A NOTE ON A PRIORI L^P-ERROR ESTIMATES FOR THE OBSTACLE PROBLEM

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Abstract. This paper is concerned with a priori error estimates for the piecewise linear finite element approximation of the classical obstacle problem. We demonstrate by means of two one-dimensional counterexamples that the L^2-error between the exact solution u and the finite element approximation u_h is typically not of order two even if the exact solution is in $H^2(\Omega)$ and an estimate of the form $\|u - u_h\|_{H^1} \leq Ch$ holds true. This shows that the classical Aubin-Nitsche trick which yields a doubling of the order of convergence when passing over from the H^1- to the L^2-norm cannot be generalized to the obstacle problem.

Key words. A priori error analysis, Aubin-Nitsche trick, Obstacle problem

1. Introduction. While H^1- and L^∞-error estimates for the piecewise linear finite element approximation of the unilateral obstacle problem

$$\begin{aligned} \min \quad & \frac{1}{2}\int_\Omega \nabla v \cdot \nabla v \,\mathrm{d}x - \langle f, v\rangle \\ \text{s.t.} \quad & v \in H_0^1(\Omega) \quad \text{and} \quad v \geq \psi \text{ a.e. in } \Omega \end{aligned}$$

are classical (see, e.g., [1,2,4,10]), there are still several open questions regarding the behavior of the finite element error in lower L^p-norms. Especially the question of whether a duality argument similar to that of the well-known Aubin-Nitsche trick can be used in the case of the obstacle problem to obtain an L^2-error estimate of order two appears frequently in the literature (see, e.g., [7,9,11,12]). In this paper, we clarify that such an estimate cannot be obtained even if the exact solution u and the obstacle ψ possess H^2-regularity and the order of convergence in the energy norm is one. We will proceed as follows:

In Section 2 we construct a first counterexample which illustrates that a general a priori error estimate of the form $\|u - u_h\|_{L^p} \leq Ch^\beta$, $1 \leq p \leq \infty$, cannot hold true for a one-dimensional obstacle problem with $u, \psi \in W^{2,q}(\Omega)$, $q \geq 2$, unless $\beta \leq 2 - 1/q$. This shows that the order of an a priori error estimate in an arbitrary L^p-space cannot be higher than that typically obtained with an a priori error estimate in $L^\infty(\Omega)$ and that $W^{2,\infty}$-regularity has to be assumed to prove an L^2-error estimate of order two. The discretization method that we employ in our first counterexample is that most commonly found in the literature: We approximate the space $H_0^1(\Omega)$ by means of piecewise linear finite elements and use the Lagrange interpolant of the obstacle ψ to discretize the inequality constraint $v \geq \psi$.

In Section 3 we demonstrate by means of a second counterexample that the results of Section 2 are still valid when the original obstacle ψ appears in the side condition of the discrete problems used for the finite element approximation, i.e., that the order $2 - 1/q$ is still optimal when the function space is discretized but the obstacle is not modified at all. This illustrates that the discretization of the obstacle ψ is not solely responsible for the behavior of the approximation error observed in our first example.

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Lastly, in Section 4 we compare our findings with known results. We will see here that L^∞-error estimates can detect the effects observed in our model problems surprisingly well.

The appendix of this paper contains a result about one-sided finite element approximations that is needed for the discussion of the example in Section 3. The theorem found there essentially goes back to Mosco and Strang [8]. We include a proof for the convenience of the reader.

In what follows, we will use the standard notation $H_0^1(\Omega)$, $W^{m,q}(\Omega)$, $C^{m,\gamma}(\overline{\Omega})$ etc. for the Sobolev and Hölder spaces on a bounded Lipschitz domain Ω (or the closure of Ω, respectively). The dual of $H_0^1(\Omega)$ with respect to the L^2-inner product and the associated dual pairing will be denoted with $H^{-1}(\Omega)$ and $\langle .,. \rangle$. In one dimension, a prime will always denote a (weak) derivative.

2. A First Counterexample. As a first counterexample, we consider an obstacle problem of the form

$$\left.\begin{aligned} &\min \quad \frac{1}{2}\int_{-1}^{1}(v')^2 \mathrm{d}x \\ &\text{s.t.} \quad v \in H_0^1(-1,1) \quad \text{and} \quad v \geq \psi_\alpha \text{ a.e. in } (-1,1), \end{aligned}\right\} \tag{P$_\alpha$}$$

i.e., $\Omega = (-1,1)$ and $f \equiv 0$. The obstacle ψ_α appearing in (P$_\alpha$) is defined by

$$\psi_\alpha(x) := \begin{cases} \phi\left(x+\frac{1}{2}\right)\left(\frac{3}{2} - 12\left|x+\frac{1}{2}\right|^{2-\alpha}\right) - \frac{1}{2}, & \text{if } x \in (-1,0] \\ \phi\left(x-\frac{1}{2}\right)\left(\frac{3}{2} - 12\left|x-\frac{1}{2}\right|^{2-\alpha}\right) - \frac{1}{2}, & \text{if } x \in (0,1), \end{cases} \tag{2.1}$$

where $\alpha \in (0, 1/2)$ is a given constant and $\phi \in C_c^\infty(\mathbb{R})$ denotes an arbitrary but fixed cut-off function satisfying

$$0 \leq \phi(x) \leq 1, \quad \phi \equiv 1 \text{ in } (-0.3, 0.3) \quad \text{and} \quad \operatorname{supp}\phi \subset [-0.4, 0.4].$$

Note that it follows from (2.1) that ψ_α is smooth in $(-1,1)\setminus\{\pm 0.5\}$, smaller than one (almost) everywhere in $(-1,1)$, and an element of $W^{2,q}(-1,1)$ for all $q \in [2, 1/\alpha)$ (cf. Figure 2.1). It is further easy to see that only the non-positive part of ψ_α is affected by the choice of ϕ in the above situation. This will ensure that our results are independent of the cut-off function appearing in the construction. Using standard results about variational inequalities, we obtain:

Proposition 2.1. *There is one and only one solution u_α to (P$_\alpha$). This solution is uniquely determined by the variational inequality*

$$u_\alpha \in K, \qquad \int_{-1}^{1} u_\alpha'(u_\alpha' - v')\mathrm{d}x \leq 0 \quad \forall\, v \in K \tag{2.2}$$

with

$$K := \{v \in H_0^1(-1,1) : v \geq \psi_\alpha \text{ a.e. in } (-1,1)\}.$$

Furthermore, it holds

$$u_\alpha \in W^{2,q}(-1,1) \quad \forall\, q \in [2, 1/\alpha) \quad \text{and} \quad u_\alpha = 1 \text{ a.e. in } (-0.5, 0.5) \tag{2.3}$$

and there exists an $\varepsilon_\alpha > 0$ such that

$$u_\alpha = \psi_\alpha \text{ a.e. in } (-0.5 - \varepsilon_\alpha, -0.5) \cup (0.5, 0.5 + \varepsilon_\alpha). \tag{2.4}$$

