ANALYSIS OF A VISCOUS TWO-FIELD GRADIENT DAMAGE MODEL PART II: PENALIZATION LIMIT

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Abstract. A viscous partial damage model is considered which features two damage variables coupled through a penalty term in the stored energy functional. While the well-posedness of the model was shown in a companion paper, this work analyses the behaviour of the model for penalization parameter tending to infinity. It turns out that in the limit both damage variables coincide and satisfy a classical viscous damage model.

Key words. Viscous damage evolution, energy identity, penalization

1. Introduction. This paper is concerned with a viscous gradient damage model involving two damage variables which are connected through a penalty term in the stored energy functional. While the well-posedness of the model was investigated in a companion paper, cf. [19], this work addresses the limit analysis for penalization parameter tending to infinity. The damage model under consideration reads

$$\left.\begin{aligned} (\boldsymbol{u}(t),\varphi(t)) \in \underset{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)}{\arg\min}\ \mathcal{E}(t,\boldsymbol{u},\varphi,d(t)),\\ -\partial_d\mathcal{E}(t,\boldsymbol{u}(t),\varphi(t),d(t)) \in \partial\mathcal{R}_\delta(\dot{d}(t)),\quad d(0)=d_0 \text{ a.e. in } \Omega \end{aligned}\right\} \tag{P}$$

for almost all $t \in (0,T)$. Herein, d and φ denote the local and non-local damage variable, respectively, and $\boldsymbol{u}$ stands for the displacement of the body occupying the domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$. Moreover, the stored energy $\mathcal{E} : [0,T] \times V \times H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is given by

$$\mathcal{E}(t,\boldsymbol{u},\varphi,d) := \frac{1}{2}\int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\,dx - \langle \ell(t),\boldsymbol{u}\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2 + \frac{\beta}{2}\|\varphi - d\|_2^2, \tag{1.1}$$

where $\varepsilon = 1/2(\nabla + \nabla^\top)$ is the linearized strain, $\mathbb{C}$ the elasticity tensor, and ℓ the load applied to the body. The function g describes the influence of the damage on the elastic behavior of the body. Furthermore, $\alpha > 0$ denotes the gradient regularization and $\beta > 0$ stands for the penalization parameter. The viscous dissipation functional $\mathcal{R}_\delta : L^2(\Omega) \to [0,\infty]$ appearing in (P) is defined as

$$\mathcal{R}_\delta(\eta) := \begin{cases} r\int_\Omega \eta\,dx + \frac{\delta}{2}\|\eta\|_2^2, & \text{if } \eta \geq 0 \text{ a.e. in } \Omega,\\ \infty, & \text{otherwise,}\end{cases} \tag{1.2}$$

where $r > 0$ stands for the threshold value which triggers the damage evolution and $\delta > 0$ is the viscosity parameter. For a more detailed description of the model as well as its motivation, we refer to [19, Section 2] and [4,5]. Throughout the paper, we will refer to (P) as "two-field model" or "penalized damage model".

Damage models containing two different damage variables coupled through a penalization of the energy functional are frequently used for numerical simulations, cf.

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e.g. [4, 5, 18, 24–26, 29]. With this work and the companion paper [19], we show that this approach is mathematically justified for two reasons. First of all, the two-field model (P) admits a unique solution in terms of the two damage variables and the displacement field, and all three quantities depend Lipschitz continuously on the data, i.e., the applied loads, so that the problem is well-posed. These questions are addressed in the companion paper [19]. Secondly, as we will show in this paper, in the limit $\beta \to \infty$, both damage variables become equal and the resulting single-field damage model falls into the category of classical viscous, partial damage models, as for instance introduced in [8].

Let us put our work into perspective. Numerous damage models have been addressed by many authors under different aspects, among these viscous, i.e., rate-dependent models, see e.g. [1–3, 7], but also rate-independent damage models, cf. for instance [6, 13, 15–17, 20, 21, 23]. Especially in the discussion of the latter, the so-called energy inequality plays a crucial role and various notions of solutions are based on it, such as for example global energetic and semi-stable solutions, see [22] for an overview. The energy inequality is also an essential tool for the limit analysis for vanishing viscosity as performed in [15]. This passage to the limit leads to another notion of solutions, the so-called parametrized solutions, see [15, Section 5]. The energy inequality also turns out to be very useful for our limit analysis, when $\beta \to \infty$. To be more precise, we re-formulate (P) as an equivalent energy inequality, which is actually an energy identity. This then allows a passage to the limit and in this way yields the results sketched above.

The paper is organized as follows. Section 2 collects the notations and standing assumptions, as well as known results from the companion paper [19], which are needed in the present paper. In Section 3 we derive the energy inequality mentioned above. Section 4 is devoted to the passage to the limit $\beta \to \infty$ in the energy inequality. In Section 5, we will reformulate the latter as a PDE system involving only one single damage variable. By means of a time-discretization we then show in Section 6 that the damage variables associated with the penalized model (P) are essentially bounded in time and space by a constant independent of the penalty parameter β so that this bound carries over to the damage variable in the limit. Based on this result, we transform the single-damage model from Section 5 into an equivalent PDE system that coincides with the viscous model in [15] and constitutes a classical, viscous partial damage model.

2. Notation, Standing Assumptions, and Known Results.

Throughout the paper, C denotes a generic positive constant. If X and Y are two linear normed spaces, the space of linear and bounded operators from X to Y is denoted by $\mathcal{L}(X, Y)$. The dual of a linear normed space X will be denoted by X^*. For the dual pairing between X and X^* we write $\langle ., . \rangle_X$ and, if it is clear from the context, which dual pairing is meant, we just write $\langle ., . \rangle$. By $\|\cdot\|_p$ we denote the $L^p(\Omega)-$norm for $p \in [1, \infty]$ and by $(\cdot, \cdot)_2$ the $L^2(\Omega)-$scalar product. If X is compactly embedded in Y, we write $X \hookrightarrow\hookrightarrow Y$. In the rest of the paper $N \in \{2, 3\}$ denotes the spatial dimension. By bold-face case letters we denote vector valued variables and vector valued spaces.

Definition 2.1. *For $p \in [1, \infty]$ we define the following subspace of $\boldsymbol{W}^{1,p}(\Omega)$:*

$$\boldsymbol{W}^{1,p}_D(\Omega) := \{v \in \boldsymbol{W}^{1,p}(\Omega) : v|_{\Gamma_D} = 0\},$$

where Γ_D is a part of the boundary of the domain Ω, see Assumption 2.2 below. The dual space of $\boldsymbol{W}^{1,p'}_D(\Omega)$ is denoted by $\boldsymbol{W}^{-1,p}_D(\Omega)$, where p' is the conjugate exponent *of p. If $p = 2$, we abbreviate $V := \boldsymbol{W}^{1,2}_D(\Omega)$.*

Before we turn to our assumptions on the data, we summarize the often used symbols in Table 2.1 for convenience of the reader.

TABLE 2.1
Functionals, operators and variables

Symbol	Meaning	Definition
$\mathcal{I}$	Reduced energy functional	Definition 3.1
$\mathcal{R}_\delta$	Viscous dissipation functional	(1.2)
$\boldsymbol{u}$	Displacement	
φ	Nonlocal damage	
d	Local damage	
$\widetilde{\mathcal{E}}$	Energy functional w/o penalty	Definition 4.10
$\widetilde{\mathcal{I}}$	Reduced energy functional w/o penalty	(4.16)
$\widetilde{\mathcal{R}}_1$	Dissipation functional in the limit	(5.10)
$\widetilde{\mathcal{R}}_\delta$	Viscous dissipation functional in the limit	Definition 4.13

Let us now state our standing assumptions. We begin with the smoothness of the computational domain.

ASSUMPTION 2.2. *The domain $\Omega \subset \mathbb{R}^N$, $N \in \{2, 3\}$, is bounded with Lipschitz boundary Γ. The boundary consists of two disjoint measurable parts Γ_N and Γ_D such that $\Gamma = \Gamma_N \cup \Gamma_D$. While Γ_N is a relatively open subset, Γ_D is a relatively closed subset of Γ with positive measure.*

In addition, the set $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, cf. [9]. That is, for every point $x \in \Gamma$, there exists an open neighborhood $\mathcal{U}_x \subset \mathbb{R}^N$ of x and a bi-Lipschitz map (a Lipschitz continuous and bijective map with Lipschitz continuous inverse) $\Psi_x : \mathcal{U}_x \to \mathbb{R}^N$ such that $\Psi_x(x) = 0 \in \mathbb{R}^N$ and $\Psi_x\big(\mathcal{U}_x \cap (\Omega \cup \Gamma_N)\big)$ equals one of the following sets:

$$
\begin{aligned}
E_1 &:= \big\{y \in \mathbb{R}^N : |y| < 1,\ y_N < 0\big\},\\
E_2 &:= \big\{y \in \mathbb{R}^N : |y| < 1,\ y_N \le 0\big\},\\
E_3 &:= \{y \in E_2 : y_N < 0 \text{ or } y_1 > 0\}.
\end{aligned}
$$

A detailed characterization of Gröger-regular sets in two and three spatial dimensions is given in [10].

ASSUMPTION 2.3. *The function $g : \mathbb{R} \to [\epsilon, 1]$ satisfies $g \in C^2(\mathbb{R})$ and $g', g'' \in L^\infty(\mathbb{R})$ with some $\epsilon > 0$. With a little abuse of notation the Nemystkii-operators associated with g and g', considered with different domains and ranges, will be denoted by the same symbol.*

ASSUMPTION 2.4. *The fourth-order tensor $\mathbb{C} \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{N\times N}_{\text{sym}}))$ is symmetric and uniformly coercive, i.e., there is a constant $\gamma_{\mathbb{C}} > 0$ such that*

$$
\mathbb{C}(x)\boldsymbol{\sigma} : \boldsymbol{\sigma} \ge \gamma_{\mathbb{C}} |\boldsymbol{\sigma}|^2 \quad \forall\, \boldsymbol{\sigma} \in \mathbb{R}^{N\times N}_{\text{sym}} \text{ and f.a.a. } x \in \Omega, \tag{2.1}
$$

where $|\cdot|$ denotes the Frobenius norm on $\mathbb{R}^{N\times N}$ and $(\cdot : \cdot)$ the scalar product inducing this norm.

ASSUMPTION 2.5. *For the applied volume and boundary load we require*

$$
\ell \in C^1([0, T]; \boldsymbol{W}^{-1,p}_D(\Omega)),
$$

where $p > N$ is specified below, see Assumption 2.7.1 and Assumption 5.4.

Moreover, the initial damage is supposed to satisfy $d_0 \in L^2(\Omega)$.

Our last assumption concerns the balance of momentum associated with the energy functional in (1.1). For its precise statement we need the following

DEFINITION 2.6. *For given $\varphi \in L^1(\Omega)$ we define the linear form $A_\varphi : V \to V^*$ as*

$$\langle A_\varphi \boldsymbol{u}, v\rangle_V := \int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, dx.$$

The operator A_φ considered with different domains and ranges will be denoted by the same symbol for the sake of convenience.

ASSUMPTION 2.7. *For the rest of the paper we require the following:*

1. *There exists $p > N$ such that, for all $\overline{p} \in [2, p]$ and all $\varphi \in L^1(\Omega)$, the operator $A_\varphi : \boldsymbol{W}_D^{1,\overline{p}}(\Omega) \to \boldsymbol{W}_D^{-1,\overline{p}}(\Omega)$ is continuously invertible. Moreover, there exists a constant $c > 0$, independent of φ and $\overline{p}$, such that*

$$\|A_\varphi^{-1}\|_{\mathcal{L}(\boldsymbol{W}_D^{-1,\overline{p}}(\Omega),\boldsymbol{W}_D^{1,\overline{p}}(\Omega))} \leq c$$

   holds for all $\overline{p} \in [2, p]$ and all $\varphi \in L^1(\Omega)$.
2. *The penalization parameter β is sufficiently large, depending only on the data.*

REMARK 2.8. *The critical assumption is Assumption 2.7.1. If $N = 2$, then this condition is automatically fulfilled, see [19, Lemma 3.2]. The situation changes however, if one turns to $N = 3$. In this case this assumption can be guaranteed by imposing additional and rather restrictive conditions on the data, in particular on the ellipticity and boundedness constants associated with $\mathbb{C}$ and g, see [19, Remark 3.20] for more details. However, as explained in [19, Remark 3.21], one could alternatively modify the energy functional in (1.1) by replacing $\|\nabla\varphi\|_2^2$ with its $H^{3/2}$-seminorm. This would allow to drop Assumption 2.7.1 in the three dimensional case, too. However, we chose not to work with the $H^{3/2}$-seminorm, as the associated bilinear form is difficult to realize in numerical computations.*

As an immediate consequence of Assumption 2.7.1 and the regularity of ℓ in Assumption 2.5 one can introduce the following

DEFINITION 2.9. *We define the operator $\mathcal{U} : [0, T] \times H^1(\Omega) \to \boldsymbol{W}_D^{1,p}(\Omega)$ by $\mathcal{U}(t, \varphi) := A_\varphi^{-1}\ell(t)$. We will frequently consider $\mathcal{U}$ with different range and domain, but denote it by the same symbol.*

Thanks to Assumptions 2.5 and 2.7.1 there exists a constant $c > 0$, independent of t and φ, such that

$$\|\mathcal{U}(t, \varphi)\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \leq c \quad \forall\, (t, \varphi) \in [0, T] \times H^1(\Omega), \tag{2.2}$$

which will be frequently used in the sequel.

In the rest of this section we recall some known results and definitions from the companion paper [19] that will be used in the upcoming analysis.

LEMMA 2.10 (Global Lipschitz continuity of $\mathcal{U}$, [19, Proposition 3.7]). *Let $p > 2$ and $r \in \big[2p/(p-2), \infty\big]$ be given. Then there exists $L > 0$ such that for all $\varphi_1, \varphi_2 \in H^1(\Omega) \cap L^r(\Omega)$ and all $t_1, t_2 \in [0, T]$ it holds*

$$\|\mathcal{U}(t_1, \varphi_1) - \mathcal{U}(t_2, \varphi_2)\|_{\boldsymbol{W}_D^{1,\pi}(\Omega)} \leq L(|t_1 - t_2| + \|\varphi_1 - \varphi_2\|_r), \tag{2.3}$$

where $1/\pi = 1/p + 1/r$.

LEMMA 2.11 (Fréchet differentiability of $\mathcal{U}$, [19, Proposition 5.6]). *It holds* $\mathcal{U} \in C^1([0,T] \times H^1(\Omega); V)$ *and at all* $t \in [0,T]$ *and* $\varphi, \delta\varphi \in H^1(\Omega)$ *we have*

$$\partial_t \mathcal{U}(t,\varphi) = A_\varphi^{-1} \dot{\ell}(t) \in \boldsymbol{W}_D^{1,p}(\Omega), \tag{2.4a}$$

$$A_\varphi\big(\partial_\varphi \mathcal{U}(t,\varphi)(\delta\varphi)\big) = \operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)\big) \quad \text{in } V^*, \tag{2.4b}$$

where $\operatorname{div} : L^2(\Omega; \mathbb{R}^{n\times n}_{sym}) \to V^*$ *denotes the distributional divergence. Moreover, there exists a constant* $c > 0$*, independent of* t *and* φ*, such that*

$$\|\partial_t \mathcal{U}(t,\varphi)\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \le c \quad \forall\, (t,\varphi) \in [0,T] \times H^1(\Omega). \tag{2.5}$$

In order to state the Euler-Lagrange equations associated with the energy minimization in (P) let us further define the mappings $B : H^1(\Omega) \to H^1(\Omega)^*$ and $F : [0,T] \times H^1(\Omega) \to H^1(\Omega)^*$ by

$$\langle B\varphi, \psi\rangle_{H^1(\Omega)} := \int_\Omega \alpha \nabla\varphi \cdot \nabla\psi + \beta\varphi\psi \, dx, \qquad \varphi,\psi \in H^1(\Omega), \tag{2.6}$$

$$\langle F(t,\varphi), \psi\rangle_{H^1(\Omega)} := \frac{1}{2}\int_\Omega g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi))\psi \, dx, \quad \varphi,\psi \in H^1(\Omega). \tag{2.7}$$

Note that F is well defined because of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^s(\Omega)$ with $s = 6$ for $N = 3$ and $s < \infty$ for $N = 2$ in combination with Assumption 2.7.1.

