Generalized quasi-Newtonian approach for modeling and simulating complex flows

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Max Planck Institute for Dynamics of Complex Technical System Magdeburg
The powder can transit from the quasi-static to the intermediate regime as the shearing rate is increased.

Shear and pressure dependent viscosity
Viscoplastic flow

Viscoplastic Lubricate Flow (Yield stress fluids)
- Dependent on the stress field
- Constitutive model is dependent on different flow regimes
- Non-smooth change in the constitutive relations

Model preserving the sharp changes of the constitutive equations w.r.t. flow regimes
Thixotropy concept
- Based on viscosity
- Flow induced by time-dependent decrease of viscosity
- The phenomena is reversible

- Aging / Build-up
  - At rest or under slow flow: fluid ages
    Increases of the viscosity in time

- Rejuvenation / Breakdown
  - “Faster” flow: fluid rejuvenates
    Decreases of viscosity with acceleration of the flow

Investigation of solid/liquid and liquid/solid transitions with non constant yield stress
Non-Newtonian phenomena

- Effects due to normal stresses
- Effects due to elongational viscosity
- The drag reduction phenomenon

Differential models

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Realization in FeatFlow

**HPC features:**
- Moderately parallel
- GPU computing
- Open source

**Non-Newtonian flow module:**
- generalized Newtonian model (Power-law, Carreau,...)
- viscoelastic differential model (Giesekus, FENE, Oldroyd,...)

**Multiphase flow module (resolved interfaces):**
- \( \text{l/l} \) – interface capturing (Level Set)
- \( \text{s/l} \) – interface tracking (FBM)
- \( \text{s/l/l} \) – combination of \( \text{l/l} \) and \( \text{s/l} \)

**Engineering aspects:**
- Geometrical design
- Modulation strategy
- Optimization

**Numerical features:**
- Higher order FEM in space & (semi-) Implicit FD/FEM in time
- Semi-(un)structured meshes with dynamic adaptive grid deformation
- Fictitious Boundary (FBM) methods
- Newton-Multigrid-type solvers

**Hardware-oriented Numerics**

**Here:** FEM-based tools for the accurate simulation of (multiphase) flow problems, particularly with complex rheology
Governing equations

- **Generalized Navier-Stokes equations**

\[
\rho \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) u - \nabla \cdot \sigma + \nabla p = \rho f,
\]

\[
\nabla \cdot u = 0,
\]

\[
\sigma = \sigma_s + \sigma_p,
\]

- **Viscous stress**

\[
\sigma_s = 2\eta_s (D_\Pi, p) D(u), \quad D_\Pi = \text{tr} \left( D(u)^2 \right).
\]

- **Elastic stress**

\[
\sigma_p + \text{We} \frac{\delta_a \sigma_p}{\delta t} = 2\eta_p D(u).
\]
Quasi-Newtonian models

- **Viscous stress**

\[
\sigma_s = 2\eta_s(D_{II}, p)D(u), \quad D_{II} = \text{tr} \left( D(u)^2 \right)
\]

- **Power law model**

\[
\eta_s(z) = \eta_0 z^{r-\frac{1}{2}} \quad (\eta_0 > 0, r > 1)
\]

- **Powder flow in the quasi-static and intermediate regimes**

\[
\begin{aligned}
\eta_s(z, p) &= \sqrt{2p} \left( \sin \phi z^{-\frac{1}{2}} + \cos \phi z^{r-\frac{1}{2}} \right) \quad \text{if } z \neq 0, r > 1 \\
\|\sigma_s\| &\leq \sqrt{2p} \sin \phi \quad \text{else}
\end{aligned}
\]

(\(\phi\) : the angle of internal friction)
Quasi-Newtonian models

- **Yield stress flow (Bingham Model)**

\[
\begin{aligned}
\eta_s(z, \lambda) &= \eta_0 + \tau_0 z^{-\frac{1}{2}} & \text{if } z \neq 0 \\
\|\sigma_s\| &\leq \tau_0 & \text{else}
\end{aligned}
\]

(\(\tau_0\) : yield stress)

- **Thixotropic model**

\[
\begin{aligned}
\eta_s(z, \lambda) &= \eta(\lambda) + \tau(\lambda) z^{-\frac{1}{2}} & \text{if } z \neq 0 \\
\|\sigma_s\| &\leq \tau(\lambda) & \text{else}
\end{aligned}
\]