Proof. The unique solvability of the problem (P_α) and the characterization of u_α by (2.2) are easy consequences of the well-known theorem of Lions-Stampacchia (see, e.g., [6, Chapter II]). Concerning the $W^{2,q}$-regularity of the solution we refer to [6, Chapter IV]. To prove $u_\alpha = 1$ a.e. in $(-0.5, 0.5)$, note that the problem (P_α) is symmetric w.r.t. the origin and that $H_0^1(-1,1) \hookrightarrow C([-1,1])$. This implies that we can evaluate u_α pointwise and that $u_\alpha(0.5) = u_\alpha(-0.5)$. Since u_α is feasible for (P_α), we know that $u_\alpha(\pm 0.5) \geq \psi_\alpha(\pm 0.5) = 1$ and from the definition of ψ_α we obtain that $\psi_\alpha \leq 1$ holds (almost) everywhere in the interval $(-1,1)$. Combining these observations yields that the function

$$v_\alpha(x) := \begin{cases} u_\alpha(0.5), & \text{if } x \in [-0.5, 0.5] \\ u_\alpha(x), & \text{else} \end{cases}$$

is an element of K. From (2.2), it now follows

$$0 \geq \int_{-1}^{1} u_\alpha'(u_\alpha' - v_\alpha')\mathrm{d}x = \int_{-0.5}^{0.5} (u_\alpha')^2 \mathrm{d}x.$$

This implies that u_α is constant in $[-0.5, 0.5]$. Using [6, Corollary II.6.5], we obtain further that u_α cannot be bigger than one in $(-1,1)$. Hence, $u_\alpha(\pm 0.5) = 1$ and u_α is identical one in $[-0.5, 0.5]$. It remains to prove (2.4). To this end, note that it follows from (2.2) and the $W^{2,q}$-regularity in (2.3) that u_α is affine linear in the open set $\{x \in (-1,1) : u_\alpha(x) \neq \psi_\alpha(x)\}$. (We, of course, use the continuous representatives of u_α and ψ_α here.) This yields that there cannot exist points $x_1, x_2 \in (-0.8, -0.5]$ satisfying $x_1 < x_2$, $u_\alpha(x_1) = \psi_\alpha(x_1)$, $u_\alpha(x_2) = \psi_\alpha(x_2)$ and $u_\alpha > \psi_\alpha$ in (x_1, x_2) (since the concavity of ψ_α in $(-0.8, -0.2)$, the linearity of u_α in (x_1, x_2) and the feasibility of u_α would otherwise yield a contradiction). On the other hand, we know that $u_\alpha(-0.5) = \psi_\alpha(-0.5)$ and it is obvious that -0.5 cannot be the only point in $(-0.8, -0.5]$ where u_α and ψ_α coincide. Consequently, there has to exist an $\varepsilon_\alpha > 0$ such that $u_\alpha = \psi_\alpha$ holds a.e. in $(-0.5 - \varepsilon_\alpha, -0.5)$. The symmetry of the problem now yields (2.4). This completes the proof. ∎

Remark 2.2. *Using a more elaborate argumentation than that employed in the proof of Proposition 2.1, it is possible to show that the solution u_α to (P_α) is given by*

$$u_\alpha(x) = \begin{cases} \psi_\alpha(-\tilde{\varepsilon}_\alpha - 0.5)\,\dfrac{x+1}{0.5 - \tilde{\varepsilon}_\alpha}, & \text{if } x \in (-1, -\tilde{\varepsilon}_\alpha - 0.5) \\ \psi_\alpha(x), & \text{if } x \in [-0.5 - \tilde{\varepsilon}_\alpha, -0.5) \\ 1, & \text{if } x \in [-0.5, 0.5) \\ \psi_\alpha(x), & \text{if } x \in [0.5, 0.5 + \tilde{\varepsilon}_\alpha) \\ \psi_\alpha(\tilde{\varepsilon}_\alpha + 0.5)\,\dfrac{x-1}{\tilde{\varepsilon}_\alpha - 0.5}, & \text{if } x \in [0.5 + \tilde{\varepsilon}_\alpha, 1), \end{cases} \tag{2.5}$$

where $\tilde{\varepsilon}_\alpha$ is the unique solution to

$$\psi_\alpha(-\tilde{\varepsilon}_\alpha - 0.5) = (0.5 - \tilde{\varepsilon}_\alpha)\psi_\alpha'(-\tilde{\varepsilon}_\alpha - 0.5), \quad \tilde{\varepsilon}_\alpha \in (0, 0.3).$$

This explicit formula for u_α allows to calculate the error of the finite element method in numerical experiments precisely. We point out that (2.5) is not needed for the construction of our counterexamples - the information in Proposition 2.1 is sufficient for this purpose.

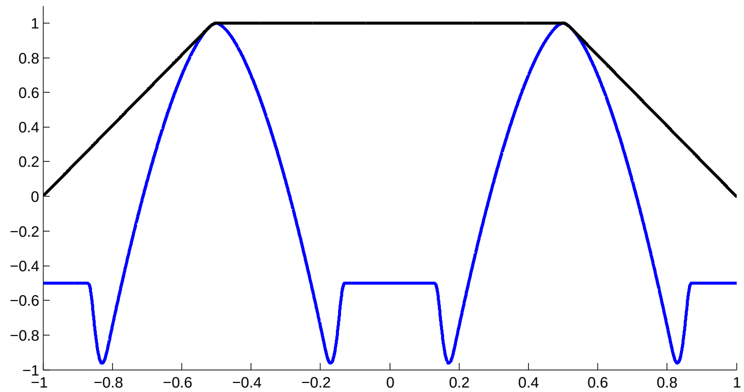

FIG. 2.1. *The obstacle ψ_α and the solution u_α for $\alpha = 0.4$ and a suitable cut-off function ϕ.*

For brevity's sake, in the following we will frequently suppress the index α and simply write u and ε for u_α and ε_α.

