Density of various subsets in spaces of places of function fields, and applications to real holomorphy rings

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Happy birthday, Eberhard!

Herzlichen Glückwunsch zum Geburtstag, Eberhard!

Und vielen Dank für die effektive und freudvolle Zusammenarbeit!

The complicated valuations

Once Eberhard said, with a smile on his face:

"Der Kuhlmann beschäftigt sich ja immer mit den komplizierten Bewertungen".

Here he was thinking not only of positive characteristic, but also of valuations of arbitrary rank, i.e., with arbitrarily large value groups, not necessarily subgroups of the ordered additive group of the reals.

So, let's see...

Notation

Following Eberhard's preference, we write places as functions ξ written in the usual way, i.e., the image of an element a under the place ξ is written as $\xi(a)$. Its valuation ring is $\mathcal{O}_\xi$ and its maximal ideal is $\mathcal{M}_\xi$. The associated valuation will be denoted by v_ξ and its value group on a field F by $v_\xi F$. As a compromise, Eberhard accepted to write the residue field my way: $F\xi$.

The Zariski space $S(F)$ is the set of all places of the field F. If $F|K$ is an algebraic function field and $\wp$ is a place of K, then $S(F|K;\wp)$ denotes the subspace of $S(F)$ of all places ξ such that $\xi|_K = \wp$. We set

$$S(F|K) \;:=\; S(F|K;\mathrm{id}_K)\,.$$

The beginnings

The subject of density of subsets of places of algebraic function fields has spannend the entire length of my career. My first published paper was written jointly with Alexander Prestel:

On places of algebraic function fields, J. reine angew. Math. **353** (1984), 181–195.

The basic idea in this paper is to use the Ax-Kochen/Ershov theorem (which works over henselian valued fields of residue characteristic 0) to prove results about density of suitable subsets of $S(F|K)$, where $F|K$ is an algebraic function field of characteristic 0.

A quick proof of the Ax-Kochen/Ershov theorem is given in the paper.

The patch topology

When we speak of "density", we mean it with respect to the patch topology (also called constructible topology) which is finer than the Zariski topology and whose basic open sets are the sets of the form

$$\{\xi \in S(F) \mid a_1, \ldots, a_k \in \mathcal{O}_\xi \,;\, b_1, \ldots, b_\ell \in \mathcal{M}_\xi\}\,,$$

where $k, \ell \in \mathbb{N} \cup \{0\}$ and $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in F$. This induces the patch topologies on the subsets $S(F|K;\wp)$ and $S(F|K)$.

The Main Theorem

For the formulation of the Main Theorem, I need a few more notions. The dimension of a place $\xi \in S(F|K)$ is

$$\dim(\xi) \; := \operatorname{trdeg}(F\xi|K) \,.$$

The rational rank of an ordered abelian group Γ is

$$\operatorname{rr}\Gamma \; := \; \dim_{\mathbb{Q}} \mathbb{Q} \otimes \Gamma \,,$$

i.e., the maximal number of its rationally independent elements. We set $\operatorname{rr}(\xi) := \operatorname{rr} v_\xi F$.

The Main Theorem

Theorem (K – Prestel 1984)

Let $F|K$ *be an algebraic function field in* n *variables with* $\mathrm{char}\, K = 0$. *Take elements* $x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+s} \in F$ *and a place* $\xi \in S(F|K)$. *Then there exists a place* $\xi' \in S(F|K)$ *with finitely generated value group and residue field finitely generated over* K *such that*

$$\begin{aligned} \xi'(x_i) &= \xi(x_i) \quad \textit{for} \quad 1 \leq i \leq m \\ v_{\xi'}(x_i) &= v_\xi(x_i) \quad \textit{for} \quad m+1 \leq i \leq m+s\,. \end{aligned}$$

Moreover, if r_1 *and* d_1 *are natural numbers satisfying*

$$\dim(\xi) \leq d_1 \leq n-1 \ \textit{and} \ \mathrm{rr}\,(\xi) \leq r_1 \leq n - d_1\,,$$

then ξ' *can be chosen in addition to satisfy* $\dim(\xi') = d_1$ *and* $\mathrm{rr}\,(\xi') = r_1$.

In addition, certain conditions on the value group and the residue field can be realized. It follows from the first part of the theorem that the places of dimension d_1 and rational rank r_1 and satisfying the additional conditions lie dense in $S(F|K)$ with respect to the patch toology.

