On a Time-Simultaneous Multigrid Method in combination with stabilization techniques for the Convection–Diffusion Equation
IMG2022, Lugano

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Motivation

Figure Convergence behavior without stabilization.

- solve convection-diffusion equation efficiently → parallelization
- use a time-simultaneous multigrid method\(^1\)
- stabilization for convection-dominated problems

The perfect algorithm is not presented here!

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\(^1\)Multigrid Waveform Relaxation (by Lubich, Ostermann (1987)).
# Table of contents

1 Motivation

2 **Time-simultaneous multigrid method: heat equation**
   - Preliminaries
   - Building up the algorithm
   - Numerical studies

3 **Time-simultaneous multigrid method: convection-diffusion equation**
   - Numerical studies: upwind discretization
   - Numerical studies: central discretization
   - Higher order stabilization
   - Numerical studies: central + stabilization

4 Conclusion and Outlook
Preliminaries

Convection-diffusion equation in 1D

\[
\frac{\partial}{\partial t} u(x, t) - \varepsilon u_{xx}(x, t) + v(x, t) u_x(x, t) = f(x, t) \quad (x, t) \in \Omega \times (0, T] \\
u(0, t) = u(1, t) = 0 \quad t \in [0, T] \\
u(x, 0) = u_0(x) \quad x \in \Omega
\]

with \( \Omega = (0, 1), T > 0 \).

- finite difference (FD) discretization in space
  \[
  \frac{\partial}{\partial t} M_h u_h(t) + \varepsilon L_h u_h(t) = f_h(t)
  \]

- Crank-Nicolson scheme for discretization in time
  \[
  A u^m + B u^{m-1} = f^m, \quad m = 1, \ldots, K
  \]
  using time step size \( \delta t \) and
  \[
  A := M_h + \frac{1}{2} \delta t \varepsilon L_h, \quad B := -M_h + \frac{1}{2} \delta t \varepsilon L_h, \quad f^m := \frac{1}{2} \delta t \left( f_h^m + f_h^{m-1} \right).
  \]
Preliminaries

Convection-diffusion equation in 1D

\[ \partial_t u(x, t) - \varepsilon u_{xx}(x, t) + v(x, t)u_x(x, t) = f(x, t) \quad (x, t) \in \Omega \times (0, T] \]
\[ u(0, t) = u(1, t) = 0 \quad t \in [0, T] \]
\[ u(x, 0) = u_0(x) \quad x \in \Omega \]

with \( \Omega = (0, 1), T > 0. \)

- finite difference (FD) discretization in space
  \[ \partial_t M_h u_h(t) + \varepsilon L_h u_h(t) = f_h(t) \]

- Crank-Nicolson scheme for discretization in time
  \[ A u^m + B u^{m-1} = f^m, \quad m = 1, \ldots, K \]

using time step size \( \delta t \) and

\[ A := M_h + \frac{1}{2} \delta t \varepsilon L_h, \quad B := -M_h + \frac{1}{2} \delta t \varepsilon L_h, \quad f^m := \frac{1}{2} \delta t \left( f_h^m + f_h^{m-1} \right). \]
Algebraic transformations

Blocking all time steps into a global linear system of equations ...

\[
\begin{pmatrix}
A & B & A \\
B & A & B \\
& & \ddots & \ddots & \ddots \\
& & & B & A
\end{pmatrix}
\begin{pmatrix}
u^1_1 \\
u^2_1 \\
\vdots \\
u^K_1 \\
\vdots \\
\vdots \\
u^1_N \\
u^2_N \\
\vdots \\
u^K_N
\end{pmatrix}
= \begin{pmatrix}
f^1_0 \\
f^2_0 \\
\vdots \\
f^K_0 \\
\vdots \\
\vdots \\
f^1_K \\
f^2_K \\
\vdots \\
f^K_K
\end{pmatrix}
\]

Note: \( K \) time steps \( t^1, t^2, \ldots, t^K \) and \( N \) spatial nodes \( x_1, \ldots, x_N \)

... and rearranging the degrees of freedom...

