Observability from sensor sets with decaying density

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Observability

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Definition

(ACP) *observable* in time $T > 0$ from ω if

$$\exists C_{\mathrm{obs}} > 0\colon \quad \|\mathcal{T}(T)g\|_{L^2(\mathbb{R}^d)}^2 \leq C_{\mathrm{obs}}^2 \int_0^T \|\mathbf{1}_\omega \mathcal{T}(t)g\|_{L^2(\mathbb{R}^d)}^2 \,\mathrm{d}t, \quad g \in L^2(\mathbb{R}^d). \tag{Obs}$$

Theorem (Lebeau-Robbiano method)

If there is a family $(P_\lambda)_{\lambda\geq 1}$ *of orthogonal projections such that for all* $g \in L^2(\mathbb{R}^d)$

$$\|P_\lambda g\|^2_{L^2(\mathbb{R}^d)} \leq \mathrm{e}^{C_1\lambda^{\gamma_1}}\|P_\lambda g\|^2_{L^2(\omega)}, \quad \gamma_1 \in (0,1),\ \lambda \geq 1, \tag{S}$$

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$$\|P_\lambda^\perp \mathcal{T}(t) g\|^2_{L^2(\mathbb{R}^d)} \leq C_2 \mathrm{e}^{-C_2\lambda t^{\gamma_3}}\|g\|^2_{L^2(\mathbb{R}^d)} \quad \gamma_3 > 0,\ 0 < t \ll 1, \tag{D}$$

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Lemma

A negative & selfadjoint $\Rightarrow$ (D) *holds with* $P_\lambda = P_{-A}(\lambda)$.

Observability for non-selfadjoint operators

Strategy using the Lebeau-Robbiano method:

- Find a suitable selfadjoint *comparison* operator H and set $P_\lambda = P_H(\lambda)$.
- Prove the dissipation estimate (D).
- Prove the spectral inequality for the spectral projections (P_λ).

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$\Rightarrow$ Comparison operator determines possible sensor sets!

A class of semigroup generators

Let $q\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ complex quadratic polynomial, $\Re q \leq 0$.

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Example

- Let $\mathcal{I} \subset \{1, \ldots, d\}$ and $q(x, \xi) = -|\xi|^2 - \sum_{i\in\mathcal{I}} x_i^2 = -|\xi|^2 - |x_{\mathcal{I}}|^2$. Then $q^{\mathrm{w}} = \Delta - |x_{\mathcal{I}}|^2 =$ (negative) *partial harmonic oscillator*, $S^{\perp} = \mathbb{R}^d_{\mathcal{I}} \times \mathbb{R}^d$.

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(ii) Let $d = 2$, $\tilde{q}(x, \xi) = -\xi_2^2 - x_2^2/4 - \mathrm{i}x_2 \cdot \xi_1$. Then $\tilde{q}^{\mathrm{w}} = \partial_{x_2}^2 - x_2^2/4 - x_2 \cdot \partial_{x_1} =$ *Kramers-Fokker-Planck operator without external potential*, $S^{\perp} = \mathbb{R}^2_{\{2\}} \times \mathbb{R}^2$.

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$\Rightarrow$ $S(q) = S(\tilde{q})$ for $d = 2$, $\mathcal{I} = \{2\}$.

Theorem (D.-Seelmann-Veselić)

Let $\mathcal{I} \subset \{1, \dots, d\}$, $S(A) \subset \mathbb{R}^d_{\mathcal{I}^\complement} \times \{0\}$, *and* $P_\lambda = P_{-\Delta+|x_{\mathcal{I}}|^2}(\lambda)$.
Then for all $0 < t \ll 1$, *some* $\gamma_3 \geq 1$ *we have*

$$\|P_\lambda^\perp \mathcal{T}(t) g\|^2_{L^2(\mathbb{R}^d)} \leq C_2 \mathrm{e}^{-C_2 \lambda t^{\gamma_3}} \|g\|^2_{L^2(\mathbb{R}^d)}, \quad g \in L^2(\mathbb{R}^d). \qquad (*)$$

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Example

Semigroup $(\mathcal{T}(t))_{t \geq 0}$ generated by Kramers-Fokker-Planck operator satisfies $(*)$ with $P_\lambda = P_{-\Delta + |x_2|^2}(\lambda)$ since

$$S\big(\text{Kramers-Fokker-Planck operator}\big) = \mathbb{R}^2_{\{1\}} \times \{0\}.$$

Question

Fast decay of $f \in \operatorname{Ran} P_{-\Delta+|x_{\mathcal{I}}|^2}(\lambda)$ in $x_{\mathcal{I}}$-direction. Do we see this in the sensor set?