LEMMA 2.12. *The mapping* F *possesses the following properties:*

- *[19, Lemma 3.14] It is Lipschitzian in the following sense: Let* $r \ge 2p/(p-2)$ *and* $1/s + 2/p + 1/r = 1$*. Then, for all* $t_1, t_2 \in [0,T]$*, all* $\varphi_1, \varphi_2 \in H^1(\Omega) \cap L^r(\Omega)$ *and all* $\psi \in L^s(\Omega)$*, there holds*

$$|\langle F(t_1,\varphi_1) - F(t_2,\varphi_2), \psi\rangle_{H^1(\Omega)}| \le C\big(\|\varphi_1 - \varphi_2\|_r + |t_1 - t_2|\big)\|\psi\|_s, \tag{2.8}$$

  with a constant $C > 0$ *independent of* $(t_i, \varphi_i)_{i=1,2}$.
- *[19, Lemma 5.9] It is continuously Fréchet differentiable from* $[0,T] \times H^1(\Omega)$ *to* $H^1(\Omega)^*$*, and for all* $(t,\varphi) \in [0,T] \times H^1(\Omega)$ *and all* $(\delta t, \delta\varphi) \in \mathbb{R} \times H^1(\Omega)$ *we have*

$$\begin{aligned}\langle F'(t,\varphi)(\delta t,\delta\varphi), z\rangle_{H^1(\Omega)} &= \frac{1}{2}\int_\Omega g''(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi))z \, dx \\ &\quad + \int_\Omega g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon\Big(\mathcal{U}'(t,\varphi)(\delta t,\delta\varphi)\Big)z \, dx\end{aligned} \tag{2.9}$$

  for all $z \in H^1(\Omega)$.
- *[19, Eq. (5.28)] For its partial derivative w.r.t.* φ *there holds*

$$|\langle \partial_\varphi F(t,\varphi)z, z\rangle_{H^1(\Omega)}| \le k\|z\|_2^2 + \widetilde{c}(k)\|z\|^2_{H^1(\Omega)} \tag{2.10}$$

  for all $z \in H^1(\Omega)$*, all* $k > 0$*, and all* $(t,\varphi) \in [0,T] \times H^1(\Omega)$*, where* $\widetilde{c} : \mathbb{R}^+ \to \mathbb{R}^+$ *is a monotonically decreasing function, which tends to* 0 *as* $k \to \infty$.

With the mappings B and F at hand we can characterize the solution to the energy minimization in (P) as follows:

Lemma 2.13 (Energy minimizer, [19, Prop. 3.12, Thm. 3.17]). *For every* $(t, d) \in [0, T] \times L^2(\Omega)$*, the optimization problem*

$$\min_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{E}(t, \boldsymbol{u}, \varphi, d)$$

admits a unique minimizer $(\boldsymbol{u}, \varphi) \in \boldsymbol{W}^{1,p}_D(\Omega) \times H^1(\Omega)$ *characterized by* $\boldsymbol{u} = \mathcal{U}(t, \varphi)$ *and* $\varphi = \Phi(t, d)$*, where* $\Phi : [0, T] \times L^2(\Omega) \to H^1(\Omega)$ *is defined by* $\Phi(t, d) := (B + F(t, \cdot))^{-1}(\beta d)$.

Lemma 2.14 (Global Lipschitz continuity of Φ, [19, Eq. (3.37)]). *There exists a constant* $K > 0$ *so that*

$$\|\Phi(t_1, d_1)-\Phi(t_2, d_2)\|_{H^1(\Omega)} \le K \big(|t_1-t_2|+\|d_1-d_2\|_2\big) \quad \forall\, t_1, t_2 \in [0, T],\ d_1, d_2 \in L^2(\Omega).$$

Lemma 2.15 (Fréchet differentiability of Φ, [19, Prop. 5.12]). *The solution operator* Φ *is continuously Fréchet differentiable from* $(0, T) \times L^2(\Omega)$ *to* $H^1(\Omega)$*. Moreover, for all* $(t, d) \in [0, T] \times L^2(\Omega)$ *and all* $(\delta t, \delta d) \in \mathbb{R} \times L^2(\Omega)$ *its derivative solves the following linearized equation*

$$B\Phi'(t, d)(\delta t, \delta d) + F'(t, \varphi)\big(\delta t, \Phi'(t, d)(\delta t, \delta d)\big) = \beta\delta d, \tag{2.11}$$

where we use the abbreviation $\varphi := \Phi(t, d)$.

Finally we turn our attention to the differential inclusion in (P). First note that the functional $\mathcal{E}$ is partially Fréchet differentiable w.r.t. d on $[0, T] \times V \times H^1(\Omega) \times L^2(\Omega)$, and its partial derivative is given by

$$\partial_d\mathcal{E}(t, \boldsymbol{u}, \varphi, d) = \beta(d - \varphi). \tag{2.12}$$

Therefore, in view Lemma 2.13, (P) reduces to the following evolutionary equation

$$-\beta\big(d(t) - \Phi(t, d(t))\big) \in \partial\mathcal{R}_\delta(\dot{d}(t)) \quad \text{f.a.a. } t \in (0, T), \quad d(0) = d_0. \tag{2.13}$$

As shown in [19, Lemma 3.22], this equation is equivalent to the following non-smooth operator differential equation:

$$\dot{d}(t) = \frac{1}{\delta}\max\{-\beta\big(d(t) - \Phi(t, d(t))\big) - r, 0\} \quad \text{f.a.a. } t \in (0, T), \quad d(0) = d_0. \tag{2.14}$$

This handy reformulation of (P) and (2.13), respectively, is a main advantage of the penalty-type regularization of partial damage models and could provide a useful starting point for a numerical solution of (P). We end this section by recalling the main result of the companion paper:

Theorem 2.16 (Existence and uniqueness for the penalized damage model, [19, Thm. 5.13]). *There exists a unique solution* $(\boldsymbol{u}, \varphi, d)$ *of the problem* (P)*, satisfying* $\boldsymbol{u} \in C^1([0, T]; V)$*,* $\varphi \in C^1([0, T]; H^1(\Omega))$*,* $d \in C^{1,1}([0, T]; L^2(\Omega))$*, which is uniquely characterized by the following system of differential equations:*

$$-\operatorname{div} g(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) = \ell(t) \quad \text{in } \boldsymbol{W}^{-1,p}_D(\Omega) \tag{2.15a}$$

$$-\alpha\Delta\varphi(t) + \beta\,\varphi(t) + \frac{1}{2}\,g'(\varphi(t))\mathbb{C}\,\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) = \beta d(t) \quad \text{in } H^1(\Omega)^* \tag{2.15b}$$

$$\dot{d}(t) - \frac{1}{\delta}\max\{-\beta(d(t) - \varphi(t)) - r, 0\} = 0, \qquad d(0) = d_0. \tag{2.15c}$$

for every $t \in [0,T]$.

Note that, thanks to the definition of B and F, the equations in (2.15a) and (2.15b) are just equivalent to $\boldsymbol{u}(t) = \mathcal{U}(t, \varphi(t))$ and $\varphi(t) = \Phi(t, d(t))$, respectively.

3. Energy Identity. As seen in Theorem 2.16, for a given $\beta > 0$ sufficiently large, there exists a unique solution to (2.14), which we denote by d_β to indicate its dependency on the parameter β. The other components are uniquely determined by d_β through $\varphi_\beta = \Phi(\cdot, d_\beta(\cdot))$ and $\boldsymbol{u}_\beta = \mathcal{U}(\cdot, \varphi_\beta(\cdot))$. The purpose of this section is to derive a characterization of the local damage d_β, which allows to find an estimate of the form $\|d_\beta\|_X \leq C$ for all $\beta > 0$, where X is a suitable reflexive Banach space and $C > 0$ is a constant independent of the penalty parameter β. Such an estimate will then allow to pass to the limit in the penalized damage model as $\beta \to \infty$, see Section 4 below. As seen in (2.13) and (2.14) above, there are various ways to describe the evolution of the local damage. However, all these descriptions have the disadvantage of containing the term $\beta(d_\beta - \varphi_\beta)$, which is not necessarily uniformly bounded w.r.t. β in suitable spaces that allow a passage to the limit. Our aim is therefore to find an alternative description of the evolution of the local damage, which only contains expressions that are bounded w.r.t. β. Such a description is given by the *energy identity* in Proposition 3.5 below.

For the rest of this section we drop the index β to shorten the notation. As already indicated above, the displacement $\boldsymbol{u}$ and the nonlocal damage φ are uniquely determined by the local damage d so that it is reasonable to reduce the whole system to the variable d only. For this purpose we define the following:

DEFINITION 3.1. *The reduced energy functional* $\mathcal{I} : [0,T] \times L^2(\Omega) \to \mathbb{R}$ *is given by*

$$\mathcal{I}(t, d) := \mathcal{E}(t, \mathcal{U}(t, \Phi(t, d)), \Phi(t, d), d).$$

The reduced energy functional will be a key ingredient for deriving the energy identity. On account of (1.1) and Definitions 2.6 and 2.9 it can be rewritten as

$$\begin{aligned}
\mathcal{I}(t, d) &= \frac{1}{2}\langle A_{\Phi(t,d)}\big(\mathcal{U}(t, \Phi(t, d))\big), \mathcal{U}(t, \Phi(t, d))\rangle_V - \langle \ell(t), \mathcal{U}(t, \Phi(t, d))\rangle_V \\
&\quad + \frac{\alpha}{2}\|\nabla\Phi(t, d)\|_2^2 + \frac{\beta}{2}\|\Phi(t, d) - d\|_2^2 \\
&= -\frac{1}{2}\langle \ell(t), \mathcal{U}(t, \Phi(t, d))\rangle_V + \frac{\alpha}{2}\|\nabla\Phi(t, d)\|_2^2 + \frac{\beta}{2}\|\Phi(t, d) - d\|_2^2. \qquad (3.1)
\end{aligned}$$

This reformulation of the reduced energy allows to show the following

LEMMA 3.2 (Fréchet differentiability of $\mathcal{I}$). *It holds* $\mathcal{I} \in C^1([0,T] \times L^2(\Omega))$ *and, at all* $(t, d) \in [0,T] \times L^2(\Omega)$*, we have*

$$\partial_t \mathcal{I}(t, d) = -\langle \dot{\ell}(t), \mathcal{U}(t, \Phi(t, d))\rangle_V, \quad \partial_d \mathcal{I}(t, d) = \beta(d - \Phi(t, d)). \qquad (3.2)$$

Proof. First note that the mapping

$$f : [0,T] \times L^2(\Omega) \to \mathbb{R}, \quad f(t, d) := \langle \ell(t), \mathcal{U}(t, \Phi(t, d))\rangle_V$$

can be seen as product of the functions ℓ and $[0,T] \times L^2(\Omega) \ni (t, d) \mapsto \mathcal{U}(t, \Phi(t, d)) \in V$. The latter one is continuously Fréchet differentiable, thanks to Lemmas 2.11 and

2.15. Together with Assumption 2.5 the product rule yields $f \in C^1([0,T] \times L^2(\Omega))$. Thus, thanks to Lemma 2.15, we deduce from (3.1) that $\mathcal{I} \in C^1([0,T] \times L^2(\Omega))$ and, for given $(t,d) \in [0,T] \times L^2(\Omega)$ and $(\delta t, \delta d) \in \mathbb{R} \times L^2(\Omega)$, it holds

$$\begin{aligned} \mathcal{I}'(t,d)(\delta t, \delta d) = -\frac{1}{2}\langle \dot{\ell}(t)\delta t, \mathcal{U}(t,\Phi(t,d))\rangle_V - \frac{1}{2}\langle \ell(t), \mathcal{U}'(t,\Phi(t,d))(\delta t, \delta\varphi)\rangle_V \\ + \alpha(\nabla\Phi(t,d), \nabla\delta\varphi)_2 + \beta(\Phi(t,d) - d, \delta\varphi - \delta d)_2, \end{aligned} \tag{3.3}$$

where we abbreviate $\delta\varphi = \Phi'(t,d)(\delta t, \delta d)$. To derive the formulas for the partial derivatives, first observe that (2.4a) tested with $\mathcal{U}(t,\varphi)$, Definitions 2.6 and 2.9, and the symmetry of $\mathbb{C}$ imply

$$\begin{aligned} \langle \dot{\ell}(t), \mathcal{U}(t,\Phi(t,d))\rangle_V &= \langle A_{\Phi(t,d)}\partial_t\mathcal{U}(t,\Phi(t,d)), \mathcal{U}(t,\Phi(t,d))\rangle_V \\ &= \langle \ell(t), \partial_t\mathcal{U}(t,\Phi(t,d))\rangle_V. \end{aligned} \tag{3.4}$$

If one tests (2.4b) with $\mathcal{U}(t,\Phi(t,d)) \in V$, one further obtains

$$\begin{aligned} &-\frac{1}{2}\langle \ell(t), \partial_\varphi\mathcal{U}(t,\Phi(t,d))\delta\varphi\rangle_V + \alpha(\nabla\Phi(t,d), \nabla\delta\varphi)_2 + \beta(\Phi(t,d) - d, \delta\varphi - \delta d)_2 \\ &\quad = -\frac{1}{2}\langle \operatorname{div}\big(g'(\Phi(t,d))(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\Phi(t,d))\big), \mathcal{U}(t,\Phi(t,d))\rangle_V \\ &\qquad + \alpha(\nabla\Phi(t,d), \nabla\delta\varphi)_2 + \beta(\Phi(t,d) - d, \delta\varphi - \delta d)_2 \\ &\quad = \langle F\big(t,\Phi(t,d)\big) + B\Phi(t,d), \delta\varphi\rangle_{H^1(\Omega)} - \beta(d,\delta\varphi)_2 + \beta(d - \Phi(t,d), \delta d)_2 \\ &\quad = \beta(d - \Phi(t,d), \delta d)_2 \quad \forall\, \delta d \in L^2(\Omega), \end{aligned} \tag{3.5}$$

where $\operatorname{div} : L^2(\Omega; \mathbb{R}^{N\times N}_{\text{sym}}) \to V^*$ denotes the distributional divergence. Note that the last two equalities follow from (2.6), (2.7), and the definition of Φ, respectively. Inserting (3.4) and (3.5) in (3.3) leads to (3.2). □

As an immediate consequence of Lemma 3.2 and the chain rule, one obtains the following

Corollary 3.3 (Total derivative of $\mathcal{I}(\cdot, d(\cdot))$). *Let $d \in C^1([0,T], L^2(\Omega))$ be given. Then the map $[0,T] \ni t \mapsto \mathcal{I}(t, d(t))$ is continuously differentiable with*

$$\frac{d}{dt}\mathcal{I}(t,d(t)) = \partial_t\mathcal{I}(t,d(t)) + (\partial_d\mathcal{I}(t,d(t)), \dot{d}(t))_2 \quad \forall\, t \in [0,T].$$

With the help of the reduced energy $\mathcal{I}$ we will deduce the energy identity from the evolutionary equation in (2.13). To this end note first that, due to the second equation in (3.2), the evolutionary equation (2.13) or equivalently (2.14) can also be written as

$$0 \in \partial\mathcal{R}_\delta(\dot{d}(t)) + \partial_d\mathcal{I}(t,d(t)) \quad \forall\, t \in [0,T], \quad d(0) = d_0. \tag{3.6}$$

Since $\mathcal{R}_\delta$ is proper and convex, this is in turn equivalent to

$$\mathcal{R}_\delta(\dot{d}(t)) + \mathcal{R}^*_\delta\big(-\partial_d\mathcal{I}(t,d(t))\big) = (-\partial_d\mathcal{I}(t,d(t)), \dot{d}(t))_2 \;\; \forall\, t \in [0,T], \quad d(0) = d_0, \tag{3.7}$$

which will be the starting point for proving the energy identity in Proposition 3.5 below. To summarize, we obtained the following four alternative, but yet equivalent formulations:

- the subdifferential formulations in (2.13) and (3.6), respectively,
- the nonsmooth operator differential equation in (2.14),
- the Fenchel-Young equality in (3.7).