\[
(u^1_{11}, u^1_{12}, \ldots, u^1_{1N}, u^2_{11}, u^2_{12}, \ldots, u^2_{1N}, \ldots, u^K_{11}, u^K_{12}, \ldots, u^K_{1N})^T
\]

\[
\downarrow
\]

\[
u := (u^1_1, u^2_1, \ldots, u^K_1, u^1_2, u^2_2, \ldots, u^K_2, \ldots, u^1_N, u^2_N, \ldots, u^K_N)^T
\]
Algebraic transformations

results in a **space-only problem** with vector-valued unknowns for each spatial node:

\[
\begin{pmatrix}
\# & \# \\
\# & \# & \ddots \\
\cdots & \cdots & \# \\
\# & \# \\
\end{pmatrix} =: S \in \mathbb{R}^{NK \times NK}
\]

solution:

\[
\mathbf{u} = \mathbf{f}, \quad \text{with} \quad \#
\begin{pmatrix}
\ast \\
\ast & \ast \\
\ast & \ast & \ddots \\
\ast \\
\end{pmatrix} \in \mathbb{R}^{K \times K}
\]

→ apply geometric multigrid method on \( S \) in space!

**Aim:** design of a highly parallelizable solution strategy!
Time-simultaneous multigrid algorithm

- smoothing
  - number of pre-smoothing and post-smoothing steps: $\nu_1$ and $\nu_2$
  - (damped) block Jacobi method: $x^{(\nu)} = x^{(\nu-1)} + \omega D^{-1}(f - Sx^{(\nu-1)})$
  - block Jacobi preconditioning embedded into GMRES method:
    $$D := \begin{pmatrix} \# & \cdots & \# \\ \# & \cdots & \# \\ \# & \cdots & \# \end{pmatrix}, \quad \# = \begin{pmatrix} * & * & * \\ * & \cdots & \cdots \\ * & * & * \end{pmatrix}_{K \times K}$$

- intergrid transfer operators
  - standard coarsening in space for each time step
  - prolongation:
    $$P^\delta_{\delta t,2h} = P^h_{2h} \otimes I_K$$
  - restriction:
    $$R^\delta_{\delta t,h} = R^h_{2h} \otimes I_K = \frac{1}{2} \left( P^\delta_{\delta t,2h} \right)^\top$$
Theoretical convergence results

- linear multi-step methods: asymptotic convergence rate is the same as in time-stepping approach [Janssen, Vandewalle (1996)]

- Fourier analysis of time-simultaneous two-grid algorithm [Lohmann et al. (2022)]
  - 1D heat equation on uniform mesh
  - damped Jacobi (waveform relaxation) smoothing
  - spectral norm of two-grid iteration matrix $J$ is uniformly bounded:

$$
\|J\|_2 < C < 1, \quad C \neq C(\delta t, h, K)
$$

for $\theta \geq \frac{1}{2}, \omega = \frac{2}{3}, \nu_1 \geq 1.$
Numerical studies on heat equation

Figure $u(x,t)$ for different time steps.

Manufactured solution

$$u(x,t) = \exp\left(-\eta\left(\frac{1}{2} - x + \frac{1}{4} \sin\left(\frac{\pi}{2} t\right)^2\right)\right) \sin(\pi x)$$

where $\eta = 100$.

homogenous Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0.$$ 

Discretization: FD in space, Crank-Nicolson scheme in time

- Level: fine level with spatial resolution $h = 2^{-\text{Level}}$
- $\delta t$: time step size
- $K$: number of blocked time steps
Numerical studies on heat equation: $\varepsilon = 1$

**Results:**
- independent of $K$ and fine mesh level
- number of iterations $\leq 5$

**MG algorithm:**
$\nu_1 = \nu_2 = 4$, V-cycle, GMRES smoother, coarse level 1.
Numerical studies on convection-diffusion equation

Convection-diffusion equation in 1D

\[
\partial_t u(x, t) - \varepsilon u_{xx}(x, t) + \frac{v(x,t)}{\varepsilon} u_x(x, t) = f(x, t) \quad (x, t) \in \Omega \times (0, T]
\]

\[
\begin{align*}
  u(0, t) &= u(1, t) = 0 \\
  u(x, 0) &= u_0(x) \\
  t &\in [0, T] \\
  x &\in \Omega
\end{align*}
\]

with \( \Omega = (0, 1), T > 0. \)

\[
\partial_t M_h u_h(t) + \varepsilon L_h u_h(t) + K_h u_h(t) = f_h(t)
\]

- Convection term: Discretized with first order upwind scheme
- Numerical studies with
  - \( v = 1 \) for convenience
  - manufactured solution as above and corresponding right hand side
Numerical studies on convection-diffusion equation