(\(\lambda\) : structure parameter)

- **Structure parameter equation**

\[
\frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda = a(1 - \lambda) - b\lambda z^{\frac{1}{2}}
\]

(\(a, b\) are structure parameters)
Constitutive models

- Elastic stress

\[
\sigma_p + \text{We} \frac{\delta a \sigma_p}{\delta t} = 2\eta_p D(u).
\]

- Upper/Lower convective derivative

\[
\frac{\delta a \sigma}{\delta t} = \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \sigma + g_a(\sigma, \nabla u)
\]

\[
g_a(\sigma, \nabla u) = \frac{1-a}{2} \left( \sigma \nabla u + (\nabla u)^T \sigma \right)
- \frac{1+a}{2} \left( \nabla u \sigma + \sigma (\nabla u)^T \right) \quad (a = \pm 1)
\]
Constitutive models

- **Generalized differential constitutive model**

\[
\sigma + \text{We} \frac{\delta \sigma}{\delta t} + G(\sigma, D) + H(\sigma) = 2\eta_p D(u)
\]

- **Oldroyd**

\[G = 0, \quad H = 0\]

- **Giesekus**

\[G = 0, \quad H = \alpha \text{tr}(\sigma^2)\]

- **Phan-Thien and Tanner**

\[H = [\exp (\alpha \text{tr}(\sigma)) - 1] \sigma\]

- **White and Metzner**

\[G = \alpha (2 D : D)^{1/2}, \quad H = 0\]
Stokes problem

- **Two-field formulation** \((u, p)\)

\[
\begin{align*}
-\nabla \cdot \left( 2\eta D(u) \right) + \nabla p &= 0 \quad \text{in} \; \Omega \\
\nabla \cdot u &= 0 \quad \text{in} \; \Omega \\
u &= g_D \quad \text{on} \; \Gamma_D
\end{align*}
\]

- **Three-field formulation** \((\sigma, u, p)\)

\[
\begin{align*}
\sigma - 2\eta D(u) &= 0 \quad \text{in} \; \Omega \\
-\nabla \cdot \left( 2\eta (1 - \alpha) D(u) + \alpha \sigma \right) + \nabla p &= 0 \quad \text{in} \; \Omega \\
\nabla \cdot u &= 0 \quad \text{in} \; \Omega \\
u &= g_D \quad \text{on} \; \Gamma_D
\end{align*}
\]
Stokes problem

- **Two-field formulation** \((u, p)\)

  - **Set** \(\mathbb{V} := \left[H_0^1(\Omega)\right]^2, \mathbb{Q} := L_0^2(\Omega)\)

  - **Find** \((u, p) \in \mathbb{V} \times \mathbb{Q}\) s.t.

    \[
    \langle \mathcal{K}(u, p), (v, q) \rangle = \langle \mathcal{L}, (v, q) \rangle, \quad \forall (v, q) \in \mathbb{V} \times \mathbb{Q}
    \]

    \[
    \mathcal{K} = \begin{pmatrix}
    A_u & B^T \\
    B & 0
    \end{pmatrix}
    \]

  - **Compatibly constraints**

    \[
    \sup_{v \in \mathbb{V}} \frac{\langle Bv, q \rangle}{|v|_\mathbb{V}} \geq \beta \|q\|_{\mathbb{Q}/\text{Ker}B^T}, \quad \forall q \in \mathbb{Q}
    \]
Stokes problem

- **Three-field formulation** \((\sigma, u, p)\)

- Set \(\mathcal{T} := (L^2(\Omega))^4_{\text{sym}}\), \(\mathcal{V} := [H^1_0(\Omega)]^2\), \(\mathcal{Q} := L^2_0(\Omega)\)

- Find \((\sigma, u, p) \in \mathcal{T} \times \mathcal{V} \times \mathcal{Q}\) s.t.

  \[
  \left\langle \mathcal{K}(\sigma, u, p), (\tau, v, q) \right\rangle = \left\langle \mathcal{L}, (\tau, v, q) \right\rangle, \quad \forall (\tau, v, q) \in \mathcal{T} \times \mathcal{V} \times \mathcal{Q}
  \]

  \[
  \mathcal{K} = \begin{pmatrix}
  A_{\sigma} & C & 0 \\
  C^T & A_u & B^T \\
  0 & B & 0
  \end{pmatrix}
  \]