In all what follows, we refer to these equivalent formulations simply as *penalized damage evolution*. Note that, since (2.13) and (2.14), respectively, are uniquely solvable by Theorem 2.16, the same holds for (3.6) and (3.7).

LEMMA 3.4. *If d satisfies the penalized damage evolution, then, for every $t \in [0,T]$, there holds $\mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(t, d(t))\big) = \frac{\delta}{2}\|\dot d(t)\|_2^2$.*

Proof. Let d satisfy the penalized damage evolution. Then it follows from (3.6) that $\partial \mathcal{R}_\delta(\dot d(t)) \neq \emptyset$ so that $\dot d \geq 0$. Hence, inserting (1.2) and (3.2) in (3.7) leads to

$$\mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(t, d(t))\big) = (-\beta(d(t)-\varphi(t)), \dot d(t))_2 - r\|\dot d(t)\|_1 - \frac{\delta}{2}\|\dot d(t)\|_2^2 \quad \forall\, t \in [0,T], \tag{3.8}$$

where we again abbreviated $\varphi = \Phi(\cdot, d(\cdot))$. From the equivalent formulation (2.14) multiplied with $\dot d(t)$ and integrated over Ω, we deduce that $\delta\|\dot d(t)\|_2^2 = (-\beta(d(t) - \varphi(t)), \dot d(t))_2 - r\|\dot d(t)\|_1$. Inserting this into (3.8) gives the assertion. □

PROPOSITION 3.5 (Energy identity). *The unique solution $d \in C^1([0,T]; L^2(\Omega))$ of the penalized damage evolution fulfills for all $0 \leq s \leq t \leq T$ the energy identity*

$$\begin{aligned}\int_s^t \mathcal{R}_\delta(\dot d(\tau))d\tau + \int_s^t \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d(\tau))\big)d\tau + \mathcal{I}(t, d(t))&\\ = \mathcal{I}(s, d(s)) + \int_s^t \partial_t \mathcal{I}(\tau, d(\tau))d\tau.&\end{aligned} \tag{3.9}$$

Proof. Corollary 3.3 combined with (3.7) yields at all $\tau \in [0,T]$ the identity

$$\mathcal{R}_\delta(\dot d(\tau)) + \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d(\tau))\big) = \partial_t \mathcal{I}(\tau, d(\tau)) - \frac{d}{dt}\mathcal{I}(\tau, d(\tau)). \tag{3.10}$$

Recall that $\dot d \geq 0$ as a result of (2.14). This implies in view of (1.2) and $\dot d \in C([0,T], L^2(\Omega))$ that the map $[0,T] \ni \tau \mapsto \mathcal{R}_\delta(\dot d(\tau)) \in \mathbb{R}$ is continuous. From Lemma 3.4 and Corollary 3.3 we deduce the continuity w.r.t. time of all terms in (3.10) and therefore, the integrability thereof. Integrating (3.10) w.r.t. time then yields (3.9). □

REMARK 3.6. *One can show that the reverse statement of Proposition 3.5 is also true such that the energy identity is actually just another equivalent formulation of the penalized damage evolution. To do so, one combines the energy identity (3.9) with Corollary 3.3 and Young's inequality and in this way obtains (3.7), which was one of the equivalent formulations of the penalized damage evolution. However, for the upcoming analysis we only need the implication stated in Proposition 3.5 so that we do not go into more details.*

4. Limit Analysis. This section proves the viability of the penalty approach in the sense that one can pass to the limit $\beta \to \infty$ and in this way obtain a single-field damage model as stated in Section 5. In Section 7 below, we will then see that the limit system is equivalent to a classical viscous partial damage model.

In the first part of this section we focus on finding bounds independent of β in suitable spaces for the local and nonlocal damage, respectively. Note that, for the displacement, such a bound is already given in (2.2). This allows us to find weakly convergent subsequences. The limiting behaviour thereof, as $\beta \to \infty$, is studied in the second and third part of this section.

4.1. Uniform Boundedness. For the rest of this subsection, we fix $\beta > 0$ arbitrary so that Assumption 2.7.2 is fulfilled and denote the solution of (2.15) by $(\boldsymbol{u}, \varphi, d)$. We aim to derive bounds independent of β for $(\boldsymbol{u}, \varphi, d)$. The starting point for doing so is the energy identity in Proposition 3.5. For this purpose we require the following additional assumption, which is rather self-evident in many practical applications:

Assumption 4.1. *From now on we assume that at the beginning of the process the body is completely sound, i.e. $d_0 \equiv 0$, and that there is no load acting upon the body at initial time, i.e. $\ell(0) \equiv 0$.*

As a first consequence of Assumption 4.1, we obtain in view of (2.15) that

$$\boldsymbol{u}(0) = \varphi(0) = \dot{d}(0) \equiv 0. \tag{4.1}$$

Lemma 4.2 (Boundedness of the local damage). *Let Assumption 4.1 hold. Then there exists a constant $C > 0$, independent of β, such that $\|d\|_{H^1(0,T;L^2(\Omega))} \leq C$.*

Proof. The result follows mainly from the energy identity in Proposition 3.5. In order to see this, set $s := 0$ and $t = T$ in (3.9) and use (1.2), Lemma 3.4, (3.1), and (3.2), as well as Assumption 4.1, and (4.1) to obtain

$$\begin{aligned}
\delta \int_0^T \|\dot{d}(\tau)\|_2^2 \, d\tau &= \frac{1}{2} \langle \ell(T), \boldsymbol{u}(T) \rangle_V + \int_0^T \langle -\dot{\ell}(\tau), \boldsymbol{u}(\tau) \rangle_V \, d\tau \\
&\qquad - \Big(r \int_0^T \|\dot{d}(\tau)\|_1 \, d\tau + \frac{\alpha}{2} \|\nabla \varphi(T)\|_2^2 + \frac{\beta}{2} \|\varphi(T) - d(T)\|_2^2 \Big) \\
&\leq \int_0^T \|\dot{\ell}(\tau)\|_{V^*} \|\boldsymbol{u}(\tau)\|_V \, d\tau + \frac{1}{2} \|\ell(T)\|_{V^*} \|\boldsymbol{u}(T)\|_V \leq C
\end{aligned}$$

with $C > 0$ independent of β. The assertion then follows from $d_0 = 0$ and Poincaré-Friedrich's inequality. ∎

Next let us turn to the uniform boundedness of φ. We will establish the existence of a constant C independent of β such that $\|\varphi\|_{H^1(0,T;H^1(\Omega))} \leq C$. In view of (4.1) and Poincaré-Friedrich's inequality, we only need to show that there is $C > 0$ independent of β such that

$$\|\dot{\varphi}\|_{L^2(0,T;H^1(\Omega))}^2 = \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + \int_0^T \|\nabla \dot{\varphi}(\tau)\|_2^2 \, d\tau \leq C. \tag{4.2}$$

The starting point herefor is the equation characterizing the time derivative of the nonlocal damage. In view of Lemmata 2.12 and 2.15 this equation is given by

$$B\dot{\varphi}(t) + \partial_t F(t, \varphi(t)) + \partial_\varphi F(t, \varphi(t)) \dot{\varphi}(t) = \beta \dot{d}(t) \quad \text{ in } H^1(\Omega)^*. \tag{4.3}$$

Testing (4.3) with $\dot{\varphi}(t)$, integrating over $[0, T]$, and using (2.6) lead to

$$\begin{aligned}
\int_0^T \alpha \|\nabla \dot{\varphi}(\tau)\|_2^2 \, d\tau &= \beta \overbrace{\int_0^T (\dot{d}(\tau) - \dot{\varphi}(\tau), \dot{\varphi}(\tau))_2 \, d\tau}^{=: I_1} \\
&\qquad \underbrace{- \int_0^T \langle \partial_t F(t, \varphi(t)) + \partial_\varphi F(t, \varphi(t)) \dot{\varphi}(t), \dot{\varphi}(t) \rangle \, d\tau}_{=: I_2} .
\end{aligned} \tag{4.4}$$

LEMMA 4.3. *Under Assumption 4.1 it holds $I_1 \leq 0$.*

Proof. We follow the ideas of [14, Proposition 4.3]. From Theorem 2.16 we recall that $\dot{d}$ and φ are Lipschitz continuous, and therefore $\dot{d} \in W^{1,\infty}(0,T;L^2(\Omega))$. Hence, by [28, Theorem 3.1.40], the mapping $f : [0,T] \to L^2(\Omega)$ defined through

$$f(t) := \delta\dot{d}(t) + \beta(d(t) - \varphi(t)) + r \tag{4.5}$$

is almost everywhere differentiable. Let now $t \in (0,T)$ be arbitrary, but fixed and $h > 0$ sufficiently small such that $t+h \in (0,T)$. From (2.14) it follows that $\dot{d}(\tau,x) \geq 0$, $f(\tau,x) \geq 0$, and $f(\tau,x)\,\dot{d}(\tau,x) = 0$ for all $\tau \in [0,T]$ and almost all $x \in \Omega$. Thus we arrive at

$$\Big(\frac{f(t \pm h) - f(t)}{h}\,,\, \dot{d}(t)\Big)_2 \geq 0.$$

Passing to the limit $h \searrow 0$ and keeping in mind the fact that f is almost everywhere differentiable implies $(\dot{f}(t), \dot{d}(t))_2 = 0$ f.a.a. $t \in (0,T)$. Thanks to (4.5) this is equivalent to $\delta(\ddot{d}(t), \dot{d}(t))_2 + (\beta(\dot{d}(t) - \dot{\varphi}(t)), \dot{d}(t))_2 = 0$, which can be continued as

$$\frac{\delta}{2}\frac{d}{dt}\|\dot{d}(t)\|_2^2 + \beta\|\dot{d}(t) - \dot{\varphi}(t))\|_2^2 + \beta(\dot{d}(t) - \dot{\varphi}(t), \dot{\varphi}(t))_2 = 0 \tag{4.6}$$

for almost all $t \in (0,T)$. Due to Theorem 2.16, $\dot{\varphi}$ and $\dot{d}$ are both continuous with values in $L^2(\Omega)$ so that we can integrate (4.6) over $[0,T]$. This finally yields

$$\frac{\delta}{2}\|\dot{d}(T)\|_2^2 - \frac{\delta}{2}\|\dot{d}(0)\|_2^2 + \beta\int_0^T \|\dot{d}(\tau) - \dot{\varphi}(\tau))\|_2^2\,d\tau + \beta\int_0^T (\dot{d}(\tau) - \dot{\varphi}(\tau), \dot{\varphi}(\tau))_2\,d\tau = 0,$$

which on account of (4.1) gives the assertion. ☐

LEMMA 4.4. *For all $k > 0$ it holds*

$$|I_2| \leq \widehat{c}(k)\int_0^T \|\dot{\varphi}(\tau)\|_{H^1(\Omega)}^2\,d\tau + k\int_0^T \|\dot{\varphi}(\tau)\|_2^2\,d\tau + C\,k,$$

where $\widehat{c} : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotonically decreasing function, independent of β, which tends to 0 as $k \to \infty$ and $C > 0$ is a constant independent of β.

Proof. Let $t \in [0,T]$ be arbitrary, but fixed. From (2.9) we deduce

$$\langle \partial_t F(t,\varphi(t)), \dot{\varphi}(t)\rangle = -\langle \operatorname{div}\big(g'(\varphi(t))\dot{\varphi}(t)\mathbb{C}\varepsilon(\boldsymbol{u}(t))\big), \partial_t\mathcal{U}(t,\varphi(t))\rangle_V.$$

Due to $p > N$ we have $H^1(\Omega) \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)$ and thus, Hölder's inequality with $(p-2)/2p + 1/p + 1/2 = 1$ in combination with (2.2) and (2.5) yields

$$\begin{aligned}\big|\langle \partial_t F(t,\varphi(t)), \dot{\varphi}(t)\rangle_{H^1(\Omega)}\big| &\leq \|g'(\varphi(t))\|_\infty \|\dot{\varphi}(t)\|_{\frac{2p}{p-2}}\|\boldsymbol{u}(t)\|_{\boldsymbol{W}_D^{1,p}(\Omega)}\|\partial_t\mathcal{U}(t,\varphi(t))\|_V \\ &\leq C\|\dot{\varphi}(t)\|_{H^1(\Omega)},\end{aligned}$$

with $C > 0$ independent of β. On account of the generalized Young inequality, this can be continued as follows

$$\big|\langle \partial_t F(t,\varphi(t)), \dot{\varphi}(t)\rangle_{H^1(\Omega)}\big| \leq \frac{1}{4k}\|\dot{\varphi}(t)\|_{H^1(\Omega)}^2 + Ck \quad \forall\, k > 0. \tag{4.7}$$

Together with (2.10) and the definition of I_2 in (4.4), this gives the assertion with $\widehat{c}(k) = \frac{1}{4k} + \widetilde{c}(k)$ so that $\widehat{c}$ is indeed independent of β, monotonically decreasing and tends to 0 as $k \to \infty$. ∎

LEMMA 4.5. *Let Assumption 4.1 hold. Then there exist constants $C_1, C_2 > 0$, independent of β, such that*

$$\int_0^T \|\nabla\dot{\varphi}(\tau)\|_2^2 \, d\tau \leq C_1 \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + C_2.$$

Proof. Applying Lemmata 4.3 and 4.4 to the right hand side in (4.4) yields

$$(\alpha - \widehat{c}(k)) \int_0^T \|\nabla\dot{\varphi}(\tau)\|_2^2 \, d\tau \leq (\widehat{c}(k) + k) \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + C\, k$$

for all $k > 0$. Since $\widehat{c}$ tends to zero as $k \to \infty$, there is a $K > 0$ such that $\widehat{c}(K) < \alpha$ holds. Choosing $k = K$ thus yields the assertion. Note that K does not depend on β, since $\widehat{c}$ is independent of β. ∎

LEMMA 4.6. *Let Assumption 4.1 hold. Then, for $\beta > 0$ sufficiently large, there holds $\|\dot{\varphi}\|_{L^2(0,T;L^2(\Omega))} \leq C$ with a constant $C > 0$ independent of β.*

Proof. From (4.4) we deduce

$$\int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau \leq \int_0^T (\dot{d}(\tau), \dot{\varphi}(\tau))_2 \, d\tau - \frac{1}{\beta} I_2. \tag{4.8}$$

Young inequality implies for the first term on the right hand side in (4.8) that

$$\int_0^T (\dot{d}(\tau), \dot{\varphi}(\tau))_2 \, d\tau \leq \frac{1}{2} \int_0^T \|\dot{d}(\tau)\|_2^2 \, d\tau + \frac{1}{2} \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau. \tag{4.9}$$

In order to estimate the second term on the right hand side in (4.8), we apply Lemma 4.4 for some fixed $k > 0$, which thanks to Lemma 4.5 gives

$$\frac{1}{\beta} |I_2| \leq \frac{C_1}{\beta} \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + \frac{C_2}{\beta} \tag{4.10}$$

with constants C_1 and C_2 independent of β on account of Lemmata 4.4 and 4.5. Inserting (4.9) and (4.10) in (4.8) then implies

$$\int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau \leq \frac{1}{2} \int_0^T \|\dot{d}(\tau)\|_2^2 \, d\tau + \Big(\frac{1}{2} + \frac{C_1}{\beta}\Big) \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + \frac{C_2}{\beta}. \tag{4.11}$$

Now, for β sufficiently large such that $C_1/\beta < 1/2$, the assertion follows from Lemma 4.2. ∎

As a consequence of Lemmata 4.5 and 4.6 and Poincaré-Friedrich's inequality together with (4.1) we can now state the main result of this section:

COROLLARY 4.7 (Boundedness of the nonlocal damage). *Under Assumption 4.1 there exists a $\beta_0 > 0$ and a constant $C > 0$ such that $\|\varphi\|_{H^1(0,T;H^1(\Omega))} \leq C$ for all $\beta \geq \beta_0$.*

4.2. Passing to the Limit in the Elliptic System. We start our limit analysis with the elliptic system (2.15a)–(2.15b). In order to emphasize the dependency on the penalty parameter, we do not longer suppress the index β and denote the unique solution of (P) and (2.15a)–(2.15c), respectively, by $(\boldsymbol{u}_\beta, \varphi_\beta, d_\beta)$.

