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Unique continuation for the gradient and applications

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Divergence-type operators

Let $L \in \mathbb{N}$, $\Lambda_L := (-L/2, L/2)^d$ and let $A : \Lambda_L \to \mathrm{Sym}(d, \mathbb{R})$ satisfy $\vartheta_E^{-1}|\xi| \le \xi \cdot A\xi \le \vartheta_E|\xi|$ for all $\xi \in \mathbb{R}^d$. Consider the non-negative form

$$\mathfrak{h}(u, v) = \int_{\Lambda_L} \overline{\nabla u} \cdot A\nabla v \text{ with } \mathrm{Dom}(\mathfrak{h}) = H_0^1(\Lambda_L).$$

Self-adjoint operators and forms

$\mathfrak{h}$ is a densely defined, closed, non-negative form and consequently there exists a unique, non-negative self-adjoint operator H on $L^2(\Lambda_L)$ with compact resolvent associated with $\mathfrak{h}$, i.e.

$$Hu = f \Leftrightarrow u \in \mathrm{Dom}(\mathfrak{h}) \text{ and } \mathfrak{h}(u, v) = (f, v) \text{ for all } v \in \mathrm{Dom}(\mathfrak{h}). \qquad (1)$$

- For the operator H as defined above we simply write $H = -\mathrm{div}A\nabla_{|L}$.
- In general the domain $\mathrm{Dom}(H)$ **does not** contain $\mathcal{C}_c^2(\Lambda_L)$-functions.
- By $(E_n(H))_{n\in\mathbb{N}}$ we denote the eigenvalues of H enumerated non-decreasingly and counting multiplicities.

Eigenvalue lifting

Setup: Let $\delta < \frac{1}{2}$, $(z_j)_{j\in\mathbb{Z}^d}$ be an $(1, \delta)$-equidistributed sequence, i.e. $B(z_j, \delta) \subseteq \Lambda_1(j)$ for all $j \in \mathbb{Z}^d$, and let $L^\infty(\Lambda_L) \ni W \ge \mathbb{1}_S$, where

$$S := \cup_{j\in\Lambda_L\cap\mathbb{Z}^d} B(z_j, \delta) \subseteq \Lambda_L \qquad (2)$$

Consider the family of operators

$$H_t = -\mathrm{div}(A + t\,W\mathrm{Id})\nabla_{|L} \text{ for } t \in [0, 1]$$

where A is a uniformly elliptic and Lipschitz-continuous matrix function.

Most interesting case: δ small, $W = \mathbb{1}_S$.

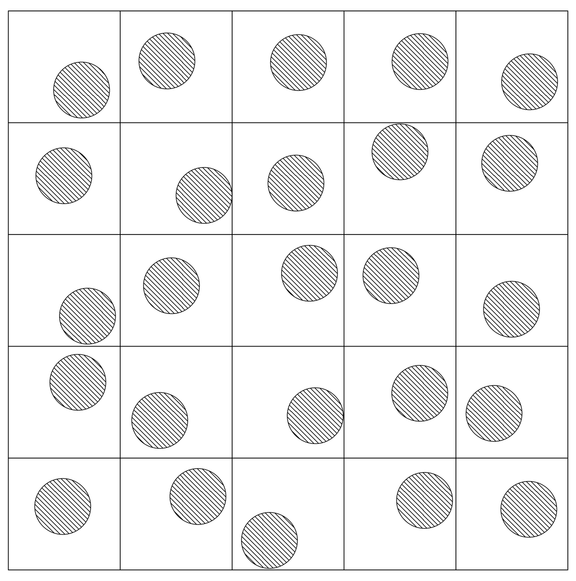

Figure 1: The set S where $W \ge 1$

How do eigenvalues move?

We want to prove that

$$E_n(H_t) \ge E_n(H_0) + t\,C_{\mathrm{evl}}. \qquad (3)$$

for all n such that $0 < E_- \le E_n(0) \le E_n(1) \le E_+ < \infty$.

Strategy: Assume that $E_n(H_t) \in [E_-, E_+]$ for all $t \in [0, 1]$. Use the fundamental theorem of calculus to write

$$E_n(H_t) = E_n(H_0) + \int_0^t \partial_r E_n(H_r)\,\mathrm{d}r.$$

The derivative exists for almost every r and satisfies

$$\partial_r E_n(H_r) = \langle W\nabla u_n(r)|\nabla u_n(r)\rangle_{L^2(\Lambda_L)} \ge \|\nabla u_n(r)\|^2_{L^2(S)}, \qquad (4)$$

where $u_n(r)$ is a normalized eigenfunction to the eigenvalue $E_n(H_r)$.