We now turn our attention to the discretization: To approximate (P_α), we employ a standard finite element method with piecewise linear continuous ansatz functions and equidistant meshes. The finite-dimensional problems that we use as discrete counterparts to (P_α) read as follows:

$$\left.\begin{aligned} &\min \quad \frac{1}{2}\int_{-1}^{1} (v_h')^2 \mathrm{d}x \\ &\text{s.t.} \quad v_h \in V_h^0 \quad \text{and} \quad v_h \geq I_h\psi_\alpha \text{ in } (-1,1). \end{aligned}\right\} \tag{$\mathrm{P}_{\alpha,h}$}$$

Here,

- $h := 1/N$ for some $N \in \mathbb{N}$,
- $\mathcal{T}_h := \{[x_l, x_{l+1}] : l = 0, ..., 2N-1\}$ with $x_l := -1 + l\,h, l = 0, ..., 2N$,
- $V_h := \{v \in C([-1,1]) : v|_T \text{ is affine for all cells } T \in \mathcal{T}_h\}$,
- $V_h^0 := V_h \cap H_0^1(-1,1)$,
- $I_h : C([-1,1]) \to V_h$: Lagrange interpolation operator associated with V_h.

Note that in the above the inequality constraint in (P_α) is discretized by replacing the continuous obstacle ψ_α with the Lagrange interpolant $I_h\psi_\alpha$. This is equivalent to imposing the constraint only in the nodes of the mesh $\mathcal{T}_h$ and constitutes the most common approach found in the literature (see, e.g., [3,5,7]). Using again the theorem of Lions-Stampacchia and a well-known variant of Céa's lemma, we obtain:

PROPOSITION 2.3. *For all $h = 1/N$, $N \in \mathbb{N}$, there is one and only one solution $u_{\alpha,h}$ to ($\mathrm{P}_{\alpha,h}$). This solution is uniquely determined by the variational inequality*

$$u_{\alpha,h} \in K_h, \quad \int_{-1}^{1} u_{\alpha,h}'(u_{\alpha,h}' - v_h')\mathrm{d}x \leq 0 \quad \forall\, v_h \in K_h \tag{2.6}$$

with

$$K_h := \{v_h \in V_h^0 : v_h \geq I_h\psi_\alpha \text{ in } (-1,1)\}.$$

Further, there exists a constant C independent of h such that

$$\|u_\alpha - u_{\alpha,h}\|_{H^1} \leq C\,h. \tag{2.7}$$

Proof. The existence of the solution and its characterization by means of the variational inequality (2.6) are obtained analogously to the continuous case. We refer to [5]. The H^1-error estimate follows from standard estimates for the Lagrange interpolant and a well-known theorem of Falk (see [4]). A complete derivation of (2.7) can be found in [3, Theorem 9.1, 9.2]. □

Analogously to the continuous setting, in what follows we drop the index α and simply write u_h instead of $u_{\alpha,h}$.

As Proposition 2.3 shows, in case of our model problem the qualitative behavior of the H^1-error is exactly the same as for the Poisson equation. The L^2-error, however, behaves differently. To see this, we observe the following:

PROPOSITION 2.4. *If $h_k := 1/(2k+1)$, $k \in \mathbb{N}$, then it holds*

$$u_{h_k} \equiv 1 - 12\left(\frac{h_k}{2}\right)^{2-\alpha} \quad \text{in} \quad \left(-0.5 - \frac{h_k}{2}, 0.5 + \frac{h_k}{2}\right). \tag{2.8}$$

Proof. Since any partition of the interval $(0,1)$ with width h_k, $k \in \mathbb{N}$, has an odd number of cells, the point 0.5 has to be the midpoint of some $[x_l, x_{l+1}] \in \mathcal{T}_{h_k}$. The same, of course, holds true for the point -0.5. This means that the maxima of the obstacle ψ_α are cut off by the Lagrange interpolation operator and that the interpolant $I_{h_k}\psi_\alpha$ satisfies

$$(I_{h_k}\psi_\alpha)(x) \leq \psi_\alpha\left(-0.5 - \frac{h_k}{2}\right) = \psi_\alpha\left(0.5 + \frac{h_k}{2}\right) = 1 - 12\left(\frac{h_k}{2}\right)^{2-\alpha} =: C(k,\alpha)$$

in $[-1,1]$. We can now proceed along the lines of the proof of Proposition 2.1 to obtain the claim: From the feasibility of u_{h_k} and the symmetry of the problem, it follows that

$$u_{h_k}\left(-0.5 - \frac{h_k}{2}\right) = u_{h_k}\left(0.5 + \frac{h_k}{2}\right) \geq C(k,\alpha)$$

and using the test function

$$v_{h_k}(x) := \begin{cases} u_{h_k}\left(0.5 + \frac{1}{2}h_k\right), & \text{if } x \in \left[-0.5 - \frac{1}{2}h_k, 0.5 + \frac{1}{2}h_k\right] \\ u_{h_k}(x), & \text{else} \end{cases}$$

in (2.6) yields that u_{h_k} is constant in $[-0.5 - h_k/2, 0.5 + h_k/2]$. Further, the constant function $g_{h_k} := C(k,\alpha)$ is a discrete supersolution of the problem $(\mathrm{P}_{\alpha,h_k})$ in the sense of [2, Definition 5]. This implies that $u_{h_k} \leq C(k,\alpha)$ holds a.e. in $(-1,1)$ (see [2, Theorem 8]). Combining the above yields (2.8) as claimed. □

From Propositions 2.1 and 2.4, it readily follows

$$\|u - u_{h_k}\|_{L^p(-1,1)} \geq \|u - u_{h_k}\|_{L^p(-0.5,0.5)} = 12\left(\frac{h_k}{2}\right)^{2-\alpha} \tag{2.9}$$

for all $1 \leq p \leq \infty$. This shows that in the situation of our model problem the order of convergence in any L^p-norm cannot be higher than $2-\alpha$ despite the optimal order of the H^1-error in (2.7) and the H^2-regularity of the exact solution.

Taking into account that $u, \psi_\alpha \in W^{2,q}(-1,1)$ for all $q \in [2, 1/\alpha)$, our findings can be summarized as follows:

THEOREM 2.5. *In case of the one-dimensional obstacle problem and the above discretization technique (i.e., linear finite elements and Lagrange interpolation of the obstacle) an a priori error estimate of the form*

$$\text{If the obstacle and the solution are functions in } W^{2,q}(\Omega),\\ \text{then it holds } \|u - u_h\|_{L^p} \le Ch^\beta$$

for some $1 \le p \le \infty$ and $q \ge 2$ cannot hold true unless $\beta \le 2 - 1/q$. In particular, an L^2-error estimate of order two can in general only be obtained if the obstacle and the solution are assumed to possess $W^{2,\infty}$-regularity.