Convection-diffusion equation in 1D

\[
\frac{\partial}{\partial t} u(x, t) - \varepsilon u_{xx}(x, t) + v(x, t) u_x(x, t) = f(x, t) \quad (x, t) \in \Omega \times (0, T]
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u(0, t) = u(1, t) = 0 \quad t \in [0, T]
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u(x, 0) = u_0(x) \quad x \in \Omega
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- Convection term: Discretized with first order upwind scheme
- Numerical studies with
  - \( v = 1 \) for convenience
  - manufactured solution as above and corresponding right hand side
Numerical studies: upwind discretization, $\varepsilon = 10^{-3}$

Figure Level 7.

Results:

- similar behavior as for heat equation, but only 1st order accuracy
- Goal: higher order of convergence
  → central difference scheme?

**MG algorithm:**

$\nu_1 = \nu_2 = 4$, V-cycle,
GMRES smoother, coarse level 1.
Numerical studies: upwind discretization, $\varepsilon = 0$

Results:

- similar behavior as for heat equation, but only 1st order accuracy
- Goal: higher order of convergence → central difference scheme?

MG algorithm:
$
u_1 = \nu_2 = 4$, V-cycle, GMRES smoother, coarse level 1.
Numerical studies: central discretization, $\varepsilon = 10^{-3}$

Convection term discretized using second order central discretization.

**Figure** Level 7.

**Figure** Level 9.

**MG algorithm:**

$\nu_1 = \nu_2 = 4$, V-cycle,
GMRES smoother, coarse level 1.
Numerical studies: central difference quotient

Figure Convergence behavior for different values of $\varepsilon$.

- fixed $\nu = 1$, varying $\varepsilon$

→ stability issues arise for convection-dominated problems

**TG algorithm:**
level = 7, $\delta t = \frac{1}{128}$, $K = 64$
$\nu_1 = \nu_2 = 4$, GMRES smoother.
Higher order stabilization

- add diffusive term with stabilization parameter $\alpha_{add} \geq 0$ and compensation term\(^2\)

\[
(\partial_t u_h, \varphi_h) + \varepsilon(\nabla u_h, \nabla \varphi_h) + (v \cdot \nabla u_h, \varphi_h) + \alpha_{add}(\nabla u_h, \nabla \varphi_h) - \alpha_{add}(g_h, \nabla \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in V_h
\]
\[
(g_h - \nabla u_h, \psi_h) = 0 \quad \forall \psi_h \in (V_h)^d
\]

- semi-discrete formulation in matrix form: $M_h \sim \text{id}$, $B_h \sim \text{grad}$, $B_h^\top \sim \text{div}$

\[
\partial_t M_h u_h(t) + \varepsilon L_h u_h(t) + K_h u_h(t) + \alpha_{add}(L_h - B_h^\top M_h^{-1} B_h) u_h(t) = f_h(t) \quad (\star)
\]
\[
\implies L_h - L_{2h} \sim \frac{1}{(2h)^2} [1, -4, 6, -4, 1]
\]

- ($\star$) 1D with linear FEM, uniform grid, quadrature based mass-lumping:

\[
M_h^{-1} B_h^\top M_h^{-1} B_h \sim -\frac{1}{(2h)^2} [1, 0, -2, 0, 1] \sim L_{2h}, \quad L_h \sim -\frac{1}{h^2} [1, -2, 1]
\]

\[
\implies L_h - L_{2h} \sim \frac{1}{(2h)^2} [1, -4, 6, -4, 1]
\]

Choice of $\alpha_{\text{add}}$ in multigrid algorithm

$$\alpha_{\text{add}} := \alpha \left( \frac{h_f}{h} \right) ^\gamma \Rightarrow \alpha \left( \frac{h_f}{h} \right) ^\gamma (L_h - L_{2h})u_h$$

where $\alpha > 0$, $\gamma = 2$, $h_f$: mesh size of fine level, $h$: mesh size of current level.