Problem: Show that (4) is bounded from below by some constant that does not depend on r.

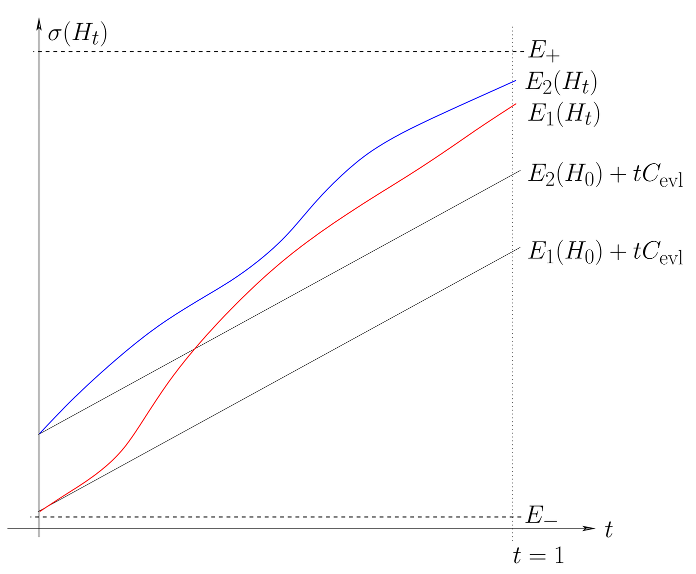

Figure 2: Possible movement of eigenvalues with the lower bound given in (3).

Simplest case: If $W \ge 1$ on the whole cube Λ_L, the inequality (3) easily follows from the the definition (1).

Conclusion: Small support of W poses a difficulty! If we can find a constant $C_{\mathrm{evl}} > 0$ (not depending on r) such that

$$\|\nabla u_n(r)\|^2_{L^2(S)} \ge C_{\mathrm{evl}}\,\|u_n(r)\|^2_{L^2(\Lambda_L)}, \qquad (5)$$

the desired inequality (3) follows.

Schrödinger vs. Divergence-type

Compare (4) with the corresponding inequality for the Schrödinger operator

$$\tilde{H}_t = -\Delta + t\,W_{|L}.$$

In this case: Derivatives satisfy

$$\partial_r E_n(\tilde{H}_r) = \langle W\tilde{u}_n(r)|\tilde{u}_n(r)\rangle_{L^2(\Lambda_L)} \ge \|\tilde{u}_n(r)\|^2_{L^2(S)},$$

where $\tilde{u}_n(r)$ is a normalized eigenfunction to the eigenvalue $E_n(\tilde{H}_r)$ – as above: easier for $W \ge 1$ on Λ_L.

Reduces to: Quantitative Unique Continuation Principle

- This is well known for Schrödinger operators, see e.g. [1].
- The eigenvalue lifting (3) is an important ingredient in the proof of the Wegner estimate for random Schrödinger operators.

Idea: Use similar arguments for divergence-type operators!

- Prove a quantitative unique continuation principle for the gradient as in (5) and
- adapt the proof of the Wegner estimate for random Schrödinger operators to suit random divergence-type operators.

Unique continuation for the gradient

Theorem

Assume that the matrix function A is uniformly elliptic, Lipschitz-continuous and that all of its non-diagonal entries vanish on $\partial\Lambda_L$. Let $H = -\mathrm{div}A\nabla_{|L}$ and let $0 < E_- < E_+ < \infty$. Then there exists a constant $C_{\mathrm{uc}}^\nabla > 0$ that does not depend on L, such that for every $u \in \mathrm{Dom}(H)$ satisfying $Hu = Eu$ for some $E \in [E_-, E_+]$ we have

$$\|\nabla u\|^2_{L^2(S)} \ge C_{\mathrm{uc}}^\nabla\,\|u\|^2_{L^2(\Lambda_L)}.$$

- The proof combines unique continuation estimate for elliptic second order operators due to [3] with a, in some sense, reverse Caccioppoli-type inequality motivated by [2]. The former is the reason we have to assume $A \in \mathrm{Lip}(\Lambda_L)$. The latter uses only the equivalence (1) but is not very flexible.
- The constant C_{uc}^∇ is explicitly known and polynomial in δ.
- Qualitative unique continuation for the gradient was examined in [2].