- **Compatibly constraints**

  \[
  \sup_{v \in \mathcal{V}} \frac{\left\langle \mathcal{B}v, q \right\rangle}{\|v\|_{\mathcal{V}}} \geq \beta \|q\|_{\mathcal{Q}/\text{Ker}B^T}, \quad \forall q \in \mathcal{Q}
  \]

  \[
  \sup_{v \in \mathcal{V}} \frac{\left\langle \mathcal{C}v, \tau \right\rangle}{\|v\|_{\mathcal{V}}} \geq \gamma \|\tau\|_{\mathcal{T}/\text{Ker}C^T}, \quad \forall \tau \in \mathcal{T}
  \]

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Approximated Stokes problem

- **Conforming approximations**

  \[ T_h \subset T, \quad \mathbf{V}_h \subset \mathbf{V} = \tilde{\mathbf{V}}, \quad \mathbf{Q}_h \subset \mathbf{Q} \]

  \[ \mathbf{A}_\sigma h = \mathbf{A}_\sigma, \quad \mathbf{A}_{uh} = \mathbf{A}_u, \quad \mathbf{B}_h = \mathbf{B}, \quad \mathbf{C}_h = \mathbf{C} \]

- **Non-conforming approximation**

  \[ T_h \subset T, \quad \mathbf{V}_h \not\subset \mathbf{V} \& \mathbf{V}_h \subset \tilde{\mathbf{V}}, \quad \mathbf{Q}_h \subset \mathbf{Q} \]

  \[ \mathbf{A}_\sigma h = \mathbf{A}_\sigma, \quad \mathbf{A}_{uh} \neq \mathbf{A}_u, \quad \mathbf{B}_h \neq \mathbf{B}, \quad \mathbf{C}_h \neq \mathbf{C} \]

- **Discrete inf-sup condition**

  \[
  \sup_{v_h \in \mathbf{V}_h} \frac{\langle \mathbf{B}_h v_h, q_h \rangle}{|v_h|_{\tilde{\mathbf{V}}}} \geq \beta_h \| q_h \|_{Q/\text{Ker}\mathbf{B}_h^T}, \quad \forall q_h \in \mathbf{Q}_h
  \]

  \[
  \sup_{v_h \in \mathbf{V}_h} \frac{\langle \mathbf{C}_h v_h, \tau_h \rangle}{|v_h|_{\tilde{\mathbf{V}}}} \geq \gamma_h \| \tau_h \|_{T/\text{Ker}\mathbf{C}_h^T}, \quad \forall \tau_h \in T_h
  \]
The family of non-/conforming FEM $\tilde{Q}_r/P_{r-1}^{\text{disc}}$, $r \geq 2$ and the family of nonconforming FEM $Q_r/P_{r-1}^{\text{disc}}$, $r \geq 2$ for $(u, p)$

- Inf-sup stable
- Arbitrary order with optimal convergence order
- Discontinuous pressure
  - Good for the solver
  - Element-wise mass conservation

The family of conforming FEM $Q_r/Q_r/P_{r-1}^{\text{disc}}$, $r \geq 2$ for $(\sigma, u, p)$ with stabilization

$$J_u(u_h, v_h) = \gamma_u \sum_{e \in \mathcal{E}_h} 2\eta \alpha h \int_e [\nabla u_h] : [\nabla v_h] d\Omega$$

- Both Inf-sup conditions are satisfied
- Highly consistent and symmetric stabilization, penelazing any spurious current, enhancing the preconditioner, which improve accuracy and efficiency
- None tensorial FEM approximation for the tensorial field
  - Robust solver w.r.t. the monolithic approach
  - Efficient solver w.r.t. multigird solver
Monolithic-multigrid linear solver

- Standard geometric multigrid solver

- Full $Q_r$ and $P_{r-1}^{\text{disc}}$ restriction and prolongation

- Local Multilevel Pressure Schur Complement via Vanka-like smoother

\[
\begin{pmatrix}
\sigma^{l+1} \\
u^{l+1} \\
p^{l+1}
\end{pmatrix} =
\begin{pmatrix}
\sigma^l \\
u^l \\
p^l
\end{pmatrix} + \omega^l \sum_{T \in \mathcal{T}_h} \begin{pmatrix}
(K_h + \mathcal{J}_u) \\
\mathcal{R}_{\sigma^l} \\
\mathcal{R}_{u^l} \\
\mathcal{R}_{p^l}
\end{pmatrix}_{|T}^{-1}
\begin{pmatrix}
\mathcal{R}_{\sigma^l} \\
\mathcal{R}_{u^l} \\
\mathcal{R}_{p^l}
\end{pmatrix}_{|T}
\]