REMARK 2.6. *It should be noted that the positive part $(u - u_h)^+ := \max(0, u - u_h)$ of the approximation error is responsible for the comparatively slow convergence in (2.9). In fact, using an approach of Mosco [9], it is possible to prove that the norm $\|(u - u_h)^-\|_{L^2}$ converges to zero with order two in our example, i.e., the rate of convergence typically obtained for the Poisson equation can be recovered if only the negative part of the error is considered.*

We conclude this section with a numerical experiment that confirms our theoretical findings: Figure 2.1 shows the experimental order of convergence in $L^2(-1,1)$, i.e., the quantity

$$(L^2\text{-EOC})_k := \frac{\log \|u - u_{h_{k+1}}\|_{L^2} - \log \|u - u_{h_k}\|_{L^2}}{\log h_{k+1} - \log h_k},$$

that is achieved when $(\mathrm{P}_{\alpha,h})$ is solved by means of an active set algorithm for the widths $h_k = 1/(2k+1)$ and $\alpha = 0.4$. It can be seen that the L^2-EOC scatters around $2 - \alpha$. This behavior agrees well with our analytical predictions.

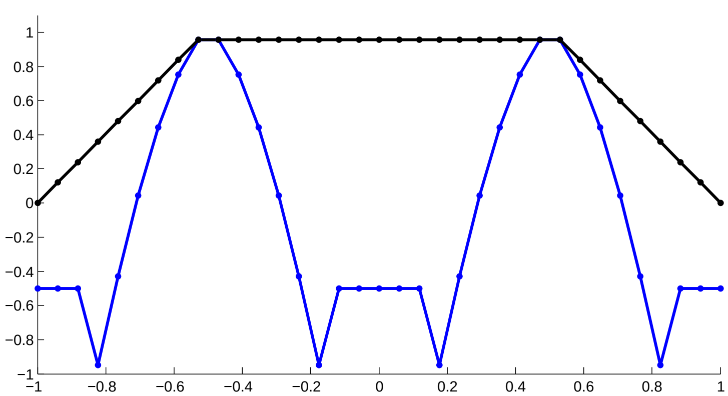

FIG. 2.2. *The obstacle $I_h\psi_\alpha$ and the approximate solution u_h for $\alpha = 0.4$ and $h = 1/17$.*

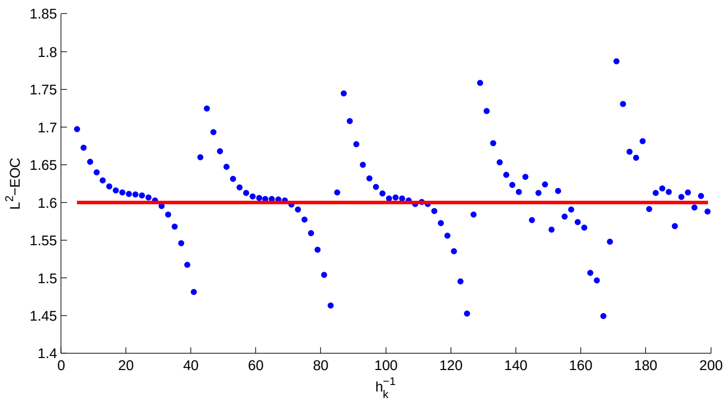

FIG. 2.3. *L^2-EOC for* (P_α) *in the case $\alpha = 0.4$. The results scatter around $2-\alpha$.*

The reason for the loss of the factor h^α observed in (2.9) is intuitively clear: The maxima of the obstacle ψ_α are not reproduced accurately enough by the Lagrange interpolants $I_{h_k}\psi_\alpha$ appearing in the discrete problems (P_{α,h_k}) and thus the finite element solutions u_{h_k} do not reach the height that would be necessary to obtain, e.g., the order two in the L^2-norm. This demonstrates that in case of the obstacle problem a special pollution effect may occur: local inaccuracies in the approximation of the obstacle - in our example the error between ψ_α and $I_{h_k}\psi_\alpha$ at ± 0.5 - can propagate and are able to affect the rate of convergence globally.

3. A Second Counterexample. In view of the analysis in the last section, it is tempting to think that an L^2-error estimate of order two can be recovered if better approximations of the obstacle are used in the discrete problems that characterize the finite element solutions u_h. But this is not the case. Even if the continuous obstacle itself is used in the inequality constraint of the approximate problems, we cannot expect the rate of convergence in any L^p-norm to be higher than the threshold $2 - 1/q$ appearing in Theorem 2.5. Note that this is a purely theoretical result since there is no way to handle a constraint of the type $v_h \geq \psi$ numerically if ψ is an arbitrary function.

To see that an L^2-estimate of order two cannot be obtained even if the obstacle is not discretized at all, we consider the following one-dimensional model problem:

$$\left.\begin{aligned} &\min \quad \frac{1}{2}\int_{-1}^{1} (v')^2 \mathrm{d}x - \langle f_\alpha, v\rangle \\ &\text{s.t.} \quad v \in H_0^1(-1,1) \quad \text{and} \quad v \geq \psi_\alpha \text{ a.e. in } (-1,1). \end{aligned}\right\} \qquad (Q_\alpha)$$

The inhomogeneity f_α appearing in (Q_α) is defined to be $-u_\alpha''$, where u_α is the solution to the problem (P_α) discussed in the last section, i.e., the function defined in (2.5). The obstacle ψ_α is the same as before. This construction ensures that the following holds:

PROPOSITION 3.1. *The problem* (Q_α) *admits a unique solution. Furthermore, the solutions to* (Q_α) *and* (P_α) *coincide.*

Proof. The unique solvability of (Q_α) can be proved analogously to (P_α). To obtain that the solution is exactly u_α, one rewrites (Q_α) as a variational inequality. The claim then follows from $f_\alpha = -u_\alpha''$ and integration by parts. ∎

The finite-dimensional problems that we will use to approximate (Q_α) are chosen to be

$$\left.\begin{aligned} &\min \quad \frac{1}{2}\int_{-1}^{1}(v_h')^2\mathrm{d}x - \langle f_\alpha, v_h\rangle \\ &\text{s.t.} \quad v_h \in V_h^0 \quad \text{and} \quad v_h \geq \psi_\alpha \text{ a.e. in } (-1,1), \end{aligned}\right\} \tag{$\mathrm{Q}_{\alpha,h}$}$$

where V_h^0 and the underlying meshes $\mathcal{T}_h$ are defined as in Section 2. Note that the exact obstacle ψ_α appears in the inequality constraint of ($\mathrm{Q}_{\alpha,h}$) - only the function space is discretized. Similarly to the proof of Proposition 2.3, we obtain:

PROPOSITION 3.2. *The problem* ($\mathrm{Q}_{\alpha,h}$) *is uniquely solvable for all* $h = 1/N$, $N \in \mathbb{N}$. *Furthermore, the solution to* ($\mathrm{Q}_{\alpha,h}$) *(which we again denote with* u_h*) is uniquely determined by the variational inequality*

$$u_h \in K_h, \quad \int_{-1}^{1} u_h'(u_h' - v_h')\mathrm{d}x \leq \langle f_\alpha, u_h - v_h\rangle \quad \forall\, v_h \in K_h \tag{3.1}$$

with

$$K_h := \{v_h \in V_h^0 : v_h \geq \psi_\alpha \text{ in } (-1,1)\}$$

and there exists a constant C *independent of* h *such that the error between the exact solution to* (Q_α) *(which we again denote with* u*) and* u_h *satisfies*

$$\|u - u_h\|_{H^1} \leq C\, h. \tag{3.2}$$

Proof. The unique solvability of the problem ($\mathrm{Q}_{\alpha,h}$) and the characterization of u_h by means of the variational inequality (3.1) again follow from the theorem of Lions-Stampacchia. To obtain the H^1-error estimate (3.2), we note that according to [4, Theorem 9.1] there exists a constant $C > 0$ independent of h such that

$$\|u - u_h\|_{H^1} \leq C\left(\|u_h - v\|_{L^2} + \|u - v_h\|_{L^2} + \|u - v_h\|_{H^1}^2\right)^{\frac{1}{2}} \tag{3.3}$$

holds for all $v_h \in K_h$ and all $v \in K$. Choosing $v = u_h$ and $v_h = z_h$ in (3.3), where $u \leq z_h \in V_h^0$ is a unilateral finite element approximation of u as constructed in Theorem A.1 in the Appendix, yields (3.2) as desired. ∎

As the above result shows, in case of the problems (Q_α) and ($\mathrm{Q}_{\alpha,h}$) the order of convergence in $H^1(-1,1)$ is exactly the same as in our first example. The following, however, can also be observed:

PROPOSITION 3.3. *Let* ε *be an arbitrary but fixed positive number such that* (2.4) *is satisfied, i.e., such that it holds*

$$u = \psi_\alpha \text{ a.e. in } (-0.5-\varepsilon, -0.5) \cup (0.5, 0.5+\varepsilon)\,. \tag{3.4}$$

Then for all mesh widths $h_k = 1/(2k+1)$, $k \in \mathbb{N}$, *with* $h_k/2 < \varepsilon$ *it is true that*

$$u_{h_k} \geq 1 + 6\left(\frac{h_k}{18}\right)^{2-\alpha} \quad \text{in} \quad \left[-0.5 + \frac{h_k}{2}, 0.5 - \frac{h_k}{2}\right]. \tag{3.5}$$

Proof. Since we consider mesh widths of the form $h_k = 1/(2k+1)$, $k \in \mathbb{N}$, there exist mesh cells $T_1 = [x_{i-1}, x_i]$ and $T_2 = [x_j, x_{j+1}]$ in $\mathcal{T}_{h_k}$ such that

$$x_i = -0.5 + \frac{h_k}{2} \quad \text{and} \quad x_j = 0.5 - \frac{h_k}{2}.$$

From the symmetry of the problem ($\mathrm{Q}_{\alpha,h}$) w.r.t. the origin, it follows further that $u_{h_k}(x_i) = u_{h_k}(x_j)$ has to hold and from the definition of the obstacle ψ_α we readily obtain $\psi_\alpha(x) \leq \psi_\alpha(x_i)$ for all $x \in [x_i, x_j]$ (cf. Figure 2.1). Combining the above yields that the function

$$v_{h_k}(x) := \begin{cases} u_{h_k}(x_i), & x \in [x_i, x_j] \\ u_{h_k}(x), & \text{else} \end{cases}$$

is feasible for (Q_{α,h_k}). Using that $f_\alpha = -u_\alpha'' \equiv 0$ in $(-0.5, 0.5)$, cf. (2.5), it now follows analogously to the proof of Proposition 2.4 that

$$0 \geq \int_{-1}^{1} u_{h_k}'(u_{h_k}' - v_{h_k}') + f_\alpha(v_{h_k} - u_{h_k})\mathrm{d}x = \int_{x_i}^{x_j} (u_{h_k}')^2 \mathrm{d}x.$$

Thus, the function u_{h_k} is constant in $[x_i, x_j]$ and the situation near the maxima of the obstacle ψ_α is that depicted in Figure 3.1.

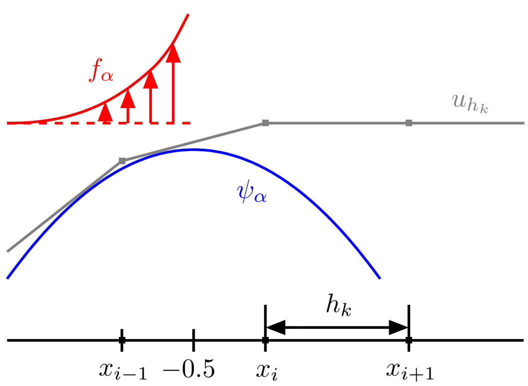

Fig. 3.1. *The functions ψ_α and u_{h_k} near the point -0.5. The situation near 0.5 is analogous.*

We now define $\varphi_{h_k}^i$ to be the unique element of $V_{h_k}^0$ with $\varphi_{h_k}^i(x_l) = \delta_{il}$ for all nodes x_l of the mesh $\mathcal{T}_{h_k}$ (i.e., $\varphi_{h_k}^i$ is the element of the nodal basis associated with x_i) and choose $v_{h_k} := u_{h_k} + \varphi_{h_k}^i$ in (3.1) to obtain

$$-\int_{-1}^{1} u_{h_k}'(\varphi_{h_k}^i)'\mathrm{d}x \leq -\int_{-1}^{1} f_\alpha \varphi_{h_k}^i \mathrm{d}x.$$

Thanks to $\operatorname{supp} \varphi_{h_k}^i = [x_{i-1}, x_{i+1}]$, $u_{h_k}' \equiv 0$ in $[x_i, x_j]$, $h_k/2 < \varepsilon$, the definition of the right-hand side f_α, (2.1), and (3.4), the above inequality yields

$$\begin{aligned}\frac{u_{h_k}(x_i)-u_{h_k}(x_{i-1})}{h_k} &= \int_{x_{i-1}}^{x_i} u_{h_k}'(\varphi_{h_k}^i)'\mathrm{d}x\\ &\geq \int_{x_{i-1}}^{-0.5} f_\alpha \varphi_{h_k}^i \mathrm{d}x\\ &= \int_{-0.5-\frac{h_k}{2}}^{-0.5} 12(2-\alpha)(1-\alpha)(-x-0.5)^{-\alpha}\frac{(x+0.5+\frac{h_k}{2})}{h_k}\mathrm{d}x\\ &= \frac{12}{h_k}\left(\frac{h_k}{2}\right)^{2-\alpha}.\end{aligned}$$

This implies

$$u_{h_k}(x_i) \geq u_{h_k}(x_{i-1}) + 12\left(\frac{h_k}{2}\right)^{2-\alpha}. \tag{3.6}$$

To prove the claim, we now consider two different cases:

1. case: $u_{h_k}(x_{i-1}) \geq 1-6\,(h_k/2)^{2-\alpha}$
In this case, we deduce from (3.6) that

$$u_{h_k}(x) = u_{h_k}(x_i) \geq u_{h_k}(x_{i-1}) + 12\left(\frac{h_k}{2}\right)^{2-\alpha} \geq 1+6\left(\frac{h_k}{2}\right)^{2-\alpha} \quad \forall\, x\in[x_i,x_j],$$

giving in turn (3.5).