Wegner estimate

Fix $C > 0$, $m > 0$, $\delta \in (0, 1/2)$ and define a random perturbation

$$V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \omega_j u_j(x)$$

where

- $(\omega_j)_{j\in\mathbb{Z}^d}$ a sequence of independent random variables with uniformly bounded probability densities ρ_j satisfying $\mathrm{supp}\,\rho_j \subseteq [0, m]$,
- $(z_j)_{j\in\mathbb{Z}^d}$ a sequence of points in $\mathbb{R}^d$ such that $B(z_j, \delta) \subseteq \Lambda_1(j)$ for all $j \in \mathbb{Z}^d$,
- $(u_j)_{j\in\mathbb{Z}^d}$ a sequence of measurable functions satisfying $\mathbb{1}_{B(z_j,\delta)} \le u_j \le C\,\mathbb{1}_{\Lambda_1(j)}$,

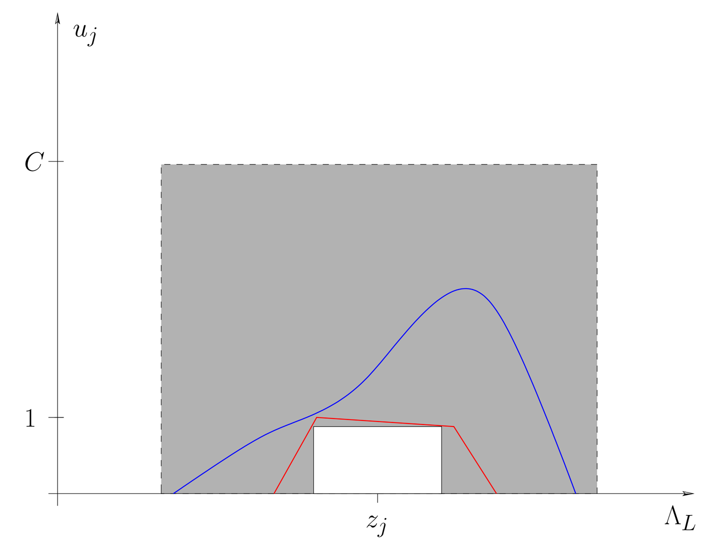

Figure 3: Two possible single-site perturbations u_j. The graphs have to stay in the grey area.

Important observation: $W := \sum_{j\in\mathbb{Z}^d} u_j \ge \mathbb{1}_S$, where S is the set defined in (2).

Random operator:

$$H_\omega = -\mathrm{div}(1 + V_\omega)\nabla_{|L}.$$

Adapting the proof of the Wegner estimate for random Schrödinger operators, we prove:

Wegner estimate

Let $0 < E_- < E_+ < \infty$. There is a constant $C_W > 0$ such that for every $\varepsilon > 0$ and $E > 0$ satisfying $[E - 3\varepsilon, E + 3\varepsilon] \subseteq [E_-, E_+]$ we have

$$\mathbb{E}\left[\mathrm{Tr}\,\chi_{[E-\varepsilon,E+\varepsilon]}(H_\omega)\right] \le C_W\,\varepsilon\,|\Lambda_L|^2.$$

Further research goal

Upgrade of the UCP for the gradient: Our method applies only to eigenfuctions! We conjecture:

Conjecture

Assume that the matrix function A is uniformly elliptic and Lipschitz-continuous. Let $H = -\mathrm{div}A\nabla_{|L}$ and let $0 < E_- < E_+ < \infty$. Then there exists a constant $C_{\mathrm{uc}}^\nabla > 0$ that does not depend on L, such that for every $u \in \mathrm{Ran}\,\chi_{[E_-,E_+]}(H)$ we have

$$\|\nabla u\|^2_{L^2(S)} \ge C_{\mathrm{uc}}^\nabla\,\|u\|^2_{L^2((\Lambda_L)}.$$

Upgrade of the Wegner estimate: To the best of our knowledge, the Wegner estimate given above is the first one for random divergence-type operators where the single-site perturbations u_j have small support. However, it is quadratic in the volume of Λ_L. We aim to prove a Wegner estimate linear in the volume of Λ_L.

References

[1] I. Nakić, M. Täufer, M. Tautenhahn and I. Veselić, *Scale-free unique continuation principle for spectral projectors, eigenvalue-lifting and Wegner estimates for random Schrödinger operators*, Anal. PDE 11, No. 4, 1049–1081 (2018).

[2] M. N. Nkashama, *Unique continuation on the gradient for second order elliptic equations with lower order terms*, J. Comput. Anal. Appl. 12, No. 1B, 293–304 (2010).

[3] M. Tautenhahn and I. Veselić, *Sampling and equidistribution theorems for elliptic second order operators, lifting of eigenvalues, and applications.* 2019. Submitted.

[4] P. Stollmann, *Localization for random perturbations of anisotropic periodic media*, Isr. J. Math. 107, 125–139 (1998).