Coupled Monolithic Multigrid Solver!
Two field formulation versus three field formulation

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<th>NL/LL</th>
<th>Lift</th>
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<tr>
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<td>8/1</td>
</tr>
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</table>

1Damanik. H "FEM Simulation of Non-isothermal Viscoelastic fluids", PhD Thesis

Consistency of the stabilization for three field formulation

<table>
<thead>
<tr>
<th>Level</th>
<th>α</th>
<th>No stabilization</th>
<th>With stabilization</th>
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<td>5</td>
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<td>0.010619</td>
<td>5.5794</td>
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</table>
Robustness and efficiency of the stabilization for three field formulation without any viscous contribution

<table>
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<th>With stabilization</th>
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<td>4</td>
<td>1</td>
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</tr>
</tbody>
</table>

Accurate, robust and efficient monolithic-multigrid Stokes solver in two-field and three-field formulations!
Incompressible N-S Equation)

\[ \rho(\Gamma) \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) u - \nabla \cdot \sigma_s + \nabla p = 0 \]

- **Viscous stress**

\[ \sigma_s = 2\eta_s(\Gamma)D(u) \]

- **Interface boundary conditions**

\[ [u]|_{\Gamma} = 0 \]

\[ -[pI + \sigma_s]|_{\Gamma} \cdot n = \sigma_k n \]
Multiphase flow problem

• New extra stress for multiphase flow

\[ \sigma_m = -\sigma \left( \frac{\nabla \psi \otimes \nabla \psi}{|\nabla \psi|} \right) \]

• Full set of equations for multiphase flow

\[
\begin{align*}
\rho(\psi) \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) - \nabla \cdot \tau + \nabla p &= 0 \\
\nabla \cdot u &= 0 \\
\tau - 2\eta_s(\psi)D(u) + \sigma \left( \frac{\nabla \psi \otimes \nabla \psi}{|\nabla \psi|} \right) &= 0 \\
\frac{\partial \psi}{\partial t} + u \cdot \nabla \psi + \nabla \cdot \left( \gamma_{nc}(1 - \psi) \frac{\nabla \psi}{|\nabla \psi|} \right) - \nabla \cdot \left( \gamma_{nd} \left( \nabla \psi \cdot \frac{\nabla \psi}{|\nabla \psi|} \right) \frac{\nabla \psi}{|\nabla \psi|} \right) &= 0
\end{align*}
\]
Monolithic approach

Oscillating bubble (by Aaqib Afaq)

Surface Area

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Robust nonlinear solver based on Newton’s method following a specific path of convergence using the residual’s convergence

- Robust w.r.t. starting guesses
- Dealing with Jacobian’s singularities using generalized derivatives or approximated one
- Full benefit from the quadratic convergence’s region of classical Newton’s method
Newton’s method

Let \( \mathcal{U} = (u, p) \), \((\sigma, u, p)\), or \((\sigma, u, p, \varphi)\) and \( \mathcal{R}_\mathcal{U}(\mathcal{U}) \) be the continuous or the discrete corresponding system’s residuum.

- Update of the nonlinear iteration with the correction \( \delta \mathcal{U} \) i.e.
  \[
  \mathcal{U}^{N} = \mathcal{U} + \delta \mathcal{U}
  \]

- The linearization of the residual provides
  \[
  \mathcal{R}_\mathcal{U}\left(\mathcal{U}^{N}\right) = \mathcal{R}_\mathcal{U}\left(\mathcal{U} + \delta \mathcal{U}\right)
  = \mathcal{R}_\mathcal{U}\left(\mathcal{U}\right) + \mathcal{J}\left(\mathcal{U}\right) \cdot \delta \mathcal{U}
  \]

- The Newton’s method assuming invertible Jacobian
  \[
  \mathcal{U}^{N} = \mathcal{U} - \mathcal{J}^{-1}\left(\mathcal{U}\right) \cdot \mathcal{R}_\mathcal{U}\left(\mathcal{U}\right)
  \]
Generalized Newton’s method