2. case: $u_{h_k}(x_{i-1}) < 1-6\,(h_k/2)^{2-\alpha}$
Define

$$\delta_\alpha := \left(\frac{1}{2^{2-\alpha}(2-\alpha)}\right)^{\frac{1}{1-\alpha}} \in \left[\frac{1}{18},\frac{1}{8}\right] \quad \forall\,\alpha\in\left(0,\frac{1}{2}\right)$$

and consider the tangent T_α to ψ_α in the point $-0.5-\delta_\alpha h_k \in [x_{i-1},-0.5]$, i.e., the function

$$T_\alpha(x) = 1-12(\delta_\alpha h_k)^{2-\alpha} + 12(2-\alpha)(\delta_\alpha h_k)^{1-\alpha}(x+0.5+\delta_\alpha h_k).$$

Then it holds

$$T_\alpha(x_{i-1}) \geq 1-12h_k^{2-\alpha}\left(\frac{(2-\alpha)}{2}\delta_\alpha^{1-\alpha}\right) = 1-6\left(\frac{h_k}{2}\right)^{2-\alpha} > u_{h_k}(x_{i-1})$$

and it follows from $u_{h_k}(-0.5-\delta_\alpha h_k) \geq \psi_\alpha(-0.5-\delta_\alpha h_k) = T_\alpha(-0.5-\delta_\alpha h_k)$ that T_α and u_{h_k} intersect in $(x_{i-1},-0.5)$ (cf. Figure 3.2). This yields $u_{h_k}(x_i) \geq T_\alpha(x_i)$ and implies

$$u_{h_k}(x_i) \geq 1+12h_k^{2-\alpha}\delta_\alpha^{2-\alpha}\left(\frac{(2-\alpha)}{2\delta_\alpha}+(1-\alpha)\right) \geq 1+6\left(\frac{h_k}{18}\right)^{2-\alpha}.$$

This completes the proof. ∎

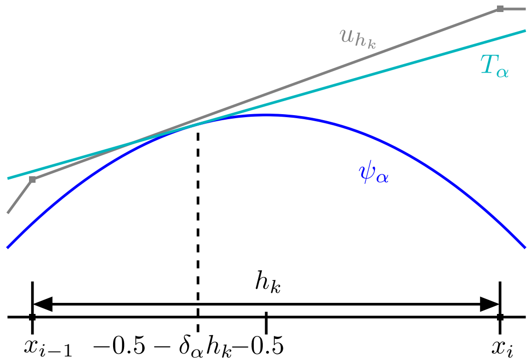

FIG. 3.2. *The situation in the second case considered in the proof of Proposition 3.3.*

Analogously to our first counterexample, it follows from (3.5) (in combination with Proposition 2.1) that for all sufficiently small h_k and all $1 \le p \le \infty$ it holds

$$\|u - u_{h_k}\|_{L^p(-1,1)} \ge \|u_{h_k} - 1\|_{L^p(-0.25,0.25)} \ge 3\left(\frac{h_k}{18}\right)^{2-\alpha}. \tag{3.7}$$

Thus, the order two is again out of reach - no matter which L^p-norm is considered. Note that, in contrast to our first example, this time the component $(u-u_h)^-$ is responsible for the slow convergence, i.e., the discrete solutions are too big to obtain the accuracy that is typically achieved in case of the Poisson equation. Taking into account the regularity of the functions u and ψ_α, our findings can be summarized as follows:

THEOREM 3.4. *In case of the one-dimensional obstacle problem and the above discretization technique (i.e., linear finite elements without any discretization of the obstacle) an a priori error estimate of the form*

If the obstacle and the solution are functions in $W^{2,q}(\Omega)$,
then it holds $\|u - u_h\|_{L^p} \le C\, h^\beta$

for some $1 \le p \le \infty$ and $q \ge 2$ cannot hold true unless $\beta \le 2 - 1/q$. In particular, an L^2-error estimate of order two can in general only be obtained if the obstacle and the solution are assumed to possess $W^{2,\infty}$-regularity.

The reason for the loss of the factor h^α in (3.7) is again intuitively clear: Since neither the contact set $\{u = \psi_\alpha\}$ nor the set $\{f_\alpha \neq 0\}$ is resolved properly by the meshes $\mathcal{T}_{h_k}$ and since the obstacles in (Q_{α,h_k}) are not piecewise linear, the error between u_{h_k} and u at the nodes x_i and x_j is affected negatively. This local perturbation propagates and spoils the rate of convergence in the L^p-norms similarly to our first counterexample.

4. Concluding Remarks and Outlook. The behavior of the error $u - u_h$ observed in Sections 2 and 3 can be explained as follows: As shown in [2], the Ritz projection $R_h u$ of the solution u to a one-dimensional obstacle problem of the form

$$\begin{aligned} \min \quad & \frac{1}{2}\int_{-1}^{1} (v')^2 \mathrm{d}x - \langle f, v\rangle \\ \text{s.t.} \quad & v \in H_0^1(-1,1) \quad \text{and} \quad v \ge \psi \text{ a.e. in } (-1,1), \end{aligned}$$

i.e., the unique element of V_h^0 satisfying

$$\int_{-1}^{1} (R_h u)' v_h' \mathrm{d}x = \int_{-1}^{1} u' v_h' \mathrm{d}x \quad \forall\, v_h \in V_h^0,$$

is exactly the solution of the discrete obstacle problem

$$\left.\begin{aligned} \min \quad & \frac{1}{2}\int_{-1}^{1} (v_h')^2 \mathrm{d}x - \langle f, v_h \rangle \\ \text{s.t.} \quad & v_h \in V_h^0 \quad \text{and} \quad v_h \geq R_h u + \psi - u \text{ a.e. in } (-1,1). \end{aligned}\right\} \tag{P$_{R,h}$}$$

This implies that the difference between the Ritz projection $R_h u$ of u and a finite element approximation u_h which is characterized by a problem of the form

$$\begin{aligned} \min \quad & \frac{1}{2}\int_{-1}^{1} (v_h')^2 \mathrm{d}x - \langle f, v_h \rangle \\ \text{s.t.} \quad & v_h \in V_h^0 \quad \text{and} \quad v_h \geq \psi_h \text{ a.e. in } (-1,1) \end{aligned}$$

can be identified with the change that occurs in the solution to (P$_{R,h}$) when the obstacle $R_h u + \psi - u$ is replaced with ψ_h. In other words, the error $R_h u - u_h$ is directly related to the sensitivity of the solution to (P$_{R,h}$) w.r.t. perturbations of the obstacle $R_h u + \psi - u$. Pointwise perturbations of the obstacle, however, can affect the solution of a (discrete) one-dimensional obstacle problem globally (cf. our first example) and thus it is only logical that the error $\|R_h u - u_h\|_{L^p}$ is typically not of higher order than the quantity $\|R_h u + \psi - u - \psi_h\|_{L^\infty}$. The pointwise error $\|R_h u + \psi - u - \psi_h\|_{L^\infty}$ that enters here is responsible for the comparatively slow convergence observed in our counterexamples.