- stabilization term turns out to be the biharmonic operator with a certain factor
  - $L_h u_h \sim \frac{1}{h^2} (h^2 u_{xx} + Ch^4 u_{xxxx} + O(h^6))$
  - $L_{2h} u_h \sim \frac{1}{4h^2} (4h^2 u_{xx} + 2^4 Ch^4 u_{xxxx} + O(h^6))$
  - $(L_h - L_{2h}) u_h \sim \tilde{C} h^2 u_{xxxx} + O(h^4)$

- solve the same continuous problem on each level, **but** less stabilization on coarser levels

$$\alpha \left( \frac{h_f}{h} \right) ^\gamma (L_h - L_{2h})u_h \sim \alpha \left( \frac{h_f}{h} \right) ^\gamma \tilde{C} h^2 u_{xxxx} = \alpha h_f^2 \tilde{C} u_{xxxx}, \quad \text{for } \gamma = 2$$

- The stabilization term is treated fully implicit!
Numerical studies: stabilization

\[ \varepsilon = 0 \]

\[ \varepsilon = 1 \]

Figure No stabilization.

- number of iterations hardly increases for small values of \( \varepsilon \)
- the plateau disappears

\[ \rightarrow \text{stabilization can help! How to choose } \alpha \text{ quantitatively?} \]

Figure Stabilization with \( \alpha = 0.1 \).
Numerical studies: choice of $\alpha$

<table>
<thead>
<tr>
<th>$\delta t = h$</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.01$</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/64</td>
<td>2.0e-03</td>
<td>2.1e-03</td>
<td>6.9e-03</td>
<td>4.8e-02</td>
<td>1.3e-01</td>
</tr>
<tr>
<td>1/128</td>
<td>4.8e-04</td>
<td>5.1e-04</td>
<td>1.7e-03</td>
<td>1.6e-02</td>
<td>7.8e-02</td>
</tr>
<tr>
<td>1/256</td>
<td>1.2e-04</td>
<td>1.3e-04</td>
<td>4.4e-04</td>
<td>4.2e-03</td>
<td>3.4e-02</td>
</tr>
</tbody>
</table>

Table Discrete $L_2$-error at final time $T = 1$, $\varepsilon = 10^{-3}$.

<table>
<thead>
<tr>
<th>$\delta t = h$</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.01$</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/64</td>
<td>5.0e-03</td>
<td>2.1e-03</td>
<td>7.0e-03</td>
<td>4.8e-02</td>
<td>1.3e-01</td>
</tr>
<tr>
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</tr>
<tr>
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<td>1.3e-04</td>
<td>4.4e-04</td>
<td>4.2e-03</td>
<td>3.4e-02</td>
</tr>
</tbody>
</table>

Table Discrete $L_2$-error at final time $T = 1$, $\varepsilon = 0$.

- the error is reduced by a factor of $\approx 4 \rightarrow$ 2nd order of convergence observed
- loss of accuracy for larger $\alpha$

$\rightarrow$ do not choose $\alpha$ too large
## Numerical studies: choice of $\alpha$ 

### Table Level 5, $\varepsilon = 10^{-3}$.

<table>
<thead>
<tr>
<th>$\delta t$</th>
<th>$K = 256$</th>
<th>$K = 512$</th>
<th>$K = 1024$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0$</td>
<td>100</td>
<td>100</td>
<td>14</td>
</tr>
<tr>
<td>$\alpha = 0.01$</td>
<td>100</td>
<td>27</td>
<td>9</td>
</tr>
<tr>
<td>$\alpha = 0.1$</td>
<td>15</td>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>7</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$\alpha = 10$</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

### Table Level 7, $\varepsilon = 10^{-3}$.

<table>
<thead>
<tr>
<th>$\delta t$</th>
<th>$K = 256$</th>
<th>$K = 512$</th>
<th>$K = 1024$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0$</td>
<td>100</td>
<td>100</td>
<td>14</td>
</tr>
<tr>
<td>$\alpha = 0.01$</td>
<td>22</td>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>$\alpha = 0.1$</td>
<td>8</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$\alpha = 10$</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

**Without stabilization**
- increasing/many iterations for large $K$
- $\delta t$- and level-dependency

**TG algorithm:**
$\nu_1 = \nu_2 = 4$, GMRES smoother, maximum number of iterations: 100.

**With stabilization**
- number of iterations decreases as $\alpha \to \infty$
- independent of number of blocked time steps $K$ and $\delta t$ for sufficiently large $\alpha$
- similar convergence behavior for different fine levels

$\rightarrow$ do not choose $\alpha$ too small
Numerical studies: choice of $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$K$</th>
<th>$\delta t$</th>
<th>$1/32$</th>
<th>$1/128$</th>
<th>$1/512$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>256</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>512</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>38</td>
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<tr>
<td>0.01</td>
<td>256</td>
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<tr>
<td>0.1</td>
<td>256</td>
<td>15</td>
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<td>1024</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

Table Level 5, $\varepsilon = 10^{-3}$.