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Corollary (Everything is comparable with the Laplacian)

(ACP) *with* $S(A) \subset \mathbb{R}^d \times \{0\}$ *observable from thick sets.*

Spectral inequalities on $\mathbb{R}^d$: Previous results

$$\forall f \in \operatorname{Ran} P_{-A}(\lambda) \colon \|f\|^2_{L^2(\mathbb{R}^d)} \leq \mathrm{e}^{C_1 \lambda^{\gamma_1}} \|f\|^2_{L^2(\omega)}, \quad \gamma_1 \in (0,1). \tag{S}$$

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- [Kov01,EV18,WWZZ19]: For $A = \Delta$ we have (S) with $\gamma_1 = 1/2$, ω *thick*, i.e.,

$$\exists \rho > 0\colon \quad \frac{|\omega \cap B(x,\rho)|}{|B(x,\rho)|} \geq \text{const.} \quad \text{for all} \quad x \in \mathbb{R}^d.$$

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- [NTTV20, DRST]: For $A = \Delta - V$ (V with mild local singularities) we have (S) with $\gamma_1 = 1/2$, ω *equidistributed*, i.e.,

$$\exists \delta \ll 1\colon \quad \omega \supset \bigcup_{k \in \mathbb{Z}^d} B(z_k, \delta), \quad z_k \in \Lambda_1(k).$$

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- [LM]: For $A = \Delta - V$ (V real-valued, suitably analytic & vanishing at ∞) we have (S) with $\gamma_1 = 1/2$, ω thick.

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Here $\gamma_1 = 1 - \varepsilon/2$.

Spectral inequality: Motivation for the harmonic oscillator

Quantification of fast decay:

Lemma

Let $f \in \operatorname{Ran} P_{-\Delta+|x|^2}(\lambda)$. *Then*

$$\|f\|_{L^2(\mathbb{R}^d \setminus B(0,7\sqrt{\lambda}))} \leq \frac{1}{2} \cdot \|f\|_{L^2(\mathbb{R}^d)}.$$

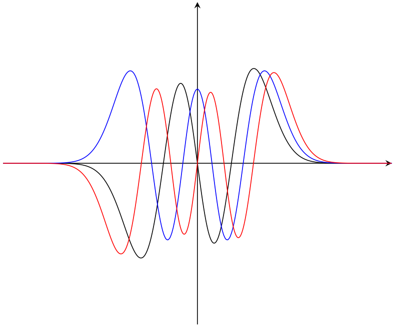

Figure: Functions in the spectral subspace (here: Hermite functions)

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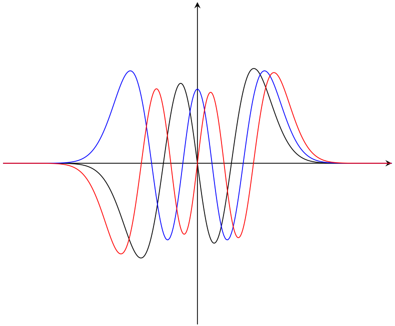

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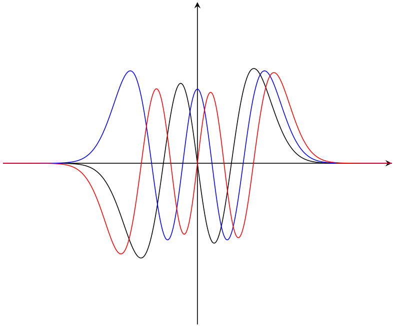

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$\Rightarrow$ Strong localization.