If the above informal discussion is made rigorous (i.e., when it is carefully analyzed which error occurs when the obstacle $R_h u + \psi - u$ in the problem (P$_{R,h}$) is replaced with a function ψ_h), then the following L^∞-error estimates can be obtained for the one-dimensional obstacle problem:

THEOREM 4.1 ([2, Theorem 11]). *Let Ω be an open bounded interval. Suppose that $f \in H^{-1}(\Omega)$ and $\psi \in C(\overline{\Omega})$ are given such that the obstacle problem*

$$\left.\begin{aligned} \min \quad & \frac{1}{2}\int_{\Omega} (v')^2 \mathrm{d}x - \langle f, v \rangle \\ \text{s.t.} \quad & v \in H_0^1(\Omega) \quad \textit{and} \quad v \geq \psi \textit{ a.e. in } \Omega \end{aligned}\right\} \tag{P}$$

admits a unique solution u. Assume that

- *$\{\mathcal{T}_h\}_{h>0}$ is a family of partitions of Ω with $\max\{\operatorname{diam} T : T \in \mathcal{T}_h\} \leq h$,*
- *$V_h^0 := H_0^1(\Omega) \cap \{v \in C(\overline{\Omega}) : v|_T$ is affine for all cells $T \in \mathcal{T}_h\}$,*
- *$\{\psi_h\}_{h>0}$ is a family of $C(\overline{\Omega})$-functions with $\psi_h \leq 0$ on $\partial\Omega$ for all $h > 0$,*
- *There exist $\gamma_1, \gamma_2 \in (0,1]$ with $\psi_h|_T \in C^{1,\gamma_1}(T)$ and $u|_T, \psi|_T \in C^{1,\gamma_2}(T)$ for all $T \in \mathcal{T}_h$ and all $h > 0$.*

Then the discrete obstacle problem

$$\left.\begin{aligned} \min \quad & \frac{1}{2}\int_\Omega (v_h')^2 \mathrm{d}x - \langle f, v_h\rangle \\ \text{s.t.} \quad & v_h \in V_h^0 \quad \textit{and} \quad v_h \geq \psi_h \ \textit{a.e. in}\ \Omega \end{aligned}\right\} \tag{P$_h$}$$

admits a unique solution u_h *for all* $0 < h < \operatorname{diam}\Omega$ *and it holds*

$$\begin{aligned} \|(u-u_h)^-\|_{L^\infty} \leq \|(u - R_h u)^-\|_{L^\infty} + \|(\psi_h - \psi + u - R_h u)^+\|_{L^\infty} \\ + \frac{1}{1+\gamma_1} h^{1+\gamma_1} \max_{T\in\mathcal{T}_h} |\psi_h|_{C^{1,\gamma_1}(T)} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \|(u-u_h)^+\|_{L^\infty} \leq \|(u - R_h u)^+\|_{L^\infty} + \|(\psi_h - \psi + u - R_h u)^-\|_{L^\infty} \\ + \frac{1}{1+\gamma_2} h^{1+\gamma_2} \max_{T\in\mathcal{T}_h} |\psi - u|_{C^{1,\gamma_2}(T)}. \end{aligned} \tag{4.2}$$

Here, $R_h u$ *again denotes the Ritz projection of* u *and*

$$|v|_{C^{1,\gamma}(T)} := \sup_{x\neq y\in T} \frac{|v'(x) - v'(y)|}{|x-y|^\gamma}.$$

It should be noted that the last error contribution in (4.1) and the second to last contributions in (4.1) and (4.2), respectively, behave contrarily. While an accurate approximation of the continuous obstacle ψ (possibly involving curved obstacles ψ_h in the discrete problems) is favorable to reduce the error $\psi_h - \psi$, the last error contribution in (4.1) becomes larger when the curvature of the function ψ_h increases. These two effects were also observed in our two counterexamples: Whereas the pointwise error in the approximation of the obstacle is responsible for the reduction of the order of convergence in our first example, the curvature of the obstacle ψ_h induces the problems in our second example, cf. Figures 3.1 and 3.2.

By employing standard finite element error estimates for the Ritz projection and Sobolev embeddings, one deduces the following result from Theorem 4.1:

Corollary 4.2 ([2, Corollary 14]). *Let* Ω *be an open bounded interval and assume that:*

- $f \in L^q(\Omega)$, $\psi \in W^{2,q}(\Omega)$ *and* $\psi|_{\partial\Omega} \leq 0$ *holds for some* $2 \leq q < \infty$,
- $\{\mathcal{T}_h\}_{h>0}$ *and* V_h^0 *satisfy the assumptions of Theorem 4.1.*

Suppose further that ψ_h *is chosen to be the Lagrange interpolant* $I_h\psi$ *or that* ψ_h *is chosen to be the equal to* ψ*. Let* $h < \operatorname{diam}\Omega$*. Then the problems* (P) *and* (P$_h$) *in Theorem 4.1 admit unique solutions* $u \in H_0^1(\Omega)\cap W^{2,q}(\Omega)$ *and* $u_h \in V_h^0$*, respectively, and there exists a constant* $C > 0$ *independent of* h *such that*

$$\|u - u_h\|_{L^\infty} \leq C h^{2-1/q} \left(\|f\|_{L^q} + \|\psi\|_{W^{2,q}}\right). \tag{4.3}$$

Note that it follows from (4.3) that the examples in Sections 2 and 3 are worst-case scenarios. Conversely, our model problems demonstrate that an a priori estimate of the form (4.3) is optimal in the sense that no general a priori L^p-error estimate,

$1 \leq p \leq \infty$, can yield an order higher than $2 - 1/q$ in the situations that we have considered. This answers the question what can (and, more importantly, what cannot) be expected when it comes to a priori L^p-error estimates for the piecewise linear finite element approximation of one-dimensional obstacle problems.