PROPOSITION 4.8 (Passing to the limit in (2.15a)). *Let Assumption 4.1 hold. Then, for every sequence $\beta_n \to \infty$, there exist a (not relabeled) subsequence $\{\varphi_{\beta_n}\}_{n\in\mathbb{N}}$ such that*

$$\varphi_{\beta_n} \rightharpoonup \varphi \qquad \text{in } H^1(0,T;H^1(\Omega)), \tag{4.12}$$

$$\boldsymbol{u}_{\beta_n} = \mathcal{U}(\cdot,\varphi_{\beta_n}(\cdot)) \to \mathcal{U}(\cdot,\varphi(\cdot)) =: \boldsymbol{u} \quad \text{in } C([0,T];V) \tag{4.13}$$

as $n \to \infty$.

Proof. Since $H^1(0,T;H^1(\Omega))$ is a reflexive Banach space, Corollary 4.7 implies the existence of a (not relabeled) subsequence of $\{\varphi_{\beta_n}\}_{n\in\mathbb{N}}$ such that (4.12) holds.

To prove the second assertion, we apply Lemma 2.10 with $\pi = 2$. Then $r = 2p/(p-2)$ and (2.3) implies that the mapping $\mathcal{U}_c : C([0,T];L^r(\Omega)) \ni \varphi \mapsto \mathcal{U}(\cdot,\varphi(\cdot)) \in C([0,T];V)$ is Lipschitz continuous with constant $L > 0$. Note that L is independent of β, since β does not appear in the elliptic equation (2.15a) associated with $\mathcal{U}$. Now, since $p > 2$ by Assumption 2.7.1, the embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact, which implies that $H^1(0,T;H^1(\Omega)) \hookrightarrow\hookrightarrow C([0,T];L^r(\Omega))$ is a compact as well, cf. [28, Corollary 3.1.42]. Consequently (4.12) leads to $\varphi_{\beta_n} \to \varphi$ in $C([0,T];L^r(\Omega))$ and the Lipschitz continuity of $\mathcal{U}_c$ then gives (4.13). ☐

For the rest of this section we denote by $\{\beta_n\}_{n\in\mathbb{N}}$ a fixed sequence such that $\{\varphi_{\beta_n}\}$ converges weakly in $H^1(0,T;H^1(\Omega))$ and by φ the limit of this particular sequence. Proposition 4.8 guarantees the existence of such a sequence. Notice however that (at this point) φ depends on the chosen subsequence. Nevertheless, as we will see in Proposition 5.6 below, under a (rather restrictive) regularity condition on the elliptic operator A_φ, the weak limit is unique so that the whole sequence converges weakly.

PROPOSITION 4.9 (Passing to the limit in (2.15b)). *Let Assumption 4.1 hold and let $\{\beta_n\}_{n\in\mathbb{N}}$ be the subsequence from Proposition 4.8 and φ the corresponding limit. Then there holds*

$$d_{\beta_n} \rightharpoonup \varphi \text{ in } H^1(0,T;L^2(\Omega)) \qquad \text{as } n \to \infty,$$

which implies in particular that both damage variables coincide in the limit.

Proof. First of all, Lemma 4.2 yields the existence of a subsequence $\{\beta_{n_k}\}_{k\in\mathbb{N}}$ so that

$$d_{\beta_{n_k}} \rightharpoonup d \text{ in } H^1(0,T;L^2(\Omega)) \qquad \text{as } k \to \infty. \tag{4.14}$$

Now, let $t \in [0,T]$ and $\psi \in H^1(\Omega)$ be arbitrary, but fixed. Testing (2.15b) with ψ gives the following estimate, where we use Corollary 4.7, the boundedness of g' from Assumption 2.3, and (2.2):

$$\begin{aligned}
&\int_\Omega \big(d_{\beta_n}(t) - \varphi_{\beta_n}(t)\big)\psi \, dx \\
&\leq \frac{1}{\beta_n}\big(\alpha\|\nabla\varphi_{\beta_n}(t)\|_2 + \|g'(\varphi_{\beta_n}(t))\|_\infty \|\mathbb{C}\varepsilon(\boldsymbol{u}_{\beta_n}(t)) : \varepsilon(\boldsymbol{u}_{\beta_n}(t))\|_{\frac{p}{2}}\big)\|\psi\|_{H^1(\Omega)} \\
&\leq \frac{C}{\beta_n}\|\psi\|_{H^1(\Omega)} \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{4.15}$$

Note that $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$, which yields the boundedness of $\|\nabla\varphi_{\beta_n}(t)\|_2$ uniformly in t. Of course the above estimate also holds for the subsequence $\{\beta_{n_k}\}_{k\in\mathbb{N}}$ and hence, (4.14) and the convergence of $\{\varphi_{\beta_n}\}$ by assumption imply

$$\int_\Omega (d(t) - \varphi(t))\psi \, dx \leq 0,$$

and, since t and ψ were arbitrary, this gives in turn $d(t) = \varphi(t)$ for all $t \in [0,T]$. Hence, we obtain $d_{\beta_{n_k}} \rightharpoonup \varphi$ in $H^1(0,T;L^2(\Omega))$ as $k \to \infty$. Thus the weak limit is unique and a well known argument implies the convergence of the whole sequence $\{d_{\beta_n}\}$ to φ. □

4.3. Passing to the Limit in the Energy Identity. We now turn our attention to the passage to the limit in (2.15c). However, as already indicated at the beginning of Section 3, the term $\beta(d-\varphi)$ involved in (2.15c) is not bounded in suitable spaces that allow a passage to the limit. Instead we will pass to the limit in (3.9), which will result in an energy inequality, that turns out to be equivalent to an evolutionary equation as shown in Section 5.

We begin by introducing the energy and dissipation functionals that will arise after passing to the limit. The energy without penalty term reads as follows

Definition 4.10 (Energy functionals without penalty). *We define the energy functional without penalty term by*

$$\begin{aligned}
&\widetilde{\mathcal{E}} : [0,T] \times V \times H^1(\Omega) \to \mathbb{R},\\
&\widetilde{\mathcal{E}}(t,\boldsymbol{u},\varphi) := \frac{1}{2}\int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}) \, dx - \langle \ell(t), \boldsymbol{u}\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2.
\end{aligned}$$

The reduced energy functional without penalty is given by

$$\widetilde{\mathcal{I}} : [0,T] \times H^1(\Omega) \to \mathbb{R}, \quad \widetilde{\mathcal{I}}(t,\varphi) := \widetilde{\mathcal{E}}(t,\mathcal{U}(t,\varphi),\varphi).$$

Remark 4.11. *Since the expressions in $\widetilde{\mathcal{E}}$ involving the displacement are not affected by the penalty term, it follows that, for a given pair $(t,\varphi) \in [0,T] \times H^1(\Omega)$, $\boldsymbol{u}$ solves*

$$\min_{\boldsymbol{u}\in V} \widetilde{\mathcal{E}}(t,\boldsymbol{u},\varphi),$$

iff $\boldsymbol{u} = \mathcal{U}(t,\varphi)$ with $\mathcal{U}$ as defined in Definition 2.9. As a consequence we obtain $\widetilde{\mathcal{I}}(t,\varphi) = \min_{\boldsymbol{u}\in V} \widetilde{\mathcal{E}}(t,\boldsymbol{u},\varphi)$.

Completely analogous to (3.1), the definitions of $\widetilde{\mathcal{E}}$ and $\mathcal{U}$ allow to rewrite the reduced energy functional without penalty as

$$\widetilde{\mathcal{I}}(t,\varphi) = -\frac{1}{2}\langle \ell(t), \mathcal{U}(t,\varphi)\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2. \tag{4.16}$$

Lemma 4.12 (Fréchet differentiability of $\widetilde{\mathcal{I}}$). *It holds $\widetilde{\mathcal{I}} \in C^1([0,T] \times H^1(\Omega))$ and its partial derivatives read*

$$\partial_t\widetilde{\mathcal{I}}(t,\varphi) = -\langle \dot{\ell}(t), \mathcal{U}(t,\varphi)\rangle_V, \quad \partial_\varphi\widetilde{\mathcal{I}}(t,\varphi) = -\alpha\triangle\varphi + F(t,\varphi), \tag{4.17}$$

where $\triangle : H^1(\Omega) \to H^1(\Omega)^*$ denotes the distributional Laplace operator.

Proof. The proof is completely along the lines of the proof of Lemma 3.2 so that we shorten the depiction. By applying the product rule to (4.16), one obtains that $\widetilde{\mathcal{I}}$ is indeed continuously Fréchet-differentiable with

$$\widetilde{\mathcal{I}}'(t,\varphi)(\delta t,\delta\varphi) = -\frac{1}{2}\langle\dot{\ell}(t)\delta t,\mathcal{U}(t,\varphi)\rangle_V - \frac{1}{2}\langle\ell(t),\mathcal{U}'(t,\varphi)(\delta t,\delta\varphi)\rangle_V + \alpha(\nabla\varphi,\nabla\delta\varphi)_2.$$

Similarly to (3.4) and (3.5), the first to addends can be reformulated by using (2.4a), the symmetry of $\mathbb{C}$, (2.4b), and the definition of F to obtain

$$\frac{1}{2}\langle\dot{\ell}(t)\delta t,\mathcal{U}(t,\varphi)\rangle_V + \frac{1}{2}\langle\ell(t),\mathcal{U}'(t,\varphi)(\delta t,\delta\varphi)\rangle_V = \langle\dot{\ell}(t)\delta t,\mathcal{U}(t,\varphi)\rangle_V - \langle F(t,\varphi),\delta\varphi\rangle_{H^1(\Omega)},$$

which completes the proof. □

Next we introduce the viscous dissipation functional corresponding to the situation without penalty:

Definition 4.13 (Viscous dissipation functional without penalty). *We define the functional $\widetilde{\mathcal{R}}_\delta$ by*

$$\widetilde{\mathcal{R}}_\delta : H^1(\Omega) \to [0,\infty], \quad \widetilde{\mathcal{R}}_\delta(\eta) := \begin{cases} r\int_\Omega \eta\,dx + \frac{\delta}{2}\|\eta\|_2^2 & \text{if } \eta \geq 0 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\widetilde{\mathcal{R}}_\delta$ coincides with $\mathcal{R}_\delta$ from (1.2) apart from its domain which is now $H^1(\Omega)$ instead of $L^2(\Omega)$.

In order to pass to the limit in (3.9) we consider a sequence $\beta_n \to \infty$ such that

$$\begin{aligned} \varphi_{\beta_n} &\rightharpoonup \varphi && \text{in } H^1(0,T;H^1(\Omega)), && (4.18)\\ d_{\beta_n} &\rightharpoonup \varphi && \text{in } H^1(0,T;L^2(\Omega)), && (4.19)\\ \boldsymbol{u}_{\beta_n} &\to \mathcal{U}(\cdot,\varphi(\cdot)) && \text{in } C([0,T];V). && (4.20) \end{aligned}$$

Recall that such a sequence exists according to Propositions 4.8 and 4.9.

Lemma 4.14. *Under Assumption 4.1 it holds for all $t \in [0,T]$ that*

$$\int_0^t \widetilde{\mathcal{R}}_\delta(\dot{\varphi}(\tau))\,d\tau \leq \liminf_{n\to\infty} \int_0^t \mathcal{R}_\delta(\dot{d}_{\beta_n}(\tau))\,d\tau.$$

Proof. Let $t \in (0,T)$ be arbitrary, but fixed. From (4.19) it follows that

$$\dot{d}_{\beta_n} \rightharpoonup \dot{\varphi} \quad \text{in } L^2(0,t;L^2(\Omega)) \tag{4.21}$$

so that $\dot{d}_{\beta_n} \geq 0$ a.e. in $\Omega \times (0,t)$, see (2.15c), implies $\dot{\varphi} \geq 0$ a.e. in $\Omega \times (0,t)$ by the weak closedness of the set of non-negative functions in $L^2(0,t;L^2(\Omega))$. Thus the definition of $\widetilde{\mathcal{R}}_\delta$ implies

$$\int_0^t \widetilde{\mathcal{R}}_\delta(\dot{\varphi}(\tau))\,d\tau = r\,\|\dot{\varphi}\|_{L^1(0,t;L^1(\Omega))} + \frac{\delta}{2}\,\|\dot{\varphi}\|^2_{L^2(0,t;L^2(\Omega))}$$

and the same obviously holds for $\mathcal{R}_\delta(\dot{d}_{\beta_n}(\tau))$, cf. (1.2). The result then follows from the weak lower semicontinuity of (squared) norms. ◻

Lemma 4.15. *Let Assumption 4.1 hold. Then for all $t \in [0,T]$ we have*

$$\partial_d \mathcal{I}(t, d_{\beta_n}(t)) \rightharpoonup \partial_\varphi \widetilde{\mathcal{I}}(t, \varphi(t)) \text{ in } H^1(\Omega)^* \quad \text{as } n \to \infty.$$

Proof. Let $t \in [0,T]$ be arbitrary, but fixed and set again $r = 2p/(p-2)$. As explained at the end of the proof of Proposition 4.8, Assumption 2.7.1 implies the compact embedding $H^1(0,T;H^1(\Omega)) \hookrightarrow\hookrightarrow C([0,T];L^r(\Omega))$ so that (4.18) results in

$$\varphi_{\beta_n}(t) \to \varphi(t) \text{ in } L^r(\Omega) \quad \text{for } n \to \infty. \tag{4.22}$$

Furthermore, since $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$, (4.18) implies

$$\nabla\varphi_{\beta_n}(t) \rightharpoonup \nabla\varphi(t) \text{ in } L^2(\Omega) \quad \text{for } n \to \infty. \tag{4.23}$$

From (3.2) and (2.15b) we moreover deduce

$$\partial_d \mathcal{I}(t, d_{\beta_n}(t)) = \beta_n(d_{\beta_n}(t) - \varphi_{\beta_n}(t)) = -\alpha \triangle \varphi_{\beta_n}(t) + F(t, \varphi_{\beta_n}(t)).$$

Together with (4.17) and (2.8) this yields for every $v \in H^1(\Omega)$ that

$$\begin{aligned}
&|\langle \partial_d \mathcal{I}(t, d_{\beta_n}(t)) - \partial_\varphi \widetilde{\mathcal{I}}(t, \varphi(t)), v\rangle_{H^1(\Omega)}| \\
&\quad \leq \alpha |(\nabla\varphi_{\beta_n}(t) - \nabla\varphi(t), \nabla v)_2| + |\langle F(t, \varphi_{\beta_n}(t)) - F(t, \varphi(t)), v\rangle| \\
&\quad \leq \alpha |(\nabla\varphi_{\beta_n}(t) - \nabla\varphi(t), \nabla v)_2| + C\, \|\varphi_{\beta_n}(t) - \varphi(t)\|_r \|v\|_r.
\end{aligned}$$

The result then follows from (4.22), (4.23), and $H^1(\Omega) \hookrightarrow L^r(\Omega)$ by Assumption 2.7.1. ◻

Lemma 4.16. *Under Assumption 4.1 it holds for all $t \in [0,T]$*

$$\int_0^t \widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi \widetilde{\mathcal{I}}(\tau, \varphi(\tau))\big)\, d\tau \leq \liminf_{n\to\infty} \int_0^t \mathcal{R}_\delta^*(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau)))\, d\tau.$$