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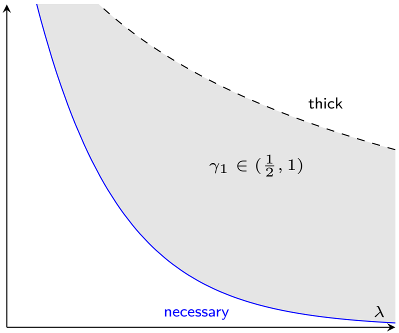

Figure: $\lambda \mapsto \inf_{f\in \mathrm{Ran}\, P_\lambda} \frac{\|f\|_{L^2(\omega)}}{\|f\|_{L^2(\mathbb{R}^d)}}$

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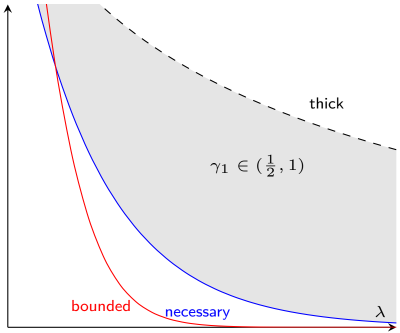

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Example

ω bounded, $\mathbb{N} \ni N \approx \lambda$,

$$f(x) = x^N \mathrm{e}^{-|x|^2/2} \in \operatorname{Ran} P_\lambda.$$

Then

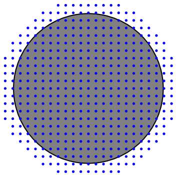
$$\frac{\|f\|^2_{L^2(\omega)}}{\|f\|^2_{L^2(\mathbb{R}^d)}} \approx \mathrm{e}^{-\lambda \log \lambda}.$$

$\Rightarrow$ No spectral inequality possible!

Generalized spectral inequality

Fix $\lambda \geq 1$ and let

$$\mathcal{K}_c(\lambda) = \{k \in \mathbb{Z}^d : |k| \leq 10 \cdot \sqrt{\lambda}\}$$
$$\approx \text{centers of covering of the region carrying half of the mass of } f \in \operatorname{Ran} P_\lambda \text{ by } B(k) = B(k, \sqrt{d}).$$



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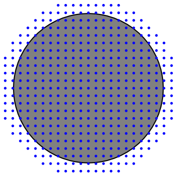

Theorem

Fix $\lambda \geq 1$. If

$$\exists \alpha \geq 0 \forall k \in \mathcal{K}_c(\lambda): \quad \frac{|\omega \cap B(k)|}{|B(k)|} \geq \exp\Big(-\mathit{const.} \cdot \lambda^{\alpha/2}\Big), \qquad (\text{Thick}_\lambda)$$

then

$$\forall f \in \operatorname{Ran} P_\lambda : \quad \frac{\|f\|^2_{L^2(\omega)}}{\|f\|^2_{L^2(\mathbb{R}^d)}} \geq \exp\Big(-\mathit{const.} \cdot \lambda^{\frac{1}{2}+\frac{\alpha}{2}}\Big).$$

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Lemma

If

$$\forall k \in \mathbb{Z}^d : \quad \frac{|\omega \cap B(k)|}{|B(k)|} \geq \exp\Big(-(1 + |k|^\alpha)\Big)$$

then

$$\forall \lambda \geq 1,\ k \in \mathcal{K}_c(\lambda) : \quad \frac{|\omega \cap B(k)|}{|B(k)|} \geq \exp\Big(-\mathit{const.} \cdot \lambda^{\alpha/2}\Big).$$

Spectral inequality for the harmonic oscillator

Theorem (D.-Seelmann-Veselić)

If there is $\alpha \in [0,1)$ *such that*

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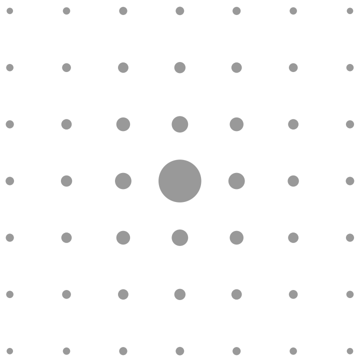

Figure: $\omega = \bigcup_{k \in \mathbb{Z}^d} B(k, \mathrm{e}^{-(1+|k|^\alpha)})$

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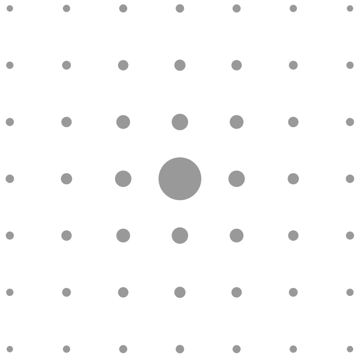

Figure: $\omega = \bigcup_{k\in\mathbb{Z}^d} B(k, \mathrm{e}^{-(1+|k|^\alpha)})$

Remark

If $\alpha > 0$, then ω has finite measure!

Spectral inequality for the harmonic oscillator

Let $f \in \operatorname{Ran} P_\lambda$.