Jacobian calculations

$$\mathbf{J}(\mathbf{U}) = \left( \frac{\partial \mathbf{R}_u(\mathbf{U})}{\partial \mathbf{U}} \right)$$

- Exact G-Newton based on a priori study of Jacobian’s properties and decompositions

$$\mathbf{J}(\mathbf{U}) = \left( \frac{\partial \hat{\mathbf{R}}_u(\mathbf{U})}{\partial \mathbf{U}} \right) + \delta \left( \frac{\partial \tilde{\mathbf{R}}_u(\mathbf{U})}{\partial \mathbf{U}} \right)$$

- Inexact G-Newton based on the residuum’s convergence

$$\left( \frac{\partial \tilde{\mathbf{R}}_u(\mathbf{U})}{\partial \mathbf{U}} \right)_{ij} \approx \left( \frac{\mathbf{R}_u(\mathbf{U} + \epsilon^+ e_j) - \mathbf{R}_u(\mathbf{U} - \epsilon^- e_i)}{\mathbf{R}_u(\mathbf{U})} \right)$$

$$= \mathbf{R} = \hat{\mathbf{R}}, \tilde{\mathbf{R}}.$$
Three-field viscoplastic application

• **Viscoplastic constitutive law**

  ➢ **Bingham constitutive law**

  \[
  \begin{aligned}
  \sigma &= 2\eta D(u) + \tau_0 \frac{D(u)}{\|D(u)\|} \quad \text{if } \|D(u)\| \neq 0 \\
  \|\sigma\| &\leq \tau_0 \quad \text{if } \|D(u)\| = 0
  \end{aligned}
  \]

  ➢ **New extra stress** $\sigma_{Y_0}$ **for viscoplastic flow s.t.**

  \[
  \|D(u)\| \sigma_{Y_0} = D(u)
  \]

• **Three-field viscoplastic set of equations**

  \[
  \begin{aligned}
  \|D(u)\| \sigma_{Y_0} - D(u) &= 0 \quad \text{in } \Omega \\
  - \nabla \cdot \left( 2\eta D(u) + \tau_0 \sigma_{Y_0} \right) + \nabla p &= 0 \quad \text{in } \Omega \\
  \nabla \cdot u &= 0 \quad \text{in } \Omega
  \end{aligned}
  \]
Generalized Newton’s method

Dynamic path versus static one w.r.t. number of iterations, and the corresponding convergence of the residium (by Arooj Fatima)
Lid driven cavity benchmark

Unyielded zone for two different yield stresses, \( \tau_0 = 2 \), and \( \tau_0 = 5 \) (by Arooj Fatima)
Constitutive models

- **Generalized differential constitutive model**

\[
\sigma + We \frac{\partial \sigma}{\partial t} + G(\sigma, D) + H(\sigma) = 2\eta_p D(u)
\]

- **Oldroyd**

  \[
  G = 0, \quad H = 0
  \]

- **Giesekus**

  \[
  G = 0, \quad H = \alpha \text{tr}(\sigma^2)
  \]

- **Phan-Thien and Tanner**

  \[
  H = [\exp(\alpha \text{tr}(\sigma)) - 1] \sigma
  \]

- **White and Metzner**

  \[
  G = \alpha (2D : D)^{1/2}, \quad H = 0
  \]
Viscoelastic benchmark

- Planar flow around cylinder Oldroyd-B (by Hogenrich Damanik)
Quasi-Newtonian model for powder flow

- Experimental and numerical results for dry, frictional powder flows in the quasi-static and intermediate regimes

The numerical method do not introduce errors!
Quasi-Newtonian thixotropic model

- **Viscosity model for thixotropic flow i.e. extended viscosity defined on all domain s.t.**

\[
\eta_s(\|D(u)\|, \lambda) = \begin{cases} 
\eta(\lambda) + \tau(\lambda)\|D(u)\|^{-\frac{1}{2}} & \text{if } \|D(u)\| \neq 0 \\
\|\sigma_s\| \leq \tau(\lambda) & \text{else}
\end{cases}
\]

(\lambda : \text{structure parameter})

- **Structure equation**

\[
\frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda = a(1 - \lambda) - b\lambda z^{\frac{1}{2}}
\]