An application to holomorphy rings

Let me present one of the applications (of a modification of the theorem). We fix a field K and consider a class $\mathcal{K}$ of all field extensions F of K in which K is existentially closed. This means, roughly speaking, that every existential sentence which holds in F also holds in K. Examples are the classes

- of all extension fields of an algebraically closed field K,
- of all formally real extension fields of a real closed field K,
- of all formally p-adic extension fields of fixed p-rank d of a p-adically closed field K with the same p-rank,
- of all totally real extension fields of a maximal PRC-field K,
- of all totally real and regular extension fields of a PRC-field K.

Applications to holomorphy rings

Take an algebraic function field $F|K$ in n variables and call $\xi \in S(F|K)$ a $\mathcal{K}$-place if $F\xi \in \mathcal{K}$. Define the $\mathcal{K}$-holomorphy ring of $F|K$ by

$$H(\mathcal{K},F) \;:=\; \bigcap \{\mathcal{O}_\xi \mid \xi \text{ a } \mathcal{K}\text{-place of } F\} \,.$$

Theorem (K – Prestel 1984)

If $0 \leq d_1 \leq n-1$, *then*

$$\begin{aligned} H(\mathcal{K},F) \;&=\; \bigcap \{\mathcal{O}_\xi \mid \xi \textit{ a } \mathcal{K}\textit{-place of } F \textit{ with} \\ &\qquad \dim(\xi) = d_1 \textit{ and } v_\xi F = \mathbb{Z}\} \,. \end{aligned}$$

Moreover, we may restrict the intersection to $\mathcal{K}$*-places* ξ *such that* $F\xi$ *is a finitely generated subfield of a purely transcendental extension of* K.

Applications to holomorphy rings

Let me present another result proven in the paper. Observe that a class $\mathcal{K}$ of all field extensions F of K in which K is existentially closed is closed under subfields and purely transcendental extensions. Let us consider a class $\mathcal{K}$ that has these two properties, but where K need not be existentially closed in all extensions $F \in \mathcal{K}$. Even then one can prove:

Proposition (K – Prestel 1984)

Take an algebraic function field $F|K$ in n variables and let $\mathcal{K}$ be as described. Then

$$H(\mathcal{K},F) \;=\; \bigcap\{\mathcal{O}_\xi \mid \xi \textit{ a } \mathcal{K}\textit{-place of } F \textit{ with } \dim(\xi) = n-1\}\,.$$

Applications to holomorphy rings

For the class $\mathcal{K}$ of formally real extensions of K, this result was proved by Eberhard in

The real holomorphy ring and sums of $2n$*th powers*, Real algebraic geometry and quadratic forms (Rennes, 1981), 139–181, Lecture Notes in Math. **959**, Springer, Berlin-New York, 1982.

Bröcker and Schülting

Using Hironaka's Resolution of singularities in characteristic 0, Ludwig Bröcker and Heinz-Werner Schülting arrived at quite similar results in their paper

Valuations of function fields from the geometrical point of view, J. Reine Angew. Math. **365** (1986), 12–32.

This gave rise to the question: What is the relation between the Ax-Kochen/Ershov Theorem and Resolution of Singularities?

At the time, Peter Roquette suspected that Local Uniformization, the local form of Resolution of Singularities, in characteristic 0 could possibly be proved using the Ax-Kochen/Ershov Theorem.

Ax-Kochen/Ershov vs. Local Uniformization

At that time, Peter had some idea on how that could work, until I showed him that this idea did not work. From that point on, Peter never came back to that subject. That was actually a pity, because if he had pushed harder, the true relation between the model theory of valued fields and local uniformization could perhaps have been clarified much earlier.

Can the Main Theorem be generalized to positive characteristic if instead of Ax-Kochen/Ershov some other model theoretical principle is used? Here, the model theory of tame valued fields comes to our rescue. A henselian valued field (K, ξ) is a tame field if

(TF1) the value group is divisible by p if $p = \mathrm{char}\, K\xi > 0$,

(TF2) the residue field is perfect,

(TF3) every finite extension $(L|K, \xi)$ is defectless, i.e.,

$$[L : K] = (v_\xi L : v_\xi K)[L\xi : K\xi] \, .$$

The next chapter

Using the model theory of tame fields, I proved a generalization of the Main Theorem in the paper

On places of algebraic function fields in arbitrary characteristic, Advances in Math. **188** (2004), 399–424.

This version also covers positive characteristic, but some of the previous conditions can only be met up to the effects of inseparability. Nevertheless, the theorem can be used to prove that the places that admit Local Uniformization lie dense in the Zariski space with respect to the patch topology.

The next and up to now final step in the development of the Main Theorem is taken in my joint paper with Eberhard and Katarzyna:
Density of Composite Places in Function Fields and Applications to Real Holomorphy Rings, Mathematische Nachrichten **296** (2023), 57–79.