Proof. Again, let $t \in [0,T]$ by arbitrary, but fixed. By definition of the Fenchel-conjugate, it holds for any $\xi \in L^2(\Omega)$ that

$$\widetilde{\mathcal{R}}_\delta^*(\xi) = \sup_{v \in H^1(\Omega)} \big((\xi, v)_2 - \widetilde{\mathcal{R}}_\delta(v)\big) \leq \sup_{v \in L^2(\Omega)} \big((\xi, v)_2 - \mathcal{R}_\delta(v)\big) = \mathcal{R}_\delta^*(\xi). \tag{4.24}$$

Notice that we used in the above estimate that $\mathcal{R}_\delta$ and $\widetilde{\mathcal{R}}_\delta$ are defined with different domains, see (1.2) and Definition 4.13. Further, $\widetilde{\mathcal{R}}_\delta^* : H^1(\Omega)^* \to (-\infty, \infty]$ is convex and lower semicontinuous and thus weakly lower semicontinuous, which thanks to Lemma 4.15 leads to

$$\widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi \widetilde{\mathcal{I}}(\tau, \varphi(\tau))\big) \leq \liminf_{n\to\infty} \widetilde{\mathcal{R}}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau))\big) \quad \forall\, \tau \in [0,t].$$

By setting $\xi := -\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau)) \in L^2(\Omega)$, see (3.2), in (4.24), the above estimate can be continued as

$$\widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi \widetilde{\mathcal{I}}(\tau, \varphi(\tau))\big) \leq \liminf_{n\to\infty} \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau))\big) = \liminf_{n\to\infty} \frac{\delta}{2}\, \|\dot{d}_{\beta_n}(\tau)\|_2^2 \tag{4.25}$$

for all $\tau \in [0,t]$, where the last equation follows from Lemma 3.4. Applying Fatou's lemma to the right hand side gives

$$\int_0^t \liminf_{n\to\infty} \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau))\big)\, d\tau \leq \liminf_{n\to\infty} \int_0^t \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau))\big)\, d\tau. \tag{4.26}$$

Furthermore, arguing analogously to the derivation of (4.23), one sees that (4.19) implies $d_{\beta_n}(\tau) \rightharpoonup \varphi(\tau)$ in $L^2(\Omega)$ for every $\tau \in [0,T]$. Thus (4.25) shows that $\widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi \widetilde{\mathcal{I}}(\tau, \varphi(\tau))\big)$ is finite for every τ. In addition, due to (4.17) and $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$, the map $[0,t] \ni \tau \mapsto \partial_\varphi \widetilde{\mathcal{I}}(\tau, \varphi(\tau)) \in H^1(\Omega)^*$ is continuous. Since $\widetilde{\mathcal{R}}_\delta^*$ is lower semicontinuous, it thus follows that the mapping

$$[0,t] \ni \tau \mapsto \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(\tau, \varphi(\tau))) \in \mathbb{R}$$

is lower semicontinuous as well and therefore, measurable. Now we can integrate (4.25) over $(0,t)$, which combined with (4.26) finally gives the assertion. ☐

Proposition 4.17 (Energy inequality without penalty). *Let Assumption 4.1 hold. Then the limit function $\varphi \in H^1(0,T;H^1(\Omega))$ fulfills for all $t \in [0,T]$ the estimate*

$$\begin{aligned} \int_0^t \widetilde{\mathcal{R}}_\delta(\dot\varphi(\tau))\, d\tau + \int_0^t \widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi \widetilde{\mathcal{I}}(\tau, \varphi(\tau))\big)\, d\tau + \widetilde{\mathcal{I}}(t, \varphi(t)) \\ \leq \widetilde{\mathcal{I}}(0, \varphi(0)) + \int_0^t \partial_t \widetilde{\mathcal{I}}(\tau, \varphi(\tau))\, d\tau. \end{aligned} \tag{4.27}$$

Proof. Let $t \in [0,T]$ be arbitrary, but fixed. Setting $s := 0$ in (3.9) yields

$$\begin{aligned} \int_0^t \mathcal{R}_\delta(\dot d_{\beta_n}(\tau))\, d\tau + \int_0^t \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau))\big)\, d\tau + \mathcal{I}(t, d_{\beta_n}(t)) \\ = \mathcal{I}(0, d_{\beta_n}(0)) + \int_0^t \partial_t \mathcal{I}\big(\tau, d_{\beta_n}(\tau)\big)\, d\tau \quad \forall\, n \in \mathbb{N}. \end{aligned} \tag{4.28}$$

In view of Lemmas 4.14 and 4.16 we only need to discuss the last three terms in the above equation. To this end, we combine (3.1) and (4.16) with (4.20), (4.23), and the weakly lower semicontinuity of $\|\cdot\|_2^2$, which gives

$$\begin{aligned} \widetilde{\mathcal{I}}(t, \varphi(t)) &= -\frac{1}{2}\langle \ell(t), \mathcal{U}(t, \varphi(t))\rangle_V + \frac{\alpha}{2}\|\nabla\varphi(t)\|_2^2 \\ &\leq \liminf_{n\to\infty} \Big(-\frac{1}{2}\langle \ell(t), \boldsymbol{u}_{\beta_n}(t)\rangle_V + \frac{\alpha}{2}\|\nabla\varphi_{\beta_n}(t)\|_2^2 + \underbrace{\frac{\beta_n}{2}\|\varphi_{\beta_n}(t) - d_{\beta_n}(t)\|_2^2}_{\geq 0}\Big) \\ &= \liminf_{n\to\infty} \mathcal{I}(t, d_{\beta_n}(t)), \end{aligned}$$

i.e., the desired convergence of the last term on the left hand side of (4.28). It remains to discuss the right hand side in (4.28). Thanks to Assumption 4.1 and (4.1) the initial value just vanishes, i.e.,

$$\mathcal{I}(0, d_{\beta_n}(0)) = 0 \quad \forall\, n \in \mathbb{N}, \tag{4.29}$$

and, in light of (4.16), $\ell(0) = 0$ by Assumption 4.1, and the pointwise convergence in (4.23), which gives $\nabla\varphi(0) = 0$, we obtain the same for the limit, i.e., $\widetilde{\mathcal{I}}(0, \varphi(0)) = 0$.

Therefore, the formulas for the partial derivatives of $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ in (3.2) and (4.17) together with the regularity of ℓ and the convergence of the displacement in (4.20) finally implies

$$\begin{aligned}\mathcal{I}(0, d_{\beta_n}(0)) &+ \int_0^t \partial_t \mathcal{I}(\tau, d_{\beta_n}(\tau)) \, d\tau \\ &= \int_0^t \langle -\dot{\ell}(\tau), \boldsymbol{u}_{\beta_n}(\tau) \rangle \, d\tau \to \int_0^t \langle -\dot{\ell}(\tau), \mathcal{U}(\tau, \varphi(\tau)) \rangle \, d\tau \\ &= \widetilde{\mathcal{I}}(0, \varphi(0)) + \int_0^t \partial_t \widetilde{\mathcal{I}}(\tau, \varphi(\tau)) \, d\tau\end{aligned}$$

which completes the proof. ◻

In the next sections we use the energy inequality in (4.27) to show that the limit in (4.18)–(4.20) satisfies a system of equations which is equivalent to a classical viscous partial damage model containing only one single damage variable. As secondary result we will also see that the inequality (4.27) is in fact equivalent to an energy identity, see Remark 5.2 below.

5. A Single-Field Gradient Damage Model.

In this section we show that every solution of the energy inequality (4.27) satisfies an evolutionary equation and vice versa. The proof mainly follows the arguments of [15, Proposition 3.2].

Proposition 5.1 (Viscous differential inclusion without penalty). *Every function $\varphi \in H^1(0,T; H^1(\Omega))$ which fulfills for all $t \in [0,T]$ the energy inequality (4.27) also satisfies the following evolutionary equation*

$$0 \in \partial \widetilde{\mathcal{R}}_\delta(\dot{\varphi}(t)) + \partial_\varphi \widetilde{\mathcal{I}}(t, \varphi(t)) \quad \text{f.a.a. } t \in (0,T). \tag{5.1}$$

The reverse assertion is true as well.

Proof. We start the proof with two auxiliary results needed for both implications stated in the Proposition. To this end let $\varphi \in H^1(0,T; H^1(\Omega))$ first be arbitrary, but fixed. Since $\widetilde{\mathcal{R}}_\delta$ is convex and proper, a classical result from convex analysis result leads to

$$\widetilde{\mathcal{R}}_\delta(\dot{\varphi}(t)) + \widetilde{\mathcal{R}}^*_\delta(-\partial_\varphi \widetilde{\mathcal{I}}(t, \varphi(t))) = -\langle \partial_\varphi \widetilde{\mathcal{I}}(t, \varphi(t)), \dot{\varphi}(t) \rangle_{H^1(\Omega)} \tag{5.2}$$

$$\Longleftrightarrow$$

$$-\partial_\varphi \widetilde{\mathcal{I}}(t, \varphi(t)) \in \partial \widetilde{\mathcal{R}}_\delta(\dot{\varphi}(t)) \tag{5.3}$$

f.a.a. $t \in (0,T)$. Further note that (4.17), combined with (2.7), (2.2), and the boundedness assumption on g', implies

$$\|\partial_\varphi \widetilde{\mathcal{I}}(t, v)\|_{H^1(\Omega)^*} \le C \, \|v\|_{H^1(\Omega)} + c \quad \forall \, (t, v) \in [0,T] \times H^1(\Omega), \tag{5.4}$$

where $C, c > 0$ are independent of (t, v). By using the density of $C^1([0,T]; H^1(\Omega))$ in $H^1(0,T; H^1(\Omega))$ and the continuous Fréchet differentiability of $\widetilde{\mathcal{I}}$ by Lemma 4.12, as well as (5.4), one shows that the function $[0,T] \ni t \mapsto \widetilde{\mathcal{I}}(t, \varphi(t)) \in \mathbb{R}$ belongs to $H^1(0,T)$ with weak derivative

$$\frac{d}{dt} \widetilde{\mathcal{I}}(., \varphi(.)) = \partial_t \widetilde{\mathcal{I}}(., \varphi(.)) + \langle \partial_\varphi \widetilde{\mathcal{I}}(., \varphi(.)), \dot{\varphi}(.) \rangle_{H^1(\Omega)} \in L^2(0,T). \tag{5.5}$$

Note that $\varphi \in H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$ and (5.4) imply $\partial_\varphi \widetilde{\mathcal{I}}(.,\varphi(.)) \in L^\infty(0,T;H^1(\Omega)^*)$, which in turn renders the L^2-regularity of $\frac{d}{dt}\widetilde{\mathcal{I}}(.,\varphi(.))$.

Let us now assume that φ fulfills (4.27) for all $t \in [0,T]$. Due to $\widetilde{\mathcal{I}}(\cdot,\varphi(\cdot)) \in H^1(0,T)$ the energy inequality implies by setting $t = T$ that

$$\begin{aligned}&\int_0^T \widetilde{\mathcal{R}}_\delta(\dot\varphi(\tau))\,d\tau + \int_0^T \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau)))\,d\tau\\ &\quad\le -\int_0^T \Big(\frac{d}{dt}\widetilde{\mathcal{I}}(\tau,\varphi(\tau)) - \partial_t \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\Big)d\tau = -\int_0^T \langle \partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau)),\dot\varphi(\tau)\rangle_{H^1(\Omega)}\,d\tau,\end{aligned}$$

where we used (5.5) for the last equality. Combining this with Young's inequality, i.e.,

$$\widetilde{\mathcal{R}}_\delta(\dot\varphi(t)) + \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t))) \ge -\langle \partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)),\dot\varphi(t)\rangle_{H^1(\Omega)} \quad \text{f.a.a. } t \in (0,T),$$

leads to (5.2) and consequently (5.3), which shows the first implication.

The reverse assertion can be concluded by following the lines of the proof of Proposition 3.5. To see this, assume that $\varphi \in H^1(0,T;H^1(\Omega))$ satisfies (5.1). From the equivalence (5.3) $\Longleftrightarrow$ (5.2) and (5.5) we then obtain

$$\widetilde{\mathcal{R}}_\delta(\dot\varphi(t)) + \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t))) = -\frac{d}{dt}\widetilde{\mathcal{I}}(t,\varphi(t)) + \partial_t \widetilde{\mathcal{I}}(t,\varphi(t)) \quad \text{f.a.a. } t \in (0,T). \tag{5.6}$$

Note that any φ, which fulfills (5.1), automatically satisfies $\dot\varphi \ge 0$ in view of Definition 4.13. The latter one then also ensures the L^1-integrability of $\widetilde{\mathcal{R}}_\delta(\dot\varphi(\cdot))$. For the right hand side in (5.6) we have due to Lemma 4.12 and (5.5) that $\partial_t \widetilde{\mathcal{I}}(\cdot,\varphi(\cdot)) \in C[0,T]$ and $\frac{d}{dt}\widetilde{\mathcal{I}}(\cdot,\varphi(\cdot)) \in L^2(0,T)$, respectively. Thus, we are allowed to integrate (5.6) in time, which implies

$$\begin{aligned}&\int_0^t \widetilde{\mathcal{R}}_\delta(\dot\varphi(\tau))\,d\tau + \int_0^t \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau)))\,d\tau\\ &\qquad = \widetilde{\mathcal{I}}(0,\varphi(0)) - \widetilde{\mathcal{I}}(t,\varphi(t)) + \int_0^t \partial_t \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\,d\tau\end{aligned} \tag{5.7}$$

for all $t \in [0,T]$. This completes the proof. □

Remark 5.2. *An inspection of the proof of Proposition 5.1 shows that, in order to prove* (5.1)*, it suffices that* (4.27) *holds only at* $t = T$*. Moreover, the proof shows that* (4.27) *implies* (5.1) *which in turn gives* (5.7)*. In this way we have shown that* (4.27) *is indeed an* energy identity*. Furthermore, integrating* (5.6) *over an arbitrary interval* $[s,t] \subset [0,T]$ *(instead of* $[0,t]$*) leads to an energy identity, completely analogous to* (3.9) *so that the passage to the limit* $\beta \to \infty$ *indeed preserves the structure of the energy identity. We also refer to [15, Proposition 3.2].*

We summarize our results so far in the following

Theorem 5.3 (Single-field damage model). *Let Assumption 4.1 hold and* $\{\beta_n\}_{n\in\mathbb{N}}$ *be a sequence with* $\beta_n \to \infty$ *as* $n \to \infty$*. Then there is a subsequence (denoted by the same symbol) and a function* $\varphi \in H^1(0,T;H^1(\Omega))$ *such that*

$$\begin{gathered}\varphi_{\beta_n} \rightharpoonup \varphi \text{ in } H^1(0,T;H^1(\Omega)), \quad d_{\beta_n} \rightharpoonup \varphi \text{ in } H^1(0,T;L^2(\Omega)),\\ \boldsymbol{u}_{\beta_n} \to \boldsymbol{u} := \mathcal{U}(\cdot,\varphi(\cdot)) \text{ in } C([0,T];V).\end{gathered} \tag{5.8}$$

Moreover, every limit $(\varphi, \boldsymbol{u}) \in H^1(0,T;H^1(\Omega)) \times C([0,T];V)$ *of such a sequence satisfies f.a.a.* $t \in (0,T)$ *the following PDE system:*

$$-\operatorname{div} g(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) = \ell(t) \quad \textit{in } V^*, \tag{5.9a}$$

$$\delta\,\dot\varphi(t) - \alpha\triangle\varphi(t) + \frac{1}{2}\,g'(\varphi(t))\mathbb{C}\,\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) + \partial\widetilde{\mathcal{R}}_1(\dot\varphi(t)) \ni 0, \quad \varphi(0) = 0, \tag{5.9b}$$

with the (non-viscous) dissipation potential $\widetilde{\mathcal{R}}_1$ *defined by*

$$\widetilde{\mathcal{R}}_1 : H^1(\Omega) \to [0,\infty], \quad \widetilde{\mathcal{R}}_1(\eta) := \begin{cases} r\int_\Omega \eta\,dx, & \textit{if } \eta \geq 0 \textit{ a.e. in } \Omega, \\ \infty, & \textit{otherwise.} \end{cases} \tag{5.10}$$

Proof. The existence of the subsequence has already been established in Propositions 4.8 and 4.9. Furthermore, (5.9a) is just equivalent to $\boldsymbol{u} := \mathcal{U}(\cdot,\varphi(\cdot))$. It remains to verify (5.9b), which follows from (5.1). To see this, just apply (4.17) and the definition of F to the left hand side of (5.1) and use the sum rule for convex subdifferentials for the right hand side. □

The above theorem shows that (5.9) admits at least one solution. Of course, it would be desirable to have the uniqueness of the solution, too, in particular, since this guarantees the uniqueness of the limit in (5.8) and thus the (weak) convergence of the whole sequence. Unfortunately, for this purpose, we have to require the following rather restrictive assumption. We underline that this assumption is only needed to show the uniqueness, while the the rest of the analysis remains unaffected, if it is not fulfilled.