Without stabilization

- increasing/many iterations for large $K$
- $\delta t$, $\varepsilon$-dependency

TG algorithm:

$\nu_1 = \nu_2 = 4$, GMRES smoother, maximum number of iterations: 100.

With stabilization

- number of iterations decreases as $\alpha \to \infty$
- independent of number of blocked time steps $K$ and $\delta t$ for sufficiently large $\alpha$
- good convergence results even for smallest $\varepsilon$

$\rightarrow$ do not choose $\alpha$ too small

Table Level 5, $\varepsilon = 0$. 
Numerical studies: stabilization multigrid

<table>
<thead>
<tr>
<th>$\alpha$ = 0</th>
<th>$\delta t$</th>
<th>$1/32$</th>
<th>$1/128$</th>
<th>$1/512$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K = 256$</td>
<td>100</td>
<td>100</td>
<td>15</td>
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<tr>
<td>$K = 512$</td>
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<td>100</td>
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</tr>
<tr>
<td>$K = 1024$</td>
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<td>100</td>
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</tr>
<tr>
<td>$\alpha$ = 0.01</td>
<td>$K = 256$</td>
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<td>$K = 1024$</td>
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<td>29</td>
</tr>
<tr>
<td>$\alpha$ = 0.1</td>
<td>$K = 256$</td>
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<td>8</td>
</tr>
<tr>
<td></td>
<td>$K = 512$</td>
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<td>10</td>
</tr>
<tr>
<td></td>
<td>$K = 1024$</td>
<td>44</td>
<td>22</td>
<td>13</td>
</tr>
<tr>
<td>$\alpha$ = 1</td>
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<td>8</td>
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<tr>
<td></td>
<td>$K = 512$</td>
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Table Level 5, $\varepsilon = 10^{-3}$.

Table Level 7, $\varepsilon = 10^{-3}$.

- Two-grid result can also be observed for multigrid.
- But stabilization on coarse grid may not be enough for stable convergence rates.  
  $\alpha_{add} := \alpha \left( \frac{h_f}{h} \right)^{\gamma} \Rightarrow$ choosing $\gamma = 0$ or 1 can help!

MG algorithm:  
$\nu_1 = \nu_2 = 4$, F-cycle,  
GMRES smoother, coarse level 1, maximum number of iterations: 100.
Numerical studies: stabilization multigrid

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Table Level 7, $\varepsilon = 10^{-3}$, $\gamma = 1$.

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Table Level 7, $\varepsilon = 10^{-3}$, $\gamma = 2$.

- two-grid result can also be observed for multigrid
- but stabilization on coarse grid may not be enough for stable convergence rates
  $\alpha_{add} := \alpha \left( \frac{h_f}{h} \right)^\gamma \Rightarrow$ choosing $\gamma = 0$ or 1 can help!

**MG algorithm:**

$\nu_1 = \nu_2 = 4$, F-cycle, GMRES smoother, coarse level 1, maximum number of iterations: 100.
From another point of view: Heaviside step function

Figure No stabilization.

Figure Stabilization with $\alpha = 0.01$.

- Level 6, $\delta t = \frac{1}{128}$
- oscillations in the numerical solution
- small $\alpha$ can lead to a smoother numerical solution
- too large $\alpha$ can make it worse
→ trade-off: solution vs. convergence behavior
From another point of view: Heaviside step function

---

**Figure** No stabilization.

**Figure** Stabilization with $\alpha = 0.1$.

- small $\alpha$ can lead to a smoother numerical solution
- too large $\alpha$ can make it worse

→ trade-off: solution vs. convergence behavior

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**Table** Level 7, $\varepsilon = 0$. 
Conclusion

Summary

- presented multigrid algorithm works fine for convection-diffusion equation if diffusion parameter is sufficiently large
- difficulties with convection-dominated problems → stabilization can help!
- choice of stabilization parameter is crucial

Outlook and more aspects

- extension to 2D and 3D problems
- studies on the reduction of the computational efficiency
- stabilization in time using $u_{tttt}$
- combination of this time-simultaneous algorithm and other parallel-in-time methods

Figure Convergence behavior with stabilization.
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