One interesting subset of the space $S(F)$ for an algebraic function field $F|K$ was previously entirely missed: the set of those places ξ that factor over a fixed place $\wp$ of K, or in other words, are composite with $\wp$, which means that there is a place $\lambda \in S(F|K)$ such that $F\lambda = K$ and $\xi = \wp \circ \lambda$.
The paper presents a version of the Main Theorem that addresses subsets of $S(F)$ consisting of such places. Here, we are only interested in its real version, which I will call the "Real New Main Theorem". To explain what we need it for, I will need some notation.

Some notation

Take a real function field F over a real closed field R. By $M(F)$ we denote the set of all $\mathbb{R}$-places of F, i.e., places ξ of F with residue field $F\xi \subseteq \mathbb{R}$. These are exactly the places associated with the natural valuations of the orderings on F. Every real closed field R has a unique $\mathbb{R}$-place ξ_R , which we call its natural $\mathbb{R}$-place. We consider the set

$$M(F|R) \;=\; \{\lambda \in S(F|R) \mid F\lambda = R\}$$

of all R-rational places. The new object we study in the paper is

$$M_R(F) \;:=\; \{\xi_R \circ \lambda \mid \lambda \in M(F|R)\} \;\subseteq\; S(F|R\,;\,\xi_R)\,,$$

the set of all $\mathbb{R}$-places of F that factor over ξ_R . The Real New Main Theorem implies that for every $\mathbb{R}$-place ξ of F there is an $\mathbb{R}$-place ξ' of F that factors over ξ_R and is “very close to ξ”.

The set $M(F)$, its subset $M_R(F)$ and the set $M(F|R)$ carry natural topologies. The topology of $M(F)$ as described by Dubois is compact and Hausdorff; we denote it by Top $M(F)$. It is a quotient topology of the space of orderings with the Harrison topology. Its basic open sets are

$$U(f_1, ..., f_m) \;:=\; \{\xi \in M(F) \mid \xi(f_i) > 0 \text{ for } 1 \le i \le m\}$$

where $f_1, ..., f_m$ lie in the real holomorphy ring

$$H(F) \;:=\; \bigcap \{\mathcal{O}_\xi \mid \xi \text{ a real place of } F\}\,,$$

where ξ is called a real place of F if $F\xi$ is formally real.

A first version of the Real New Main Theorem

When Eberhard started to work with Katarzyna, one of his interests, inspired by earlier work of Katarzyna, was to study the spaces $M(F)$, $M_R(F)$ and $M(F|R)$ and their topologies. The latter two had previously found little, if any, attention in the literature. I was called in for help and produced a first version of the Real New Main Theorem:

Theorem

Let F be an algebraic function field over a nonarchimedean real closed field R, and ξ_R the natural $\mathbb{R}$-place of R. Take $\xi \in M(F)$ and a corresponding ordering $<$ on F. Further, take elements $a_1, \ldots, a_m \in F$. Then there is a place $\lambda \in M(F|R)$ such that

a) if $a_i > 0$, then $\lambda(a_i) > 0$,
b) if $a_i \in \mathcal{M}_\xi$, then $\lambda(a_i) \in \mathcal{M}_{\xi_R}$.

Here are some applications.

Proposition (Eberhard Becker – Katarzyna Kuhlmann – K 2023)

Let F be an algebraic function field over a real closed field R. Then

1) The set $M_R(F)$ is dense in $M(F)$ with respect to $\mathrm{Top}\, M(F)$.

2) If in addition R is non-archimedean, then every nonempty intersection of an open set in the Zariski patch topology of $M(F)$ with an open set in $\mathrm{Top}\, M(F)$ *contains infinitely many places from $M_R(F)$.*

Results

Theorem (Eberhard Becker – Katarzyna Kuhlmann – K 2023)

Let F be an algebraic function field over a real closed field R, and ξ_R be the natural $\mathbb{R}$-place of R. Then:
1) The mapping

$$\iota_{F|R} : M(F|R) \to M_R(F),\ \lambda \mapsto \xi_R \circ \lambda$$

is a bijection.
2) $\iota_{F|R}$ is a topological embedding of $M(F|R)$ into $M(F)$.
3) All nonempty open sets in $M(F|R)$, $M(F)$ and $M_R(F)$ are infinite.
4) In particular, none of the spaces $M(F|R)$, $M(F)$ and $M_R(F)$ admit any isolated points.

I will now present the full Real New Main Theorem. Take a real closed field R and an ordered function field $(F,<)$ over R. Assume that $\wp$ is a place on R compatible with its ordering and ξ is a place on F compatible with $<$. Take elements $a_1,\ldots,a_m \in F$. Choose $r \in \mathbb{N}$ such that

$$1 \;\leq\; r \;\leq\; s \;:=\; \operatorname{trdeg} F|K\,,$$

and an arbitrary ordering on $\mathbb{Z}^r$; denote by Γ the so obtained ordered abelian group. Then the following holds.