Assumption 5.4. *To ensure uniqueness of the solution of* (5.9)*, we require that there exists some* $p > 4$ *in the two-dimensional case and* $p \geq 6$ *in the three-dimensional case such that the operator* $A_\varphi : \boldsymbol{W}^{1,p}_D(\Omega) \to \boldsymbol{W}^{-1,p}_D(\Omega)$ *is continuously invertible for every* $\varphi \in H^1(\Omega)$ *and the norm of its inverse is bounded uniformly w.r.t.* φ.

Remark 5.5. *Assumption 5.4 is fulfilled, provided that no mixed boundary conditions are present, the domain is smooth enough, and the difference between the boundedness and ellipticity constants of the stress strain relation is sufficiently small, cf. [19, Remark 3.20] and [9, 11]. Adapted to our situation this means that the values* $\epsilon\,\gamma_{\mathbb{C}}$ *and* $\|\mathbb{C}\|_\infty$ *have to be sufficiently close to each other, which is clearly rather restrictive (beside the smoothness assumption on the domain), cf. also Remark 2.8. These assumptions on the data can be weakened, if one uses* $H^s(\Omega)$ *with* $s > N/2$ *instead* $H^1(\Omega)$ *as function space for the nonlocal damage in the penalized model* (P)*. We refer to [15, Sections 2.4 and 3.2] for details. Since the bilinear form associated with* $H^s(\Omega)$ *is harder to realize in numerical practice, we do not follow this approach.*

Proposition 5.6 (Uniqueness under additional assumptions). *Under Assumptions 5.4, system* (5.9) *admits a unique solution* $(\varphi, \boldsymbol{u}) \in H^1(0,T;H^1(\Omega)) \times C([0,T];V)$.

Proof. Let $(\varphi_i, \boldsymbol{u}_i) \in H^1(0,T;H^1(\Omega)) \times C([0,T];V)$, $i = 1,2$ be two solutions of (5.9). First note that, from the definition of F in (2.7) and $\boldsymbol{u}_i(\cdot) = \mathcal{U}(\cdot,\varphi(\cdot))$ imply that φ_i satisfies

$$\delta\dot\varphi_i(t) - \alpha\triangle\varphi_i(t) + F(t,\varphi_i(t)) \in -\partial\widetilde{\mathcal{R}}_1(\dot\varphi_i(t)), \quad i = 1,2, \tag{5.11}$$

f.a.a. $t \in (0,T)$. Therefore, $\partial\widetilde{\mathcal{R}}_1(\dot\varphi_i(t)) \neq \emptyset$, which gives $\dot\varphi_1, \dot\varphi_2 \geq 0$ f.a.a. $t \in (0,T)$. By testing (5.11) for $i = 1$ with $\dot\varphi_2 - \dot\varphi_1$ and vice versa and adding the arising inequalities, we arrive at

$$\begin{aligned}\delta\|\dot\varphi_1(t)-\dot\varphi_2(t)\|_2^2 &+ \alpha(\nabla\varphi_1(t)-\nabla\varphi_2(t),\nabla\dot\varphi_1(t)-\nabla\dot\varphi_2(t))_2\\ &\le \langle F(t,\varphi_2(t))-F(t,\varphi_1(t)),\dot\varphi_1(t)-\dot\varphi_2(t)\rangle_{H^1(\Omega)} \quad \text{f.a.a. } t\in(0,T).\end{aligned}$$

Then, adding $\alpha(\varphi_1(t)-\varphi_2(t),\dot\varphi_1(t)-\dot\varphi_2(t))_2$ on both sides of this estimate and applying (2.8) with $r=2p/(p-4)$ (such that $s=2$) lead to

$$\begin{aligned}\delta\|\dot\varphi_1(t)-\dot\varphi_2(t)\|_2^2 &+ \alpha(\varphi_1(t)-\varphi_2(t),\dot\varphi_1(t)-\dot\varphi_2(t))_{H^1(\Omega)}\\ &\le C\|\varphi_1(t)-\varphi_2(t)\|_{H^1(\Omega)}\|\dot\varphi_1(t)-\dot\varphi_2(t)\|_2\\ &\le \frac{C}{4\varepsilon}\|\varphi_1(t)-\varphi_2(t)\|_{H^1(\Omega)}^2 + C\varepsilon\,\|\dot\varphi_1(t)-\dot\varphi_2(t)\|_2^2 \quad \forall\,\varepsilon>0,\end{aligned}$$

where the last estimate follows from the generalized Young inequality. Moreover, we used $H^1(\Omega)\hookrightarrow L^{2p/(p-4)}(\Omega)$ by Assumption 5.4. By choosing $\varepsilon := \delta/(2C)$ we conclude

$$\alpha(\varphi_1(t)-\varphi_2(t),\dot\varphi_1(t)-\dot\varphi_2(t))_{H^1(\Omega)} \le \frac{C^2}{2\delta}\|\varphi_1(t)-\varphi_2(t)\|_{H^1(\Omega)}^2 \tag{5.12}$$

f.a.a. $t\in(0,T)$. On account of [28, Lemma 3.1.43] we have for all $t\in[0,T]$

$$\begin{aligned}\int_0^t (\varphi_1(\tau)-\varphi_2(\tau),\dot\varphi_1(\tau)-\dot\varphi_2(\tau))_{H^1(\Omega)}\,d\tau&\\ = \frac12\|\varphi_1(t)-\varphi_2(t)\|_{H^1(\Omega)}^2 &- \frac12\|\varphi_1(0)-\varphi_2(0)\|_{H^1(\Omega)}^2\end{aligned}$$

and, due to $\varphi_1(0)=\varphi_2(0)$, we obtain after integrating (5.12) that

$$\frac{\alpha}{2}\|\varphi_1(t)-\varphi_2(t)\|_{H^1(\Omega)}^2 \le \frac{C^2}{2\delta}\int_0^t \|\varphi_1(\tau)-\varphi_2(\tau)\|_{H^1(\Omega)}^2\,d\tau \quad \forall\,t\in[0,T],$$

which by means of Gronwall's lemma leads to

$$\|\varphi_1(t)-\varphi_2(t)\|_{H^1(\Omega)}^2 \le 0 \quad \forall\,t\in[0,T] \tag{5.13}$$

and thus completes the proof. □

As an immediate consequence of the uniqueness result we obtain the following

Corollary 5.7. *If Assumption 5.4 is fulfilled, then the convergence in (5.8) is not only valid for a subsequence, but for the whole sequence* $\{(d_{\beta_n},\varphi_{\beta_n},\boldsymbol{u}_{\beta_n})\}$.

6. Uniform Pointwise Bounds for the Damage Variable. This section is devoted to derive L^∞-bounds for the damage variables d_β and φ_β that are *independent of the penalty parameter* β. In view of (4.18) and (4.19), respectively, this boundedness result carries over to the limit damage variable φ. This bound is crucial for the transformation of our damage model in (5.9) into another one that can be seen as a classical viscous damage model, which will be performed in the upcoming Section 7. We emphasize that the Lipschitz continuity result in [19, Theorem 4.6] together with Sobolev embeddings already implies that the nonlocal damage φ_β is pointwisely bounded in space and time, at least in two-dimensions, but these bounds may depend on β. To show the desired boundedness independent of β, we mostly follow the ideas of [15], where the authors derive a similar result in two dimensions via time discretization, see [15, Proposition 4.5].

We start with the operator differential equation for the local damage variable in (2.14), which serves as the basis for the time discretization:

$$\dot{d}(t) = \frac{1}{\delta} \max\big\{ -\beta\big(d(t) - \Phi(t, d(t))\big) - r, 0\big\} \quad \forall\, t \in [0,T], \quad d(0) = d_0, \tag{6.1}$$

where we again suppressed the dependency of d on β and simply write d instead of d_β. To introduce a time-discrete incremental problem, let a number of time-steps $n \in \mathbb{N}^+$ be given, set $\tau := T/n$, and denote by $\{t_k^\tau = k\tau\}_{k=0,\dots,n}$ the corresponding partition of the time interval $[0,T]$. Further we define, beginning with $d_0^\tau := d_0$, the approximation of the local damage at time point t_{k+1}^τ, $k \in \{0,...,n-1\}$, as the unique solution of the fixed-point equation

$$d_{k+1}^\tau = d_k^\tau + \frac{\tau}{\delta} \max\big\{ -\beta(d_{k+1}^\tau - \Phi(t_{k+1}^\tau, d_{k+1}^\tau)) - r, 0\big\}. \tag{P$_k^\beta$}$$

Note that the unique solvability of (P$_k^\beta$) follows from Banach fixed-point theorem. To see this, we observe that

$$\begin{aligned} \frac{\tau}{\delta}\big\| \max\big\{ -\beta(z_1 - \Phi(t_{k+1}^\tau, z_1)) - r, 0\big\} - \max\big\{ -\beta(z_2 - \Phi(t_{k+1}^\tau, z_2)) - r, 0\big\}\big\|_2 \\ \leq \frac{\tau}{\delta}\, \beta(1+K)\|z_1 - z_2\|_2 \quad \forall\, z_1, z_2 \in L^2(\Omega), \end{aligned}$$

by the Lipschitz continuity of max and Φ, cf. Lemma 2.14. Hence, for every $k \in \{0,...,n-1\}$, the fixed-point mapping associated with (P$_k^\beta$) is contractive, provided that $\tau > 0$ is sufficiently small (depending on β), which is no real restriction, as we aim to pass to the limit $\tau \searrow 0$ anyway. This is performed in the next subsection.

6.1. Passage to the Limit in the Time-Discrete Problem. In the sequel let $\beta > 0$ be fixed, but arbitrary so that Assumption 2.7 is fulfilled. Moreover, we always tacitly assume that $\tau > 0$ is chosen sufficiently small so that (P$_k^\beta$) is uniquely solvable. Similarly to [15, Section 4], we introduce the notations

$$\bar{t}_\tau(t) := t_{k+1}^\tau \ \text{ for } t \in (t_k^\tau, t_{k+1}^\tau], \quad \underline{t}_\tau(t) := t_k^\tau \ \text{ for } t \in [t_k^\tau, t_{k+1}^\tau), \quad k \in \{0,...,n-1\},$$

$\bar{t}_\tau(0) := 0$, and $\underline{t}_\tau(T) := T$. Moreover, we define the piecewise constant interpolating functions $\bar{d}_\tau, \underline{d}_\tau : [0,T] \to L^2(\Omega)$ by

$$\bar{d}_\tau(t) := d_{k+1}^\tau \ \text{ for } t \in (t_k^\tau, t_{k+1}^\tau], \quad \underline{d}_\tau(t) := d_k^\tau \ \text{ for } t \in [t_k^\tau, t_{k+1}^\tau), \quad k \in \{0,...,n-1\},$$

$\bar{d}_\tau(0) := d_0$, and $\underline{d}_\tau(T) := d_n$, as well as the piecewise linear interpolation $d_\tau : [0,T] \to L^2(\Omega)$ by

$$d_\tau(t) := d_k^\tau + \frac{t - t_k^\tau}{\tau}(d_{k+1}^\tau - d_k^\tau) \ \text{ for } t \in [t_k^\tau, t_{k+1}^\tau], \quad k \in \{0,...,n-1\}.$$

Notice that d_τ is differentiable on $[0,T] \setminus \{t_0, t_1, ..., t_n\}$ with

$$\dot{d}_\tau(t) = \frac{d_{k+1}^\tau - d_k^\tau}{\tau} \quad \text{for } t \in (t_k^\tau, t_{k+1}^\tau), \quad k \in \{0,...,n-1\} \tag{6.2}$$

so that (P_k^β) implies for $t \in [0,T] \setminus \{t_0, t_1, ..., t_n\}$ that

$$\dot{d}_\tau(t) = \frac{1}{\delta} \max\{-\beta\big(\bar{d}_\tau(t) - \Phi(\bar{t}_\tau(t), \bar{d}_\tau(t))\big) - r, 0\}. \tag{6.3}$$

To prove convergence of the above time-discretization scheme, we need the following

Lemma 6.1. *For all $\tau > 0$ sufficiently small, there exists a constant $C > 0$, independent of τ, such that for all $t \in [0,T] \setminus \{t_0, t_1, ..., t_n\}$ it holds $\|\dot{d}_\tau(t)\|_2 \le C$.*

Proof. The proof is inspired by the proof of [15, Proposition 4.2] and employs the the (discrete version of) Gronwall's lemma. For this purpose let $k \in \{1, ..., n-1\}$ be arbitrary, but fixed. Then by testing (6.3) for fixed $t \in (t_k^\tau, t_{k+1}^\tau)$ and fixed $s \in (t_{k-1}^\tau, t_k^\tau)$ with $\dot{d}_\tau(t)$, subtracting the arising equations, and employing Young's inequality and the Lipschitz continuity of max and Φ according to Lemma 2.14, we arrive at

$$\frac{\delta}{2}\big(\|\dot{d}_\tau(t)\|_2^2 - \|\dot{d}_\tau(s)\|_2^2\big) \le \big(\beta K\,\tau + \beta(K+1)\|\bar{d}_\tau(t) - \bar{d}_\tau(s)\|_2\big)\|\dot{d}_\tau(t)\|_2. \tag{6.4}$$

Note that for the last inequality we used that $\bar{t}_\tau(t) - \bar{t}_\tau(s) = \tau$. Relying on (6.2), $\bar{d}_\tau(t) = d_{k+1}^\tau$, $\bar{d}_\tau(s) = d_k^\tau$, and Young's inequality, (6.4) can be continued as

$$\frac{\delta}{2}\big(\|\dot{d}_\tau(t)\|_2^2 - \|\dot{d}_\tau(s)\|_2^2\big) \le C\tau\big(1 + \|\dot{d}_\tau(t)\|_2^2\big) \quad \forall\, t \in (t_k^\tau, t_{k+1}^\tau),\ \forall\, s \in (t_{k-1}^\tau, t_k^\tau), \tag{6.5}$$

with a generic constant $C > 0$, independent of τ. Further, as a consequence of (6.2), we have for all $j = 0, ..., n-1$ that

$$\dot{d}_\tau(\rho) = \dot{d}_\tau\Big(\frac{t_j^\tau + t_{j+1}^\tau}{2}\Big) \quad \forall\, \rho \in (t_j^\tau, t_{j+1}^\tau). \tag{6.6}$$

Now let $t \in [t_1, T] \setminus \{t_1, ..., t_n\}$ be arbitrary, but fixed, which implies that there exists some $m \in \{1, ..., n-1\}$ such that $t \in (t_m, t_{m+1})$. By rewriting (6.5) with (6.6) and summing the resulting inequalities for $k = 1, ..., m$ up, we arrive at

$$\Big\|\dot{d}_\tau\Big(\frac{t_m + t_{m+1}}{2}\Big)\Big\|_2^2 \le \Big\|\dot{d}_\tau\Big(\frac{\tau}{2}\Big)\Big\|_2^2 + C\Big(m\,\tau + \tau\sum_{k=1}^m \Big\|\dot{d}_\tau\Big(\frac{t_k^\tau + t_{k+1}^\tau}{2}\Big)\Big\|_2^2\Big). \tag{6.7}$$