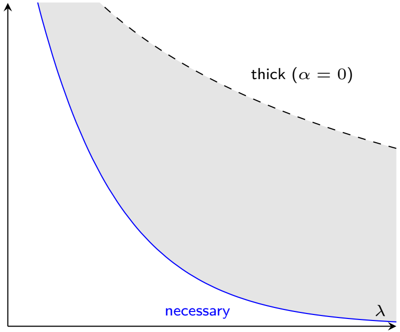

Figure: $\lambda \mapsto \inf_{f \in \operatorname{Ran} P_\lambda} \frac{\|f\|_{L^2(\omega)}}{\|f\|_{L^2(\mathbb{R}^d)}}$

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$\Rightarrow$ Spectral inequality with sublinear dependence on λ possible.

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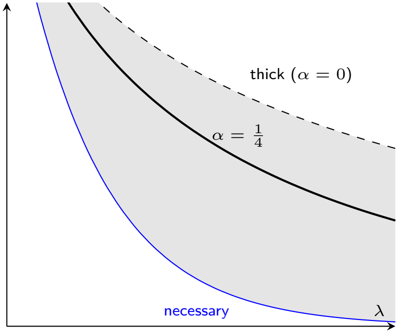

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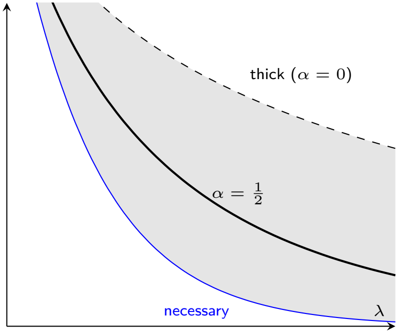

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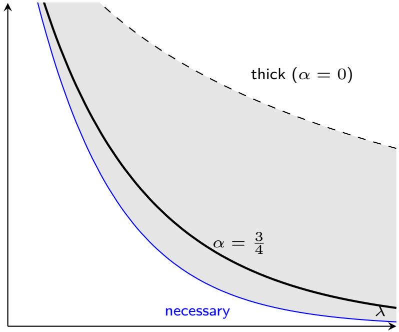

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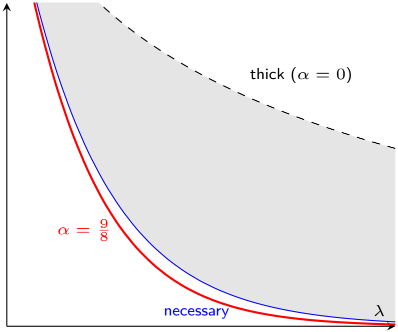

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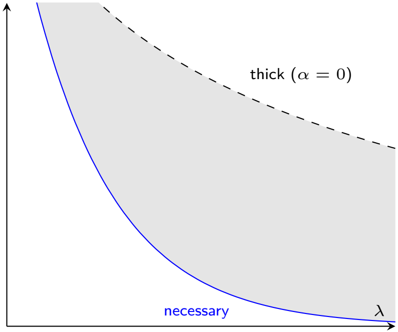

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$\Rightarrow$ Spectral inequality with sublinear dependence on λ possible.

Question

$$\omega = \bigcup_{k \in \mathbb{Z}^d} B(k, \mathrm{e}^{-(1+|k|)}) \quad \overset{?}{\Longrightarrow} \quad \inf_{f \in \operatorname{Ran} P_\lambda} \frac{\|f\|^2_{L^2(\omega)}}{\|f\|^2_{L^2(\mathbb{R}^d)}} \approx \mathrm{e}^{-\text{const.}\cdot\lambda}$$

Spectral inequality for the partial harmonic oscillator

For partial harmonic oscillator decay in encoded by $\mathcal{I}$.

Theorem

If there is $\alpha \in [0,1)$ *such that*

$$\frac{|\omega \cap B(k)|}{|B(k)|} \geq \mathrm{e}^{-(1+|k_{\mathcal{I}}|^\alpha)}, \quad k \in \mathbb{Z}^d,$$

Then

$$\frac{\|f\|^2_{L^2(\omega)}}{\|f\|^2_{L^2(\mathbb{R}^d)}} \geq \exp\left(-const. \cdot \lambda^{\frac{1}{2}+\frac{\alpha}{2}}\right)$$

for all $\lambda \geq 1$, $f \in \operatorname{Ran} P_{-\Delta+|x_{\mathcal{I}}|^2}(\lambda)$.