The Real New Main Theorem

Theorem (Eberhard Becker – Katarzyna Kuhlmann – K 2023)

There is a place $\lambda \in M(F|R)$ *such that, with*

$$\xi' := \wp \circ \lambda \in S(F|K;\,\wp),$$

we have $v_\lambda F = \Gamma$ *and for* $1 \leq i \leq m$,

(i) *if* $a_i \in \mathcal{O}_\xi$, *then* $\lambda(a_i) \in \mathcal{O}_\wp$ *and* $a_i \in \mathcal{O}_{\xi'}$,
(ii) *if* $\infty \neq \xi(a_i) > 0$, *then* $\xi'(a_i) > 0$.

The following assertions can also be realized for $1 \leq i \leq m$ *if, in case* $\wp$ *is trivial, we assume that also* ξ *is trivial:*

(iii) *if* $a_i \in \mathcal{M}_\xi$, *then* $\lambda(a_i) \in \mathcal{M}_\wp$ *and* $a_i \in \mathcal{M}_{\xi'}$,
(iv) *if* $\xi(a_i) \in R\wp$, *then* $\xi'(a_i) = \xi(a_i)$,
(v) $\lambda(a_i) \neq 0, \infty$,
(vi) *if* $a_i > 0$, *then* $\lambda(a_i) > 0$.

If $\wp$ *is trivial while* ξ *is not, then in addition to assertions (i) and (ii), we can realize also (iii) and (iv), or alternatively, (v) and (vi).*

In the paper, also real relative holomorphy rings

$$H(F|D) := \bigcap\{\mathcal{O}_\xi \mid \xi \text{ a real place of } F \text{ such that } D \subseteq \mathcal{O}_\xi\}$$

are studied for any subring D of F. Let me present some of the results. We call D a real valuation ring if it is the valuation ring of a real place ξ_D . In this case,

$$H(F|D) = \bigcap\{\mathcal{O}_{\xi_D\circ\lambda} \mid \lambda \in M(F|R)\} .$$

The following result is proved in the paper.

More applications

Theorem (Eberhard Becker – Katarzyna Kuhlmann – K 2023)

Let B, C be two real valuation rings of R. Then we have:

1) If $H(F|B) = H(F|C)$, then $B = C$.

2) If $B \neq R$, then the following statements are equivalent for each subset $\mathcal{F}$ of $M(F|R)$:

(a) $H(F|B) \;=\; \bigcap_{\lambda\in\mathcal{F}} \mathcal{O}_{\xi_B\circ\lambda}$,

(b) $\mathcal{F}$ is dense in $M(F|R)$.

3) If $B \neq R$, then there is no representation of the form (a) with a minimal $\mathcal{F}$.

4) $H(F|C)$ admits a representation of the form (a) with a minimal $\mathcal{F}$ if and only if $C = R$ and $\operatorname{trdeg} F|R = 1$. In the case of a minimal representation we necessarily have that $\mathcal{F} = M(F|R)$.

More applications

Here is another application, which is proven using the real spectra of $H(F|R)$ and $H(F)$:

Proposition (Eberhard Becker – Katarzyna Kuhlmann – K 2023)

$M(F)$ has only finitely many connected components.

One more result

Take an algebraic function field F over a non-archimedean real closed field R. Consider any smooth projective model X of F. For $f \in F$, by X_f we denote the set of those rational points for which f is in the local ring $\mathcal{O}_x$, i.e., f is defined in x. We set

$$H_X := \{f \in F \mid f(x) \in H(R) \text{ for every } x \in X_f\} .$$

The following is proven using Hironaka's Resolution of Singularities:

Theorem (Eberhard Becker – Katarzyna Kuhlmann – K 2023)

Take a function field F over a non-archimedean real closed field R. Then $H(F)$ is the intersection of the sets H_X where X runs through all smooth projective models of F.

Note that this theorem is not true for an archimedean real closed field R since in this case $H_X = F$ for every smooth projective model X of F.

While I was discussing with Katarzyna the contents of our paper with Eberhard, she said:

“And Eberhard was extending and extending, coming with new ideas all the time, while we couldn’t await to publish our paper.”

You see, such are the risks when you write a paper with Eberhard. But we greatly enjoyed it.

THE END

Thank you for your attention!

More detailed information

Preprints and further information:

https://www.valth.eu/Valth.html.

My new personal homepage:

https://www.fvkuhlmann.de/