To conclude the assertion from (6.7) with the discrete version of Gronwall's lemma, one needs an estimate for $\big\|\dot{d}_\tau\big(\frac{\tau}{2}\big)\big\|_2$, independent of τ. To this end, we test (6.3) at time point $\tau/2$ with $\dot{d}_\tau(\tau/2)$ and deduce

$$\begin{aligned}\|\dot{d}_\tau(\tau/2)\|_2^2 = \frac{1}{\delta}\Big(&\max\big\{-\beta\big(\bar{d}_\tau(\tau/2) - \Phi\big(\bar{t}_\tau(\tau/2), \bar{d}_\tau(\tau/2)\big)\big) - r, 0\big\} \\ &\qquad - \max\big\{-\beta\big(d_0 - \varphi_0\big) - r, 0\big\}, \dot{d}_\tau(\tau/2)\Big)_2 \\ &+ \frac{1}{\delta}\Big(\max\big\{-\beta(d_0 - \varphi_0) - r, 0\big\}, \dot{d}_\tau(\tau/2)\Big)_2\end{aligned}$$

where $\varphi_0 := \Phi(0, d_0)$. The same arguments used already for the derivation of (6.5), together with the Cauchy-Schwarz inequality, then yield

$$\|\dot{d}_\tau(\tau/2)\|_2^2 \le C\big(\tau + \tau\|\dot{d}_\tau(\tau/2)\|_2^2 + \|\beta\big(d_0 - \varphi_0\big) + r\|_2\,\|\dot{d}_\tau(\tau/2)\|_2\big). \tag{6.8}$$

Therefore, for all $\tau \leq 1$ (which is of course no restriction), Young's inequality implies the existence of a constant $C > 0$ independent of τ so that $\|\dot{d}_\tau(\tau/2)\|_2 \leq C$. Since in addition $m \leq n-1$ and $\tau = T/n$, (6.7) leads to

$$\Big\|\dot{d}_\tau\Big(\frac{t_m + t_{m+1}}{2}\Big)\Big\|_2^2 \leq C\Big(1 + \tau \sum_{k=1}^{m} \Big\|\dot{d}_\tau\Big(\frac{t_k^\tau + t_{k+1}^\tau}{2}\Big)\Big\|_2^2\Big).$$

For $\tau > 0$ sufficiently small (which is again no restriction), one can compensate the last addend on the right hand side for $k = m$ with the left hand side so that

$$\Big\|\dot{d}_\tau\Big(\frac{t_m + t_{m+1}}{2}\Big)\Big\|_2^2 \leq C\Big(1 + \tau \sum_{k=1}^{m-1} \Big\|\dot{d}_\tau\Big(\frac{t_k^\tau + t_{k+1}^\tau}{2}\Big)\Big\|_2^2\Big) \tag{6.9}$$

is obtained, where we used the convention $\sum_{k=1}^{0} = 0$. This holds for any $m \in \{1, ..., n-1\}$, since $t \in [t_1, T] \setminus \{t_1, ..., t_n\}$ was arbitrary. Thus, the discrete version of Gronwall's lemma tells us that

$$\Big\|\dot{d}_\tau\Big(\frac{t_m + t_{m+1}}{2}\Big)\Big\|_2^2 \leq C \exp\Big(\sum_{k=1}^{m-1} C\tau\Big) \leq C \exp(CT) \quad \forall\, m \in \{1, ..., n-1\},$$

which, thanks to (6.6), in turn implies $\|\dot{d}_\tau(t)\|_2 \leq C$ for all $t \in [t_1, T] \setminus \{t_1, ..., t_n\}$. Together with $\|\dot{d}_\tau(\tau/2)\|_2 \leq C$ (see above), this gives the assertion. □

The main result in this subsection is given by

Proposition 6.2 (Convergence of the time-discretization). *There holds*

$$\begin{aligned} d_\tau &\to d && \text{in } W^{1,\infty}(0, T; L^2(\Omega)), \\ \bar{d}_\tau &\to d && \text{in } L^\infty(0, T; L^2(\Omega)), \\ \bar{\varphi}_\tau := \Phi\big(\bar{t}_\tau(\cdot), \bar{d}_\tau(\cdot)\big) &\to \varphi := \Phi\big(\cdot, d(\cdot)\big) && \text{in } L^\infty(0, T; H^1(\Omega)), \end{aligned}$$

as $\tau \searrow 0$, *where* $d \in C^{1,1}([0, T]; L^2(\Omega))$ *is the solution to* (6.1).

Proof. By construction of d_τ, $\bar{d}_\tau$, and $\underline{d}_\tau$, we have for all $t \in [0, T]$ that

$$d_\tau(t) - \bar{d}_\tau(t) = \Big(1 - \frac{t - \underline{t}_\tau(t)}{\tau}\Big)\big(\underline{d}_\tau(t) - \bar{d}_\tau(t)\big),$$

which, due to $\frac{t - \underline{t}_\tau(t)}{\tau} \in [0, 1]$ and (6.2), implies

$$\begin{aligned} \|d(t) - \bar{d}_\tau(t)\|_2 &\leq \|d(t) - d_\tau(t)\|_2 + \|d_\tau(t) - \bar{d}_\tau(t)\|_2 \\ &\leq \|d(t) - d_\tau(t)\|_2 + \|\bar{d}_\tau(t) - \underline{d}_\tau(t)\|_2 \\ &= \|d(t) - d_\tau(t)\|_2 + \tau\|\dot{d}_\tau(t)\|_2 \\ &\leq \|d(t) - d_\tau(t)\|_2 + C\,\tau \quad \text{f.a.a. } t \in (0, T), \end{aligned} \tag{6.10}$$

where $C > 0$ is the constant from Lemma 6.1. Subtracting (6.3) from (6.1) yields f.a.a. $t \in (0, T)$

$$\begin{aligned} &\dot{d}(t) - \dot{d}_\tau(t) \\ &\quad = \frac{1}{\delta}\big(\max\{-\beta\big(d(t) - \Phi(t, d(t))\big) - r, 0\} - \max\{-\beta\big(\bar{d}_\tau(t) - \Phi(\bar{t}_\tau(t), \bar{d}_\tau(t))\big) - r, 0\}\big), \end{aligned}$$

which, together with the Lipschitz-continuity of max and Φ and (6.10) gives

$$\begin{aligned}\|\dot d(t) - \dot d_\tau(t)\|_2 &\leq \frac{\beta}{\delta}\Big(\|d(t) - \bar d_\tau(t)\|_2 + K\big(|t - \bar t_\tau(t)| + \|d(t) - \bar d_\tau(t)\|_2\big)\Big) \\ &\leq C\big(\|d(t) - d_\tau(t)\|_2 + \tau\big) \qquad \text{f.a.a. } t \in (0,T),\end{aligned} \tag{6.11}$$

where $C > 0$ is independent of τ. Because of $d(0) = d_\tau(0)$, this implies for all $t \in [0,T]$ that

$$\|d(t) - d_\tau(t)\|_2 \leq \int_0^t \|\dot d(s) - \dot d_\tau(s)\|_2 \, ds \leq C\Big(\int_0^t \|d(s) - d_\tau(s)\|_2 \, ds + T\,\tau\Big).$$

Note that, by construction, d_τ is piecewise smooth and therefore an element of $W^{1,\infty}(0,T;L^2(\Omega))$ so that Lebegue's differentiation theorem is applicable. Gronwall's Lemma now yields

$$\|d(t) - d_\tau(t)\|_2 \leq \tau\, TC \exp(TC) \qquad \text{for all } t \in [0,T]. \tag{6.12}$$

Combining (6.11) with (6.12) gives $d_\tau \to d$ in $W^{1,\infty}(0,T;L^2(\Omega))$ as $\tau \searrow 0$, and from (6.10) we then conclude $\bar d_\tau \to d$ in $L^\infty(0,T;L^2(\Omega))$. The Lipschitz continuity of Φ finally leads to

$$\|\Phi\big(\bar t_\tau(t), \bar d_\tau(t)\big) - \Phi\big(t, d(t)\big)\|_{H^1(\Omega)} \leq K\big(\tau + \|\bar d_\tau - d\|_{L^\infty(0,T;L^2(\Omega))}\big) \quad \forall\, t \in [0,T],$$

whence $\bar\varphi_\tau \to \varphi$ in $L^\infty(0,T;H^1(\Omega))$ as $\tau \searrow 0$, which completes the proof. □

6.2. Uniform Estimates for the Discrete Damage variables. In this subsection we first prove that the piecewise constant interpolation function $\bar d_\tau$ is bounded a.e. in $(0,T) \times \Omega$ by a *constant independent of* β *and* τ. Because of Proposition 6.2 and Theorem 5.3, this bound carries over to the limit function φ in (5.9), which is the ultimate goal of this section. With a little abuse of notation, we again suppress the subscript β for the solution of (6.1), i.e., the problem with penalty, in order to shorten the notation. When it comes to the pointwise boundedness of the "penalty limit" φ, i.e., the solution of (5.9), we will introduce the index β again, see Proposition 6.8 below.

To prove the result, we follow the ideas of the proof of [15, Proposition 4.5], that is, we show the pointwise boundedness of d_{k+1}^τ by induction on the index k, see proof of Lemma 6.7 below. Therefor we first rewrite (P_k^β) as an equivalent minimization problem:

LEMMA 6.3. *The solution* d_{k+1}^τ *of* (P_k^β) *is the unique minimizer of*

$$\min_{v \in \mathcal{C}_k} \; f_k(v) := \mathcal{I}(t_{k+1}^\tau, v) + r\int_\Omega v - d_k^\tau \, dx + \frac{\delta}{2}\frac{\|v - d_k^\tau\|_2^2}{\tau} \tag{6.13}$$

with $\mathcal{C}_k := \{v \in L^2(\Omega) : v \geq d_k^\tau$ *a.e. in* $\Omega\}$.

Proof. We aim to show that the necessary optimality conditions for (6.13) are equivalent to (P_k^β). However, due to the nonlinear structure of $\mathcal{U}$ and Φ, the objective f_k is in general not convex so that it is a priori not clear that its necessary conditions are also sufficient. It does therefore not suffice to prove the equivalence of (P_k^β) to the necessary conditions of (6.13) in order to establish the claim of the lemma. Instead one first has to verify the existence of solutions for (6.13). To this end, first observe that, in view of (3.1),

$$\begin{aligned} f_k(v) = -\frac{1}{2}\langle \ell(t^\tau_{k+1}), \mathcal{U}(t^\tau_{k+1}, \Phi(t^\tau_{k+1}, v))\rangle_V + \frac{\alpha}{2}\|\nabla\Phi(t^\tau_{k+1}, v)\|_2^2 \\ + \frac{\beta}{2}\|\Phi(t^\tau_{k+1}, v) - v\|_2^2 + r\int_\Omega v - d^\tau_k \, dx + \frac{\delta}{2}\frac{\|v - d^\tau_k\|_2^2}{\tau}. \end{aligned} \tag{6.14}$$

The existence of solutions for (6.13) follows from the classical direct method of variational calculus. First $\mathcal{C}_k$ is convex and closed, thus, weakly closed. Moreover, in light of (6.14) and (2.2) f is radially unbounded. It remains to show that it is also weakly lower semicontinuous. To see this, we first prove that $L^2(\Omega) \ni v \mapsto \Phi(t^\tau_{k+1}, v) \in H^1(\Omega)$ is weakly continuous. To this end, consider a sequence $\{v_j\} \subset L^2(\Omega)$ such that $v_j \rightharpoonup v$ in $L^2(\Omega)$ as $j \to \infty$ and abbreviate $\varphi_j := \Phi(t^\tau_{k+1}, v_j)$ for the sake of convenience. The boundedness of $\{v_j\}$ and the global Lipschitz continuity of Φ by Lemma 2.14 imply the existence of a subsequence, for simplicity denoted by the same symbol, such that $\varphi_j \rightharpoonup \widetilde{\varphi}$ in $H^1(\Omega)$ as $j \to \infty$. Since $p > N$ by Assumption 2.7.1, $H^1(\Omega)$ is compactly embedded in $L^{2p/(p-2)}(\Omega)$ so that (2.8) implies

$$\|F(t^\tau_{k+1}, \varphi_j) - F(t^\tau_{k+1}, \widetilde{\varphi})\|_{H^1(\Omega)^*} \le C\|\varphi_j - \widetilde{\varphi}\|_{2p/(p-2)} \to 0 \quad \text{as } j \to \infty.$$

Together with the linearity of B, see (2.6), and the definition of Φ in Lemma 2.13, this yields

$$0 = \beta\, v_j - B\varphi_j - F(t^\tau_{k+1}, \varphi_j) \rightharpoonup \beta\, v - B\widetilde{\varphi} - F(t^\tau_{k+1}, \widetilde{\varphi}) \quad \text{in } H^1(\Omega)^* \text{ as } j \to \infty$$

so that $\widetilde{\varphi} = \Phi(t^\tau_{k+1}, v)$. Thus the weak limit is unique, which in turn gives the weak convergence of the whole sequence φ_j and, since $\{v_j\}$ was an arbitrary weakly converging sequence, we thus obtain the desired weak continuity of $\Phi(t^\tau_{k+1}, \cdot)$. Together with the weak lower semicontinuity of squared norms this gives the weak lower semicontinuity of all terms on the right hand side of (6.14), except the first addend. However, since the mapping $L^{2p/(p-2)}(\Omega) \ni \varphi \mapsto \mathcal{U}(t^\tau_{k+1}, \varphi) \in V$ is continuous by Lemma 2.10, the compactness of $H^1(\Omega) \hookrightarrow L^{2p/(p-2)}(\Omega)$ guarantees the convergence of this term, too. Therefore, $f_k : L^2(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous and consequently, (6.13) admits solutions.

Next we derive first-order necessary optimality conditions for (6.13). To this end, let z be an arbitrary, but fixed, solution thereof. Thanks to Lemma 3.2, f_k is Fréchet-differentiable so that, due to convexity of $\mathcal{C}_k$, the necessary optimality conditions for (6.13) are given by

$$f_k'(z)(v - z) \ge 0 \quad \forall\, v \in \mathcal{C}_k,$$

which, in view of (3.2), reads

$$\big(\beta(z - \Phi(t^\tau_{k+1}, z)) + r + \delta\,\tau^{-1}(z - d^\tau_k)\,,\, v - \delta\,\tau^{-1}(z - d^\tau_k)\big)_2 \ge 0 \qquad \text{for all } v \in L^2(\Omega) \text{ with } v \ge 0 \text{ a.e. in } \Omega. \tag{6.15}$$

As a variational inequality in $L^2(\Omega)$ subject to pointwise inequality constraints, (6.15) can equivalently be reformulated as a pointwise complementarity relation (cf. e.g. [27, Section 2.8.2]), which reads

$$0 \le \frac{\delta}{\tau}(z - d^\tau_k) \quad \perp \quad \beta(z - \Phi(t^\tau_{k+1}, z)) + r + \frac{\delta}{\tau}(z - d^\tau_k) \ge 0 \quad \text{a.e. in } \Omega.$$

Since the max-function is a well-known complementarity function, we infer that z solves (P_k^β) and, since the latter is uniquely solvable, this gives the uniqueness of the minimizer of (6.13) as well as the equivalence of (6.13) and (P_k^β). □

In order to obtain the desired boundedness result, we impose the following

ASSUMPTION 6.4. *There exists a constant $M > 0$ such that $g(x) \geq g(M)$ for all $x \geq M$.*

REMARK 6.5. *Recall that the function $g : \mathbb{R} \to [\epsilon, 1]$ measures the material rigidity of the body, where, according to Assumption 2.3, $\epsilon > 0$ (which turns the model into a partial damage model). Thus there is always a minimum rigidity remaining, and Assumption 6.4 is for instance fulfilled, if $g(x) \equiv \epsilon$ for all $x \geq M$, which means that this minimum rigidity is already achieved with finite values of the (nonlocal) damage variable.*

Next let us introduce the Nemytskii operator associated with $\min\{\cdot, M\} : \mathbb{R} \to \mathbb{R}$, denoted by $L^2(\Omega) \ni z \mapsto \min(z, M) \in L^2(\Omega)$. Clearly, this operator is Lipschitz continuous with constant one. We will also consider this operator with domain and range in $H^1(\Omega)$, for simplicity denoted by the same symbol. It is well known that $\min(\cdot, M)$ maps $H^1(\Omega)$ to $H^1(\Omega)$ with

$$\nabla \min(v, M) = \chi_{\{v<M\}} \nabla v \quad \forall\, v \in H^1(\Omega), \tag{6.16}$$

where $\chi_{\{v<m\}} \in L^\infty(\Omega; \{0,1\})$ denotes the characteristic function of $\{x \in \Omega : v(x) < M\}$ (defined up to sets of measure zero), see e.g. [12, Theorem II.A.1].