Unfortunately, the situation is much less clear in higher dimensions. If a d-dimensional obstacle problem with a $W^{2,q}$-obstacle, $q > \max(d, 2)$, is approximated with piecewise linear finite elements, then the order of convergence obtained with an a priori L^∞-error estimate is typically $2 - d/q$ (modulo logarithmic factors). See, e.g., [2] for a higher-dimensional analogue of Theorem 4.1. On the other hand, it is easy to construct examples similar to those in Sections 2 and 3 (e.g., by rotation) which demonstrate that $2 - 1/q$ is an upper bound for the order of an a priori L^p-error estimate in d dimensions when the obstacle is assumed to be in $W^{2,q}(\Omega)$. There is thus a gap between what can be proved with counterexamples and what can be obtained from the L^∞-error analysis and, to the best of our knowledge, it is still on open question whether an a priori estimate of the form

$$\|u - u_h\|_{L^p} \leq C\, h^\alpha$$

can be obtained for some $1 \leq p < \infty$ and some $\max(1, 2 - d/q) < \alpha \leq 2 - 1/q$. This gives rise to further research.

Appendix A. Unilateral Finite Element Approximation in One Dimension.

In this section, we construct the unilateral finite element approximations that are needed in the proof of Proposition 3.2. The underlying analysis essentially goes back to Mosco and Strang, cf. [8,13]. For the convenience of the reader we shortly recall the arguments in the following.

Theorem A.1. *Let Ω be an open bounded interval and assume that $\{\mathcal{T}_h\}_{0<h<h_0}$ is a family of partitions of Ω such that $ch < \operatorname{diam} T < \min(Ch, \operatorname{diam}\Omega)$ holds for all $T \in \mathcal{T}_h$ and all $0 < h < h_0$ with constants $c, C > 0$ independent of h. Let*

$$V_h^0 := H_0^1(\Omega) \cap \{v \in C(\overline{\Omega}) : v|_T \text{ is affine for all cells } T \in \mathcal{T}_h\}$$

and suppose that $z \in H_0^1(\Omega) \cap W^{2,q}(\Omega)$, $1 < q < \infty$, is a given function. Then there exist constants C_1, C_2, C_3 independent of h and a family $\{z_h\}_{0<h<h_0} \subset V_h^0$ satisfying $z \leq z_h$ for all $0 < h < h_0$ such that

$$\|z - z_h\|_{L^q} \leq C_1 h^2 |z|_{W^{2,q}}, \quad \|z - z_h\|_{W^{1,q}} \leq C_2 h |z|_{W^{2,q}} \tag{A.1}$$

and

$$\|z - z_h\|_{L^\infty} \leq C_3 h^{2-1/q} |z|_{W^{2,q}} \tag{A.2}$$

holds for all $0 < h < h_0$.

Proof. In what follows, we will always identify z with its $C^1(\overline{\Omega})$-representative. To prove Theorem A.1, we consider for an arbitrary but fixed mesh size $0 < h < h_0$ the optimization problem

$$\min \sum_{x \in \mathcal{C}_h} v_h(x) \quad \text{s.t. } z \leq v_h \text{ in } \Omega \text{ and } v_h \in V_h^0, \tag{A.3}$$

where $\mathcal{C}_h$ denotes the set of all vertices of the partition $\mathcal{T}_h$. Using standard techniques from finite-dimensional optimization, it is easy to see that (A.3) admits at least one global minimum z_h. From the definition of (A.3), it follows that the function values $z_h(x)$, $x \in \mathcal{C}_h$, of such a minimum cannot be decreased without violating the constraint $z \leq z_h$. This implies that for every node $x \in \mathcal{C}_h \setminus \partial\Omega$ with adjacent mesh cells $T_l = [x_l, x]$ and $T_r = [x, x_r]$ one of the following has to be true:

a) It holds $z_h(x) = z(x)$.

b) There exists an $a \in [x_l, x_r] \setminus \{x\}$ such that $z_h(a) = z(a)$ and $z_h'(a) = z'(a)$. If $a \in \{x_l, x_r\}$, we mean the left (resp., right) limit of the derivative here.

If b) is the case, then the fundamental theorem of calculus yields

$$\begin{aligned} z_h(x) - z(x) &= \left| \int_a^x z''(t)(x-t)\mathrm{d}t \right| \\ &\leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} (Ch)^{2-\frac{1}{q}} \left(\max_{T \in \mathcal{T}_h : x \in T} |z|_{W^{2,q}(T)} \right) \end{aligned}$$

and if a) is true (or $x \in \partial\Omega$), it trivially holds $z_h(x) - z(x) = 0$. Thus, we obtain that z_h satisfies

$$\begin{aligned} 0 &\leq z_h(x) - I_h z(x) \\ &\leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} (Ch)^{2-\frac{1}{q}} \left(\max_{T \in \mathcal{T}_h : x \in T} |z|_{W^{2,q}(T)} \right) \quad \forall\, x \in \mathcal{C}_h. \end{aligned} \tag{A.4}$$

Here, $I_h : H_0^1(\Omega) \to V_h^0$ again denotes the Lagrange interpolation operator. From (A.4) and the piecewise linearity of the functions in V_h^0, it readily follows

$$\|z_h - I_h z\|_{L^\infty(\Omega)} \leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} (Ch)^{2-\frac{1}{q}} |z|_{W^{2,q}(\Omega)}. \tag{A.5}$$

Combining (A.5) with the triangle inequality and standard error estimates for the Lagrange interpolant yields (A.2). Further, we obtain from (A.4) that

$$\|z_h - I_h z\|_{L^q(T)} \leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} (Ch)^2 \left(\max_{\tilde{T} \in \mathcal{T}_h : T \cap \tilde{T} \neq \emptyset} |z|_{W^{2,q}(\tilde{T})} \right) \quad \forall\, T \in \mathcal{T}_h.$$

Summation over all $T \in \mathcal{T}_h$ now yields

$$\|z_h - I_h z\|_{L^q(\Omega)} \leq 3^{\frac{1}{q}} \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} (Ch)^2 |z|_{W^{2,q}(\Omega)}.$$

Using again the triangle inequality and standard interpolation error estimates, we obtain the first estimate in (A.1). To prove the $W^{1,q}$-estimate, note that

$$\|z_h' - I_h' z\|_{L^\infty(T)} \leq \frac{2}{c\,h} \|z_h - I_h z\|_{L^\infty(T)} \quad \forall\, T \in \mathcal{T}_h.$$

Proceeding as in case of the L^q-error now gives the second estimate in (A.1). □

It should be noted that in higher dimensions, it is still possible to prove L^∞-error estimates for unilateral approximations provided the function z under consideration possesses enough regularity. We refer to [2] for details.

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