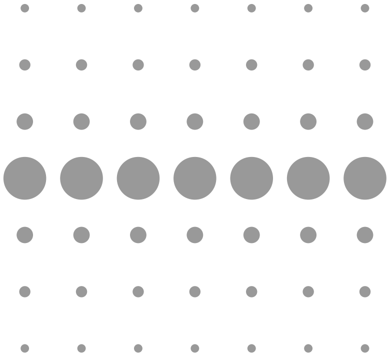

Figure: $\omega = \bigcup_{k\in\mathbb{Z}^d} B(k, \mathrm{e}^{-(1+k_2^\alpha)})$.

Extensions

Theorem

Let $R > 0$ and let $0 < \rho(x) \leq R(1+|x|^2)^{(1-\varepsilon)/2}$. If for some $\varepsilon \in (0,1]$ and $\alpha \in [0,\varepsilon)$

$$\frac{|\omega \cap B(x,\rho(x))|}{|B(x,\rho(x))|} \geq \exp\Big(-(1+|x|^\alpha)\Big) \quad \text{for all} \quad x \in \mathbb{R}^d,$$

Then

$$\frac{\|f\|^2_{L^2(\omega)}}{\|f\|^2_{L^2(\mathbb{R}^d)}} \geq \exp\Big(-const. \cdot \lambda^{1-(\varepsilon-\alpha)/2}\Big)$$

for all $\lambda \geq 1$, $f \in \mathrm{Ran}\, P_{-\Delta+|x|^2}(\lambda)$.

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Let $R > 0$ *and let* $0 < \rho(x) \leq R(1 + |x|^2)^{(1-\varepsilon)/2}$. *If for some* $\varepsilon \in (0, 1]$ *and* $\alpha \in [0, \varepsilon)$

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$\Rightarrow$ Similar result for the partial harmonic oscillator...

Corollary

If $S(A) \subset \mathbb{R}^d_{\mathcal{I}^\complement} \times \{0\}$, then (ACP) *is observable from any set ω satisfying*

$$\frac{|\omega \cap B(k)|}{|B(k)|} \geq \mathrm{e}^{-(1+|k_{\mathcal{I}}|^\alpha)}, \quad k \in \mathbb{Z}^d. \tag{$*$}$$

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Remark

Can be adapted to sets satisfying $(*)$ with general bounded, convex & open covering elements.

What happens for more general "*confinement potentials*"?

Confinement potentials

Consider the selfadjoint Schrödinger operator

$$H = -\Delta + |x|^{\tau} \quad \text{for} \quad \tau > 0.$$

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Theorem (Miller, Alphonse, Martin, D.-Seelmann)

If $\tau = 2k$ with $k \in \mathbb{N} \setminus \{1\}$ then the (ACP) *is null-controllable from*

- *cones of the form*

$$\omega = \{x \in \mathbb{R}^d \colon |x| \geq r_0 \text{ and } x/|x| \in \Omega_0\} \quad \text{for some open} \quad \Omega_0 \subset \mathbb{S}^{d-1},\ r_0 > 0,$$

- *thick sets with variable scale (upper bound depends on τ).*

Confinement potentials

We expect that

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Question

Are there numbers $p, q > 0$ such that

$$\|\mathrm{e}^{|x|^p} f\|_{L^2(\mathbb{R}^d)} \lesssim \mathrm{e}^{\lambda^q} \|f\|_{L^2(\mathbb{R}^d)} \qquad (*)$$

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$$(*) \quad \Rightarrow \quad \|f\|_{L^2(\mathbb{R}^d \setminus B(0, C\lambda^{q/p}))} \leq \frac{1}{2} \cdot \|f\|_{L^2(\mathbb{R}^d)}.$$

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Remark

For $\tau = 2$ we have $\frac{q}{p} = \frac{1}{\tau} = \frac{1}{2}$ (with $p = 2$ and $q = 1$).

Conjecture

If there is $0 \leq \alpha < \tau/2$ *with*

$$\frac{|\omega \cap B(k)|}{|B(k)|} \geq \exp\Big(-(1 + |k|^{\alpha})\Big), \quad k \in \mathbb{Z}^d,$$

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Question

Class of semigroups where $-\Delta + |x|^\tau$ is the selfadjoint comparison operator?

Thank you!