LEMMA 6.6. *Under Assumption 6.4, we have*

$$\mathcal{I}(t, \min(z, M)) \leq \mathcal{I}(t, z) \quad \forall\, (t, z) \in [0, T] \times L^2(\Omega).$$

Moreover, if $z \in L^2(\Omega)$ satisfies $\min(z, M) = z$, then $\min(\Phi(t, z), M) = \Phi(t, z)$ for every $t \in [0, T]$.

Proof. Let $(t, z) \in [0, T] \times L^2(\Omega)$ be arbitrary, but fixed and let us for simplicity abbreviate $\bar\varphi := \Phi(t, z)$ in what follows. From Lemma 2.13, Definition 3.1, and (1.1), it follows that

$$\begin{aligned} &\mathcal{I}(t, \min(z, M)) \\ &\quad \leq \mathcal{E}(t, \mathcal{U}(t, \bar\varphi), \min(\bar\varphi, M), \min(z, M)) \\ &\quad = \frac{1}{2} \int_\Omega g(\min(\bar\varphi, M)) \mathbb{C} \varepsilon(\mathcal{U}(t, \bar\varphi)) : \varepsilon(\mathcal{U}(t, \bar\varphi))\, dx - \langle \ell(t), \mathcal{U}(t, \bar\varphi) \rangle_V \\ &\qquad + \frac{\alpha}{2} \|\nabla \min(\bar\varphi, M)\|_2^2 + \frac{\beta}{2} \|\min(\bar\varphi, M) - \min(z, M)\|_2^2, \end{aligned} \tag{6.17}$$

By Assumption 6.4, we further have $g(\min(\bar\varphi, M)) \leq g(\bar\varphi)$ a.e. in Ω. Together with (2.1), (6.16), and the Lipschitz continuity of $\min(\cdot, M) : L^2(\Omega) \to L^2(\Omega)$, this implies for (6.17)

$$\begin{aligned} \mathcal{I}(t, \min(z, M)) \leq \frac{1}{2} \int_\Omega & g(\bar\varphi) \mathbb{C} \varepsilon(\mathcal{U}(t, \bar\varphi)) : \varepsilon(\mathcal{U}(t, \bar\varphi))\, dx \\ & - \langle \ell(t), \mathcal{U}(t, \bar\varphi) \rangle_V + \frac{\alpha}{2} \|\nabla \bar\varphi\|_2^2 + \frac{\beta}{2} \|\bar\varphi - z\|_2^2 = \mathcal{I}(t, z), \end{aligned} \tag{6.18}$$

where we again employed Definition 3.1. This proves the first assertion.

Now, if z satisfies $\min(z, M) = z$, then (6.18) holds with equality and thus, the inequality in (6.17) is an equality, too. However, since the minimization problem in (P) is uniquely solvable, cf. Lemma 2.13, equality in (6.17) implies $\min(\bar{\varphi}, M) = \Phi(t, \min(z, M)) = \Phi(t, z)$, which completes the proof. □

LEMMA 6.7 (Uniform estimates for the discrete local and nonlocal damage). *Suppose that Assumption 6.4 hold true and that $d_0 \in L^\infty(\Omega)$ with $\|d_0\|_{L^\infty(\Omega)} \leq M$. Then for all $t \in [0, T]$ there holds*

$$\bar{d}_\tau(t, x) \leq M, \quad \bar{\varphi}_\tau(t, x) \leq M \text{ a.e. in } \Omega,$$

where $\bar{\varphi}_\tau$ again stands for $\Phi(\bar{t}_\tau(\cdot), \bar{d}_\tau(\cdot))$.

Proof. We follow the lines of the proof of [15, Proposition 4.5] and show that $d_k^\tau \leq M$ for all $k \in \{0, ..., n\}$ by induction on the index k. Note that the assertion is fulfilled for $k = 0$, since $d_0(x) \leq M$ a.e. in Ω by assumption. Now let $k \in \{0, ..., n-1\}$ be fixed, but arbitrary and assume that

$$d_k^\tau(x) \leq M \quad \text{ a.e. in } \Omega. \tag{6.19}$$

The idea of the proof is to show that $(d_{k+1}^\tau)^- := \min(d_{k+1}^\tau, M)$ solves the problem (6.13) so that Lemma 6.3 implies $d_{k+1}^\tau = \min(d_{k+1}^\tau, M)$. From ($\mathrm{P}_k^\beta$) it is clear that $d_{k+1}^\tau \geq d_k^\tau$ and, as a result of (6.19), thus

$$(d_{k+1}^\tau)^- \geq d_k^\tau \text{ a.e. in } \Omega \quad \iff \quad (d_{k+1}^\tau)^- \in \mathcal{C}_k$$

so that $(d_{k+1}^\tau)^-$ is feasible for (6.13). For the objective of (6.13) we obtain by Lemma 6.6 that

$$\begin{aligned} f_k((d_{k+1}^\tau)^-) = \mathcal{I}(t_{k+1}^\tau, (d_{k+1}^\tau)^-) + r \int_\Omega (d_{k+1}^\tau)^- - d_k^\tau \, dx + \frac{\delta}{2} \frac{\|(d_{k+1}^\tau)^- - d_k^\tau\|_2^2}{\tau} \\ \leq \mathcal{I}(t_{k+1}^\tau, d_{k+1}^\tau) + r \int_\Omega d_{k+1}^\tau - d_k^\tau \, dx + \frac{\delta}{2} \frac{\|d_{k+1}^\tau - d_k^\tau\|_2^2}{\tau} = f_k(d_{k+1}^\tau) \end{aligned}$$

where we also used that $0 \leq (d_{k+1}^\tau)^- - d_k^\tau \leq d_{k+1}^\tau - d_k^\tau$. On the other side, Lemma 6.3 tells us that d_{k+1}^τ is the unique solution of (6.13) and, since $(d_{k+1}^\tau)^-$ is feasibe as seen above, we conclude that $(d_{k+1}^\tau)^- = d_{k+1}^\tau$. Hence, $d_{k+1}^\tau \leq M$, which ends the induction step. Therefore, the piecewise constant interpolant satisfies $\bar{d}_\tau(t, x) \leq M$ for all $t \in [0, T]$ and almost all $x \in \Omega$. This, together with the second assertion of Lemma 6.6, implies $\bar{\varphi}_\tau(t, x) = \Phi(\bar{t}_\tau(t), \bar{d}_\tau(t))(x) \leq M$ for all $t \in [0, T]$ and almost all $x \in \Omega$, which completes the proof. □

From the above results it easily follows that the local and the nonlocal damage associated with the penalized problem (2.15), as well as their limit as $\beta \to \infty$, are bounded a.e. in space and time by the constant M from Assumption 6.4. To distinguish between the solutions of the penalized problem and the single-field model in (5.9), we no longer suppress the index β.

PROPOSITION 6.8 (Uniform L^∞-bound for the penalized damage variable). *Let Assumptions 4.1 and 6.4 be fulfilled. Then, the local and nonlocal damage associated with the penalized model (2.15) fulfill $0 \leq d_\beta(t, x) \leq M$ and $\varphi_\beta(t, x) \leq M$ a.e. in $(0, T) \times \Omega$.*

Proof. The lower bound for d_β is a direct consequence of (2.15c) and Assumption 4.1, i.e., $d_0 \equiv 0$. The upper bounds immediately follow from Proposition 6.2 and the weak closedness of $\{v \in L^2((0,T)\times\Omega) : v \le M \text{ a.e. in } (0,T)\times\Omega)\}$. ☐

By analogous arguments, we finally obtain the following

THEOREM 6.9 (L^∞-bound for the limit damage variable). *Suppose that Assumptions 4.1 and 6.4 are fulfilled. Let $\beta_n \to \infty$ be a sequence so that $d_{\beta_n} \rightharpoonup \varphi$ in $H^1(0,T;L^2(\Omega))$ (whose existence is guaranteed by Theorem 5.3). Then there holds $0 \le \varphi \le M$ a.e. in $(0,T)\times\Omega$, where $M > 0$ is the constant from Assumption 6.4.*

7. Comparison to a Classical Partial Damage Model. The aim of this section is to transfer the single-field damage model (5.9) arising in the penalty limit to a system that allows a comparison with the damage model in [15] that falls into the category of classical partial damage models as for instance introduced in [8]. Throughout this section, we suppose that, beside our standing assumptions in Section 2, Assumptions 4.1 and 6.4 are fulfilled, too. Let us pick a fixed, but arbitrary solution $(\varphi, \boldsymbol{u}) \in H^1(0,T;H^1(\Omega)) \times C([0,T];V)$ of (5.9), which arises as a penalty limit for $\beta \to \infty$. Recall again that the existence of such a solution is guaranteed by Theorem 5.3. Moreover, we know from Theorem 6.9 that $\varphi \in L^\infty((0,T)\times\Omega)$ with

$$0 \le \varphi(t,x) \le M \quad \text{a.e. in } (0,T)\times\Omega. \tag{7.1}$$

The transformation of the single-field model (5.9) is based on the introduction of a new damage variable, defined by

$$z := 1 - \frac{\varphi}{M} \in H^1(0,T;H^1(\Omega)). \tag{7.2}$$

Because of (7.1) and (7.2), it holds

$$z(t,x) \in [0,1], \quad z(t,x) = 1 \iff \varphi(t,x) = 0, \quad z(t,x) = 0 \iff \varphi(t,x) = M$$

a.e. in $(0,T)\times\Omega$. Physically interpreted, this means that the body is completely sound, if $z = 1$, whereas the maximum damage is reached, if $z = 0$, cf. also Remark 6.5. We now transform the system (5.9) into a set of equations in the variables $(z, \boldsymbol{u})$, i.e., the displacement remains the same. For this purpose, we define the following transformation of the elasticity coefficient function

$$\overline{g} : \mathbb{R} \ni x \mapsto g(M(1-x)) \in \mathbb{R}, \tag{7.3}$$

as well as the following modified model parameters

$$\bar{\alpha} := \alpha\, M^2, \quad \bar{\delta} := \delta\, M^2, \quad \kappa := r\, M. \tag{7.4}$$

Given these definitions, (5.9) is equivalent to

$$-\operatorname{div} \overline{g}(z(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) = \ell(t) \quad \text{in } V^*, \tag{7.5a}$$

$$\bar{\delta}\,\dot{z}(t) - \bar{\alpha}\triangle z(t) + \frac{1}{2}\,\overline{g}'(z(t))\mathbb{C}\,\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) \in -\partial\overline{\mathcal{R}}_1(\dot{z}(t)), \quad z(0) = 1 \tag{7.5b}$$

with the dissipation functional

$$\overline{\mathcal{R}}_1 : H^1(\Omega) \to [0,\infty], \quad \overline{\mathcal{R}}_1(\eta) := \begin{cases} \kappa \int_\Omega (-\eta)\, dx, & \text{if } \eta \le 0 \text{ a.e. in } \Omega, \\ \infty, & \text{otherwise.} \end{cases} \tag{7.6}$$

Here we used that $\partial\overline{\mathcal{R}}_1(\dot z(t)) = -M\partial\widetilde{\mathcal{R}}_1(\dot\varphi(t))$ f.a.a. $t \in (0,T)$. The energy functional associated with (7.5) reads

$$\overline{\mathcal{E}}(t,\boldsymbol{u},z) := \frac{1}{2}\int_\Omega \overline{g}(z)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\,dx - \langle \ell(t),\boldsymbol{u}\rangle_V + \frac{\bar\alpha}{2}\|\nabla z\|_2^2. \tag{7.7}$$

Remark 7.1. *The energy in (7.7) and the dissipation functional in (7.6) in principle coincide with those in [15, Eq. (1.1) and (1.3)]. The major differences between $\overline{\mathcal{E}}$ and the energy in [15, Eq. (1.1)] concern the H_0^1-semi-norm in (7.7) and an additional nonlinearity in [15, Eq. (1.1)]. This nonlinearity is just a smooth Nemytskii-operator acting on z and can easily be incorporated into our analysis. We however decided not to consider this nonlinearity and to rely on the energy from [4] instead, as this reference serves as a basis for our two-field damage model.*

In the two-dimensional case, the energy in [15] also contains the H_0^1-semi-norm, while, in three spatial dimensions, the H_0^1-semi-norm is replaced by a bilinear form inducing the $H^{3/2}$-semi-norm in order to avoid Assumption 2.7.1, cf. Remark 2.8 and [15, Section 2.1].

Similarly to Lemma 2.13 one shows that, for given $(t,z) \in [0,T]\times H^1(\Omega)$, the energy $\overline{\mathcal{E}}(t,\cdot,z)$ is minimized by the unique solution of (7.5a), see also [15, Lemma 2.4]. Analogously to Definitions 2.9 and 3.1 we introduce the solution operator $\overline{\mathcal{U}} : [0,T]\times H^1(\Omega) \ni (t,z) \mapsto \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ associated with (7.5a) and the reduced energy functional $\overline{\mathcal{I}}(t,z) := \overline{\mathcal{E}}(t,\overline{\mathcal{U}}(t,z),z)$. Following the lines of the Lemmata 3.2 and 4.12, one shows that $\overline{\mathcal{I}}$ is Fréchet-differentiable, cf. also [15, Corollary 2.9]. If we moreover introduce a viscous dissipation functional analogously to Definition 4.13 by

$$\overline{\mathcal{R}}_{\bar\delta} : H^1(\Omega) \ni \eta \mapsto \overline{\mathcal{R}}_1 + \frac{\bar\delta}{2}\|\eta\|_2^2 \in \mathbb{R}\cup\{\infty\},$$

then (7.5) is equivalent to

$$\partial\overline{\mathcal{R}}_{\bar\delta}(\dot z(t)) + \partial_z\overline{\mathcal{I}}(t,z(t)) \ni 0 \quad \text{f.a.a. } t \in (0,T). \tag{7.8}$$

This is exactly the viscous model of [15], see [15, Eq. (3.1)], except for the differences in the energy functional mentioned in Remark 7.1. This shows that the limit system (5.9) indeed coincides with the "classical" viscous partial damage model from [15], which may be seen as an ultimate argument showing that the penalization procedure makes sense from a mathematical point of view.

Remark 7.2. *Interestingly, the additional nonlinearity in the energy mentioned in Remark 7.1 is needed in [15] in order to prove existence for the viscous model (7.8). Our penalization limit analysis thus shows that the existence of viscous solutions can also be establishes without this additional term.*

A crucial result in [15] concerns the vanishing viscosity analysis for $\bar\delta \searrow 0$. It turns out that the vanishing viscosity limit satisfies the following re-parametrized system w.r.t. an artificial time $s \in [0,S]$

$$\begin{gathered}\partial\overline{\mathcal{R}}_1(\hat z'(s)) + \lambda(s)\,\hat z'(s) + \partial_z\overline{\mathcal{I}}(\hat t(s),\hat z(s)) \ni 0 \quad \text{f.a.a. } s \in (0,S)\\ \hat t(S) = T, \quad \hat t'(s) \ge 0, \quad \hat t'(s)\,\lambda(s) = 0, \quad \hat t'(s) + |\hat z'(s)| \le 1 \quad \text{a.e. in } (0,S).\end{gathered}$$

For details, we refer to [15, Section 5]. It is open question what happens to the penalized model (P) and (2.15), respectively, when the viscosity vanishes, i.e., $\delta \searrow 0$

for fixed $\beta > 0$. This is of particular interest, since the non-viscous two-field model is frequently considered in computational mechanics, see [4, 5]. The vanishing viscosity limit analysis for (P) however would go beyond the scope of this paper and gives rise to future research.

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