we introduced again new grid points.

**Fig. 5.7. $\bar{u}$ at $h = 0.04$**

**Fig. 5.8. $\bar{u}$ at $h = 0.02$**

The following figures illustrate that we obtain the same convergence results for this Neumann boundary example.

**Fig. 5.9. $\|\bar{u} - u_h\|_{L^2(\Omega)}$**

**Fig. 5.10. $\|\bar{u} - u_h\|_{L^2(\Omega)}$**

Comparable with the first example, the absolute error is considerably reduced by the projection, as the following table shows.
<table>
<thead>
<tr>
<th>( h/\sqrt{2} )</th>
<th>( | u - u_h|_{L^2(\Omega)} )</th>
<th>( | \bar{u} - \bar{u}<em>h|</em>{L^2(\Omega)} )</th>
<th>( | u_h - w_h|_{L^2(\Omega)} )</th>
<th>( | u - u_h|_{L^\infty(\Omega)} )</th>
<th>( | \bar{u} - \bar{u}<em>h|</em>{L^\infty(\Omega)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.36168</td>
<td>0.10517</td>
<td>0.10659</td>
<td>1.84963</td>
<td>0.21285</td>
</tr>
<tr>
<td>0.02</td>
<td>0.17610</td>
<td>0.02652</td>
<td>0.02657</td>
<td>0.89765</td>
<td>0.05352</td>
</tr>
<tr>
<td>0.01</td>
<td>0.08744</td>
<td>0.00656</td>
<td>0.00663</td>
<td>0.44174</td>
<td>0.01340</td>
</tr>
<tr>
<td>0.005</td>
<td>0.04366</td>
<td>0.00164</td>
<td>0.00166</td>
<td>0.21905</td>
<td>0.00336</td>
</tr>
</tbody>
</table>

The convergence behaviour of the \( L^\infty(\Omega) \)-errors is illustrated in the next two figures.

![Fig. 5.11. \( \| u - u_h\|_{L^\infty(\Omega)} \)](image1)

![Fig. 5.12. \( \| \bar{u} - \bar{u}_h\|_{L^\infty(\Omega)} \)](image2)

The last two figures show the convergence behaviour of \( \| u_h - w_h\|_{L^2(\Omega)} \) (analyzed in Theorem 2.3) for the two examples.

![Fig. 5.13. \( \| u_h - w_h\|_{L^2(\Omega)} \) for Ex. 1](image3)

![Fig. 5.14. \( \| u_h - w_h\|_{L^2(\Omega)} \) for Ex. 2](image4)

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**REFERENCES**