(\(a, b\) are structure parameters)

- **Full set of equations**

\[
\begin{cases}
\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) u - \nabla \cdot \left(2\eta_s(\|D(u)\|, \lambda)D(u)\right) + \nabla p = 0 & \text{in } \Omega \\
\nabla \cdot u = 0 & \text{in } \Omega \\
\frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda - a(1 - \lambda) + b\lambda\|D(u)\| = 0 & \text{in } \Omega
\end{cases}
\]
Thixotropic flow

Shear rate in a couette w.r.t. breakdown parameter (by Naheed Begum)

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Structure parameter in a couette w.r.t. breakdown (by Naheed Begum)

Boundary of the solid zone
✓ shear history effect
✓ time history effect
✓ Hysteresis
✓ stress overhoots

A quasi-Newtonian model for thixotropic phenomena via a time and shear dependent viscosity
Generalized quasi-Newtonian approach

- Include the non-Newtonian stress or any extra stress in diffusion operator
- Get rid of a tensorial field
  - Less constraints for the choices of FE approximation
  - Robust and efficient numerical algorithms
  - Simple evolution equations!
- Proof of the concept and validation
Laplacian operator

- **Divergence form**

\[ \mathcal{L} u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} u \frac{\partial}{\partial x_i} \right) \]

- **Weak form**

\[ \mathcal{L}_W u = \sum_{i,j=1}^{N} A_{ij} : \left( \nabla \cdot e_j \otimes \nabla \cdot e_i \right) u \]

Benefit of the weak form representation!
Stokes problem

Weak form representation 2D

\[ \mathcal{L}_W u = \sum_{k,l=1}^{2} \sum_{i,j=1}^{N} [A_{kl}]_{ij} : \left( \nabla \cdot e_j \otimes \nabla \cdot e_i \right) u_j^l, \quad k = 1, 2 \]

- **Gradient formulation**

  \[ A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

  \[ A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \]

- **Deformation formulation**

  \[ A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix} \]

  \[ A_{21} = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \]

Different derivatives combinations accessibilities!
Generalized quasi-Newtonian approach

Weak form representation 2D

$$\mathcal{L}_W u = \sum_{k,l=1}^{2} \sum_{i,j=1}^{N} [A_{kl}]_{ij} : \left( \nabla \cdot e_j \otimes \nabla \cdot e_i \right) u^l_j, \quad k = 1, 2$$

- Generalized formulation I

$$A_{11} = \begin{bmatrix}
    a_{11} & \frac{1}{2} a_{21} \\
    \frac{1}{2} a_{12} & \frac{1}{4} (a_{11} + a_{22})
\end{bmatrix}, \quad A_{12} = \begin{bmatrix}
    \frac{1}{2} a_{12} & \frac{1}{4} (a_{11} + a_{22}) \\
    0 & \frac{1}{2} a_{12}
\end{bmatrix},$$

$$A_{21} = \begin{bmatrix}
    \frac{1}{2} a_{21} & 0 \\
    \frac{1}{4} (a_{11} + a_{22}) & \frac{1}{2} a_{21}
\end{bmatrix}, \quad A_{22} = \begin{bmatrix}
    \frac{1}{4} (a_{11} + a_{22}) & \frac{1}{2} a_{21} \\
    \frac{1}{2} a_{12} & a_{22}
\end{bmatrix}$$

More derivatives combinations accessibilities are allowed!
Viscoelastic benchmark

- Planar flow around cylinder Oldroyd-B (by Hogenrich Damanik)

Drag coefficient planar flow around cylinder

We number

Drag coefficient

Genalized quasi-Newtonian approach for non-Newtonian problem i.e. Oldroyd-B!

A. Ouazzi | Generalized quasi-Newtonian approach for complex flows
New generalized quasi-Newtonian approach for modeling and simulating complex flows is introduced and validated. Based on new numerical and algorithmic tools using

- Monolithic FEM two-field and three-field Stokes solver
- Generalized Newton’s method w.r.t. singularities with global convergent property
- Edge Oriented stabilization (EO-FEM)
- Fast Multigrid Solver with local MPSC smoother

Extensively tested from numerical and physical perspectives via the simulations of different flow problems in different formulations to motivate the newly introduced generalized quasi-Newtonian approach.